Lecture 18: Riemann integral

In this lecture we define the integral $\int_0^x f(t)\, dt$ if f is a differentiable function and compute it for some basic functions.

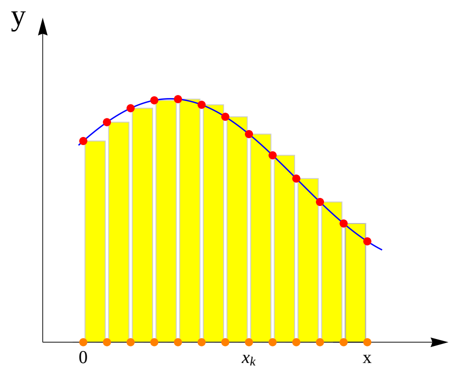
First a reminder. We have defined the **Riemann sums**

$$Sf(x) = h[\, f(0) + f(h) + f(2h) + ... + f(kh)]\ ,$$

where k is the largest integer such that $kh < x$. Lets write S_n if we want to stress that the parameter $h = 1/n$ was used in the sum. We define the integral as the limit of these sums $S_n f$ when the **mesh size** $h = 1/n$ goes to zero.

Define

$$\int_0^x f(t)\, dt = \lim_{n\to 0} S_n f(x)\ .$$



For any differentiable function, the limit exists

Proof: Lets first assume $f \geq 0$ on $[0, x]$. Let M be such that $f \leq M$ and $f' \leq M$ on $[0, x]$. The Riemann sum $S_n f(x)$ is the total area of k rectangles. Let $Sf(x)$ denote the area under the curve. If M is the maximal slope of f on $[0, x]$, then on each interval $[j/n, (j + 1)/n]$, we have $|f(x) - f(j/n)| \leq M/n$ so that the area error is smaller than M/n^2. Additionally, we have a piece above the interval $[kh, x]$ with area $\leq M/n$. If we add all the $k \leq xn$ "roof area errors" and the "side area" up, we get

$$Sf(x) - S_n f(x) \leq \frac{kM}{n^2} + \frac{M}{n} \leq \frac{xnM}{n^2} + \frac{M}{n} = \frac{xM + M}{n}\ .$$

This converges to 0 for $n \to \infty$. The limit is therefore the area $Sf(x)$. For a general, not necessarily nonnegative function, we write $f = g - h$, where g, h are nonnegative (see homework) and have $\int_0^x f(x)\, dx = \int_0^x g(x)\, dx - \int_0^x h(x)\, dx$.

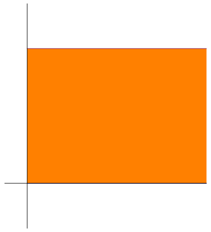
For nonnegative f, the value $\int_0^x f(x)\, dx$ is the **area between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

Remark: the Riemann integral is defined here as the limit $h \sum_{x_k = kh \in [0,x)} f(x_k)$. It converges to the area under the curve for all **continuous** functions but since we work with differentiable functions in calculus we restricted to that. Not **all** bounded functions can be integrated naturally

like this. There are discontinuous functions like the **salt and pepper function** which is defined to be $f(x) = 1$ if x is rational and zero else. Now $Sf(x) = 1$ for rational h and $Sf(x) = 0$ if h is irrational. Therefore, an other integral, the **Lebesgue integral** is used too: it can be defined as the limit $\frac{1}{n}\sum_{k=1}^n f(x_k)$ where x_k are **random points** in $[0, x]$. This **Monte-Carlo integral** definition of the Lebesgue integral gives the integral 0 for the salt and pepper function because rational numbers have zero probability.

Remark: Many calculus books define the Riemann integral using partitions $x_0 < x_1 < ..., x_n$ of points of the interval $[0, x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring x_j goes to zero. The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$, where y_k is arbitrarily chosen inside the interval (x_k, x_{k+1}). For continuous functions, the limiting result is the same the $Sf(x)$ sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated and keep a wide mesh, where the function does not change much. This leads to **numerical analysis** of integrals.

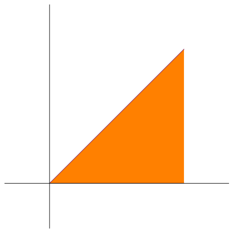
1 Let $f(x) = c$ be constant everywhere. Now $\int_0^x f(t)\, dt = cx$. We can see also that $cnx/n \leq S_n f(x) \leq c(n + 1)x/n$.



2 Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Remark: we could also have added up the Riemann sum but thats more painful: for every $h = 1/n$, let k be the largest integer smaller than $xn = x/h$. Then (remember Gauss's punishment?)

$$S_n f(x) = \frac{1}{n}\sum_{j=1}^k \frac{cj}{n} = \frac{ck(k + 1)/2}{n^2}\ .$$

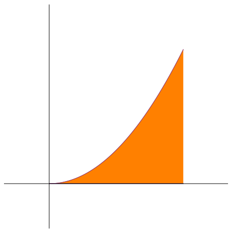
Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\, dt = cx^2/2$.

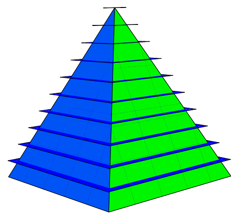


3 Let $f(x) = x^2$. In this case,we can not see the numerical value of the area geometrically. But since we have computed $S[x^2]$ in the first lecture of this course and seen that it is $[x^3]/3$ and since we have defined $S_h f(x) \to \int_0^x f(t)\, dt$ for $h \to 0$ and $[x^k] \to x^k$ for $h \to 0$, we know that

$$\int_0^x t^2\, dt = \frac{x^3}{3}\ .$$

This example actually computes the **volume of a pyramid** which has at distance t from the top an area t^2 cross section. Think about t^2dt as a slice of the pyramid of area t^2 and height dt. Adding up the volums of all these slices gives the volume.

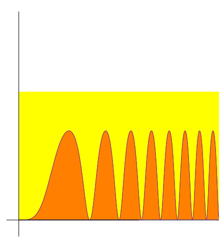




Linearity of the integral (see homework) $\int_0^x f(t)+g(t)\;dt = \int_0^t f(t)\;dt + \int_0^x g(t)\;dt$ and $\int_0^x \lambda f(t)\;dt = \lambda \int_0^x f(t)\;dt$.

Upper bound: If $0 \leq f(x) \leq M$ for all x, then $\int_0^x f(t)\;dt \leq Mx$.

4 $\int_0^x \sin^2(\sin(\sin(t)))/x\;dt \leq x$. **Solution**. The function $f(t)$ inside the interval is nonnegative and smaller or equal to 1 The graph of f is therefore contained in a rectangle of width x and height 1.



We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If f and g satisfy $|f(x)-g(x)| \leq c$, then
$$\int_0^x |f(x)-g(x)|\;dx \leq cx\;.$$

We know identities like $S_n[x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$ already. Since $[x]_h^k - [x]^k \to 0$ we have $S_n[x]_h^k - S_n[x]^k \to 0$ and from $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$.
The other equalities are the same since $\exp_h(x) = \exp(x) \to 0$. This gives us:

$\int_0^x t^n\;dt = \frac{x^{n+1}}{n+1}$ $\quad$ $\int_0^x \cos(t)\;dt = \sin(x)$
$\int_0^x e^t\;dt = e^x - 1$ $\quad$ $\int_0^x \sin(t)\;dt = 1-\cos(x)$

Homework

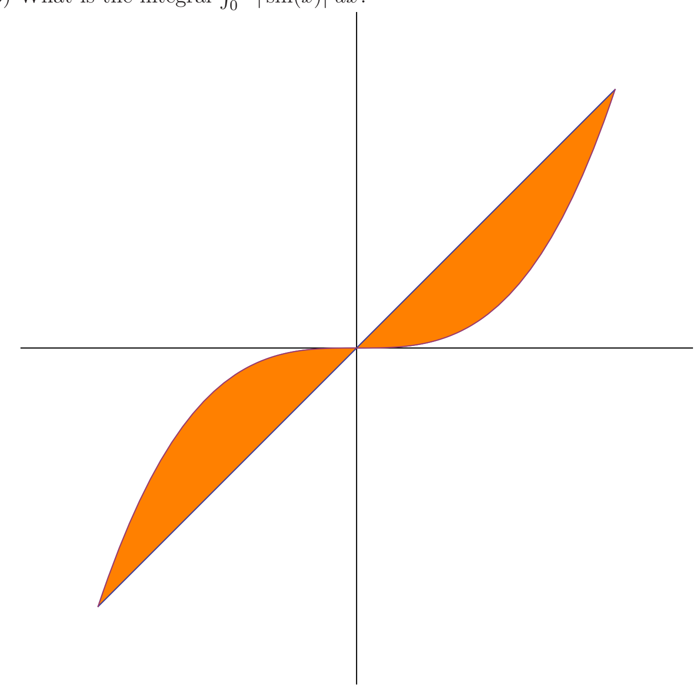
1 a) Find the integral $\int_0^x t^5 + 4t^3 + e^t\;dt$.
b) Calculate $\int_0^{10} t^3 - t + t^2\;dt$.
c) Find $\int_{-5\pi}^{3\pi} \cos(t)\;dt$.

2 Verify that the following statements hold for differentiable functions f, g and $a < b < c$ and any real number λ. You can argue geometrically with areas.

- $\int_a^b f(x)\;dx + \int_b^c f(x)\;dx = \int_a^c f(x)\;dx$.
- $\int_a^b f(x)\;dx + \int_a^b g(x)\;dx = \int_a^b f(x) + g(x)\;dx$.
- $\int_a^b \lambda f(x)\;dx = \lambda \int_a^b f(x)\;dx$.
- $0 \leq m \leq f(x) \leq M$ implies $(b-a)m \leq \int_a^b f(x)\;dx \leq (b-a)M$.

3 a) Verify that every differentiable function f can be written as a difference of two nonnegative functions. To do so, show that $g(x) = \max(f(x), 0)$ and $h(x) = \max(-f(x), 0)$ have the property that $f(x) = g(x) - h(x)$ and that $g(x) \geq 0$ and $h(x) \geq 0$.
b) Draw the graphs of the two functions $g(x), h(x)$ in the case $f(x) = \sin(3x)$ where $0 \leq x \leq 2\pi$.

4 a) The region enclosed by the graph of x and the graph of x^3 has a propeller type shape as seen in the picture. Find its (positive) area.
b) What is the integral $\int_0^{2\pi} |\sin(x)|\;dx$?



5 a) Find $\int_0^3 |x-1|\;dx$. Distinguish cases.
b) Find $\int_0^3 f(x)\;dx$ for $f(x) = |x - |x-1|| \cdot |x-2|$.