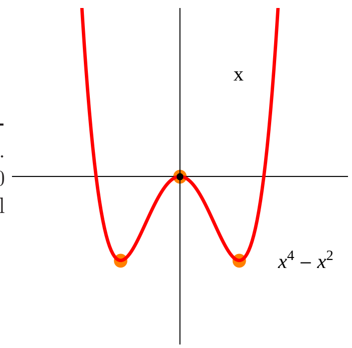
Lecture 17: Catastrophes

In this lecture, we once more cover extrema problems. We are interested how extrema change when a parameter changes. Nature, economies, processes favor extrema. Extrema change smoothly with parameters. How come that the outcome is often not smooth? What is the reason that political change can go so fast once a tipping point is reached? One can explain this with mathematical models. We look at a simple example, which explains it. In reality, the situation is more complicated. In the "New York Times" of February 24, 2011, **Jennifer E. Sims**, the director of intelligence studies at Georgetown University's School of Foreign Service and senior fellow at the Chicago Council on Global Affairs asked: *Why, with the U.S. spending 80 billion dollars on intelligence, were we apparently surprised by recent regime changes in the Middle East?* Why did change happen at all? These are complex questions. Obviously, some tipping point has been reached and the smallest event like the confiscation of a fruit stand in Tunesia or increasing food prizes in Egypt has produced change. In these complex examples, it will never be possible to understand everything. Lets look here at a simple mathematical model which illustrates the general principle that:

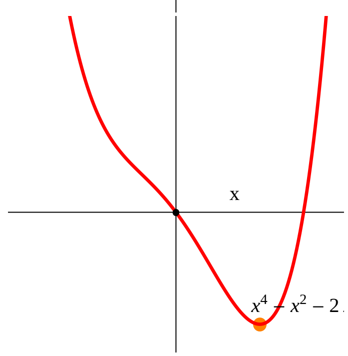
> If a local minimum seizes to become a local minimum, a new stable position is favored. This can be far away from the original situation.

To get started, lets look at an extremization problem

1 Find all the extrema of the function $f(x) = x^4 - x^2$. **Solution:** $f'(x) = 4x^3 - 2x$ is zero for $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$. The second derivative is $12x^2 - 2$. It is negative for $x = 0$ and positive at the other two points. We have two local minima and one local maximum.



2 Now find all the extrema of the function $f(x) = x^4 - x^2 - 2x$. There is only one critical point. It is $x = 1$.

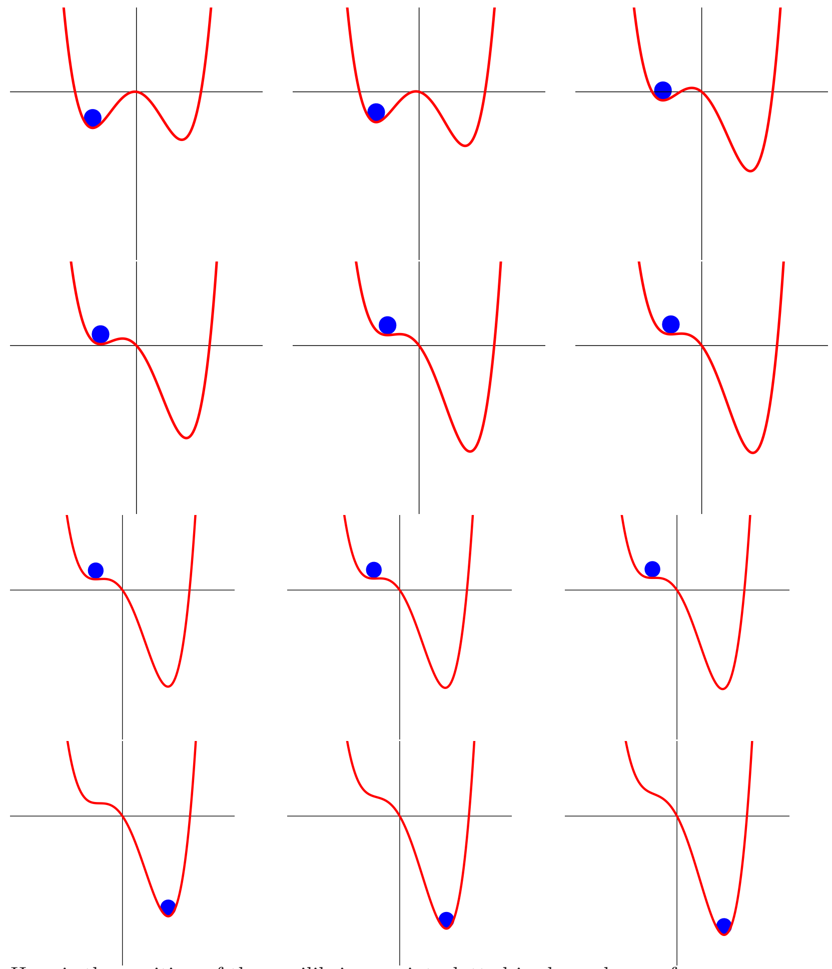


Something has happened from the first example to the second example. The local minimum to the left has disappeared. Assume the function f measures the prosperity of some kind and c is

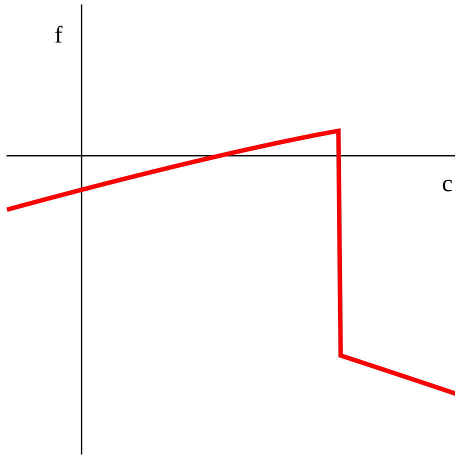
a parameter. We look at the position of the first equilibrium point of the function. Catastroph theorists usually assume the so called **Delay assumption**.

> A **stable equilibrium** is here used as an other name for a local minimum. A system state remains in a stable equilibrium until it disappears. If that happens, the system settles in a neighboring stable equilibrium.

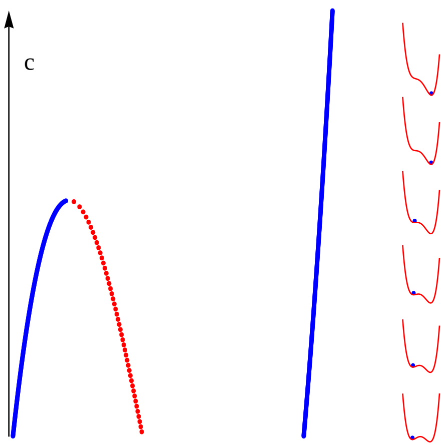
> A parameter value for which a stable minimum disappears is called a **catastrophe**.



Here is the position of the equilibrium point plotted in dependence of c.



A parameter value for which a local minimum disappears is called a **catastrophe**.



Bifurcation diagram: The picture shows the equilibrium points as they change in dependence of the parameter c. The vertical axes is the parameter c, the horizontal axes is x. At the bottom for $c = 0$, we have three equilibrium points, two local minima and one local maximum. At the top for $c = 1$ we have only one local minimum.

Catastrophes always go for the worse in the sense that the value decreases. It is not possible to reverse the process and have a catastrophe, where the minimum jumps up.

Look again at the above "movie" of graphs. But run it backwards and use the same principle. We do not end up at the position we started with. The new equilibrium stays the equilibrium. Decreasing the food prizes again did not reverse the process of change in Egypt for example.

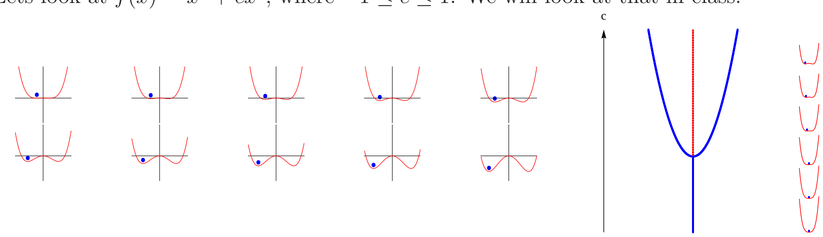
Catastrophes are in general irreversible.

We see that in real life: It is easy to screw up a relationship, get sick, have a ligament torn or lose trust. Building up a relationship, getting healthy or gaining trust on the other hand happens

slowly. Ruining a country or a company or losing a good reputation of a brand is very easy. It takes a long time to regain it.

Local minima can change discontinuously, when a parameter is changed. This can happen with perfectly smooth functions and smooth parameter changes.

3 Lets look at $f(x) = x^4 + cx^2$, where $-1 \leq c \leq 1$. We will look at that in class.



Homework

In this homework, we study a catastrophe for the function

$$f(x) = x^6 - x^4 + cx^2 ,$$

where c is a parameter between 0 and 1.

1 a) Find all the critical points in the case $c = 0$ and analyze their stability. b) Find all the critical points in the case $c = 1$ and analyze their stability.

2 Plot the graph of f for at least 10 values of c between 0 and 1. You can of course use software, a graphing calculator or Wolfram alpha. Mathematica code is below.

3 If you change from $c = -0.3$ to 0.6 pinpoint the value for the catastrophe and show a rough plot of $c \to f(x_c)$, the value at the first local minimum x_c in dependence of c. The text above provides this graph for an other function. It is the graph with a discontinuity.

4 If you change back from $c = 0.6$ to 0.3 pinpoint the value for the catastrophe (it will be different from the one in the previous question).

5 Sketch the bifurcation diagram. That is, if $x_k(c)$ is the k'th equilibrium point, then draw the union of all graphs of $x_k(c)$ as a function of c (the c-axes pointing upwards). As in the two example provided, draw the local maximum with dotted lines.

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Manipulate[ Plot[x^6 - x^4 + c x^2, {x, -1, 1}], {c, 0, 1}]
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