Lecture 19: Fundamental theorem

In this lecture we prove the **fundamental theorem of calculus** for differentiable functions. This will allow us in general to compute integrals of functions which appear as derivatives.

We have seen earlier that with $Sf(x) = h(f(0) + \cdots + f(kh))$ and $Df(x) = (f(x+h) - f(x))/h$ we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$. This becomes now:

> **Fundamental theorem of calculus:** Assume f is differentiable. Then
>
> $\int_0^x f'(t)\, dt = f(x) - f(0)$ and $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$

Proof. Using notation of Euler we write $A \sim B$ for "A and B are close" meaning $A - B \to 0$ for $h \to 0$. From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k+1)h$ because f is continuous. We also know $\int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$ because $Df(t) \sim f'(t)$ uniformly for $0 \leq t \leq x$ by the definition of the derivative and the assumption that f' is continuous. We also know $SDf(x) = f(x) - f(0)$ for $x = kh$. By definition of the Riemann integral $Sf(x) \sim \int_0^x f(t)\, dt$ and so $SDf(x) \sim \int_0^x Df(t)\, dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$$

as well as

$$f(x) \sim DSf(x) \sim D \int_0^x f(t)\, dt \sim \frac{d}{dx} \int_0^x f(t)\, dt\ .$$

1 $\int_0^5 3t^7\, dt = \frac{x^8}{8}|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

2 $\int_0^{\pi/2} \cos(t)\, dt = \sin(x)|_0^{\pi/2} = 1$. This is an important example which should become routine in a while.

3 $\int_0^x \sqrt{1+t}\, dt = \int_0^x (1+t)^{1/2}\, dt = (1+t)^{3/2}(2/3)|_0^x = [(1+x)^{3/2} - 1](2/3)$. Here the difficulty was to see that the $1+t$ in the interior of the function does not make a big difference. Keep such examples in mind.

4 Also in this example $\int_0^2 \cos(t+1)\, dt = \sin(x+1)|_0^2 = \sin(3) - \sin(1)$ the additional term $+1$ does not make a big dent.

5 $\int_{\pi/6}^{\pi/4} \cot(x)\, dx$. This is an example where the anti derivative is difficult to spot. It is easy if we know where to look for: the function $\log(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.

6 The example $\int_1^2 1/(t^2 - 9)\, dt$ is a bit challenging. We need a hint and write $-6/(x^2-9) = 1/(x+3) - 1/(x-3)$. The function $f(x) = \log|x+3| - \log|x-3|$ has therefore $-6/(x^2-9)$ as a derivative. We know therefore $\int_1^2 -6/(t^2-9)\, dt = \log|3+x| - \log|3-x||_1^2 = \log(5) - \log(1) - \log(4) + \log(2) = \log(5/2)$. The original task is now $(-1/6)\log(5/2)$.

7 $\int_0^x \cos(\sin(x))\cos(x)\, dx = \sin(\sin(x))$ because the derivative of $\sin(\sin(x))$ is $\cos(\sin(x))\cos(x)$. The function $\sin(\sin(x))$ is called the **antiderivative** of f. If we differentiate this function, we get $\cos(\sin(x))\cos(x)$.

8 Find $\int_0^\pi \sin(x)\, dx$. **Solution**: This has a very nice answer.

Here is an important notation, which we have used in the example and which might at first look silly. But it is a handy intermediate step when doing the computation.

> $F|_a^b = F(b) - F(a)$.

We give reformulations of the fundamental theorem in ways in which it is mostly used:

> If f is the derivative of a function F then
>
> $$\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a)\ .$$

In some textbooks, this is called the "second fundamental theorem" or the "evaluation part" of the fundamental theorem of calculus. The statement $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$ is the "antiderivative part" of the fundamental theorem. They obviously belong together and are two different sides of the same coin.

Here is a version of the fundamental theorem, where the boundaries are functions of x.

> Given functions g, h and if F is a function such that $F' = f$, then
>
> $$\int_{h(x)}^{g(x)} f(t)\, dt = F(g(x)) - F(h(x))\ .$$

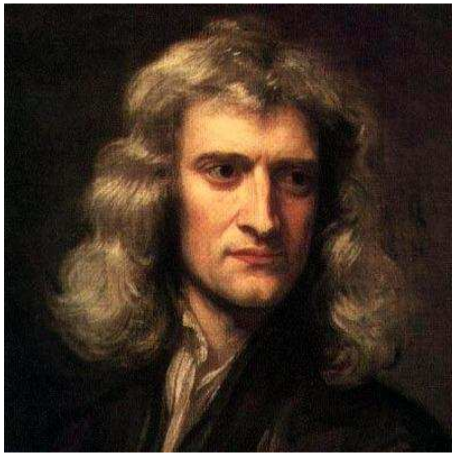
9 $\int_{x^4}^{x^2} \cos(t)\, dt = \sin(x^2) - \sin(x^4)$.

The function F is called an antiderivative. It is not unique but the above formula does always give the right result.

Lets look at a list of important antiderivatives. You should have as many antideriatives "hard wired" in your brain. It really helps. Here are the core functions you should know. They appear a lot.

function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$\frac{-\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x\log(x) - x$

Make your own table!



Meet **Isaac Newton** and **Gottfried Leibniz**. They have discovered the fundamental theorem of calculus. You can see from the expression of their faces how honored they are to find themselves on the same handout with **Austin Powers** and **Doctor Evil**. Culture clash ...

Homework

1 For any of the following functions f, find a function F such that $F' = f$.
a) $e^x + \sin(3x) + x^3 + 5x$.
b) $(x + 4)^3$.
c) $1/x + 1/(x - 1)$.
d) $\cos(x^2)2x + \sin(x^3)3x^2 + 1/\sqrt{x}$

2 Find the following integrals by finding a function g satisfying $g' = f$. We will learn techniques to find the function. Here, we just use our knowledge about derivatives:
a) $\int_2^3 5x^4 + 4x^3 \; dx$.
b) $\int_{\pi/4}^{\pi/2} \sin(3x) + \cos(x) \; dx$.
c) $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2(x)} \; dx$.
d) $\int_2^3 \frac{1}{x-1} \; dx$.

3 Evaluate the following integrals:
a) $\int_1^2 2^x \; dx$.
b) $\int_{-1}^1 \cosh(x) \; dx$. (Remember $\cosh(x) = (e^x + e^{-x})/2$.)
c) $\int_0^1 \frac{1}{1+x^2} \; dx$.
d) $\int_{1/3}^{2/3} \frac{1}{\sqrt{1-x^2}} \; dx$.

4 a) Compute $F(x) = \int_0^{x^3} \sin(t) \; dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \; dt$ then find $G'(x)$

5 a) **Be clever:** Evaluate the following integral: $\int_0^{2\pi} \sin(\sin(x)) \; dx$
Give the answer and the reason in a short sentence.

b) **Be evil:** Take a function F of your choice. Find its derivative and call it f. Now pose an integration problem to find $\int_a^b f(x) \; dx$. Submit this problem to knill@math.harvard.edu I will select the most evil one. A good problem should lead to a short function f but the integral F should be difficult to find or guess. These problems will make perfect exam problems for the second midterm .... (evil laugh).

You can submit your version of Problem 5b) electronically by email (knill@math.harvard.edu. Just send the function in the subject line. Mail can otherwise be empty). For any submission, independent how clever or evil, 10 points maximal will be added to your score (maxing up at 50). So, if your HW score of Lecture 19 is 45 and you submitted a function, it will be bumped to 50. If your HW score is 50 already you get nothing ... (more evil laugh).