Math 1A: introduction to functions and calculus **Oliver Knill, 2011**

Lecture 24: Applications of integration

You have seen these integration applications:

- the computation of area
- the computation of volume
- position from acceleration
- cost from marginal cost

Here are some more:

- probabilities and distributions
- averages and expectations
- finding moments of inertia
- work from power

Probability

In **probability theory** functions are used as observables or to define probabilities.

Assuming our probability space to be the real line, an interval $[a, b]$ is called an **event**. Given a nonnegative function $f(x)$ which has the property that $\int_{-\infty}^{\infty} f(x)\, dx = 1$, we call

$$\mathrm{P}[A] = \int_0^b f(x)\, dx$$

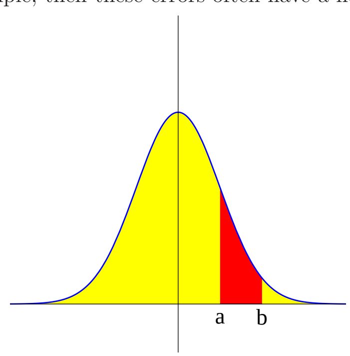
the **probability** of the event. The function $f(x)$ is called the **probability density function**.

The most famous and most important probability density is the normal distribution:

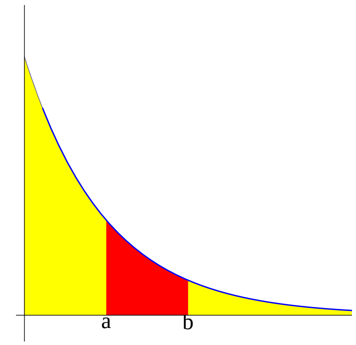
The **normal distribution** has the density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/x} .$$

It is the distribution which appears most often if data can take both positive and negative values. The reason why it appears so often is that if one observes different unrelated quantities with the same statistical properties, then their sum, suitably normalized becomes the normal distribution. If we measure **errors** for example, then these errors often have a normal distribution.



1 The probability density function of the **exponential distribution** is defined as $f(x) = e^{-x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. It is used to used measure lengths of arrival times like the time until you get the next phone call. The density is zero for negative x because there is no way we can travel back in time. What is the probability that you get a phone call between times $x = 1$ and times $x = 2$ from now? The answer is $\int_1^2 f(x)\, dx$.



Assume f is a probability density function (PDF). The antiderivative $F(x) = \int_{-\infty}^{x} f(t)\, dt$ is called the **cumulative distribution function** (CDF).

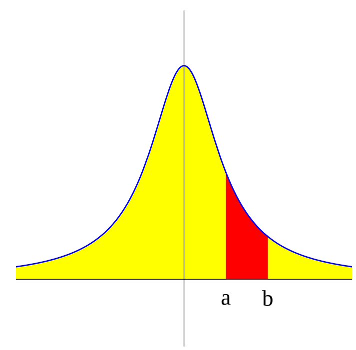
2 For the exponential function the cumulative distribution function is

$$\int_{-\infty}^{x} f(x)\, dx = \int_0^x f(x)\, dx = -e^{-x}|_0^x = 1 - e^{-x} .$$

The probability density function $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ is called the Cauchy distribution.

3 Find its cumulative distribution function. **Solution:**

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \frac{1}{\pi} \arctan(x)|_{-\infty}^{x} = (\frac{1}{\pi} \arctan(x) + \frac{1}{2}) .$$

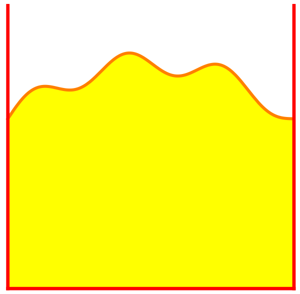


Average

Here is an example for computing the **average**.

4 Assume the level in a **honey jar** over $[0, 2\pi]$ containing crystallized honey is given by a function $f(x) = 3 + \sin(3x)/5 + x(2\pi - x)/10$. In order to restore the honey, it is placed into hot water. The honey melts to its normal state. What height does it have? **Solution:** The average height is $\int_0^{2\pi} f(x)\, dx/(2\pi)$ which is the area divided by the base length.

In probability theory we would call $f(x)$ a **random variable** and the average of f with $\mathrm{E}[f]$ the **expectation**.



Moment of inertia

If we spin a wire of radius L of mass density $f(x)$ around an axes, the **moment of inertia** is defined as $I = \int_0^L x^2 f(x)\, dx$.

The significance is that if we spin it with angular velocity w, then the energy is $Iw/2$.

5 Assume a wire has density $1+x$ and length 3. Find its moment of inertia. **Solution:**

6 **Flywheels** have a comeback for **powerplants** to absorb energy. If there is not enough power, the flywheels are charged, in peak times, the energy is recovered. They work with 80 percent efficiency. Assume a flywheel is a cylinder of radius 1, density 1 and height 1, then the moment of inertia integral is $\int_0^1 z^2 f(z)\, dx$, where $f(z)$ is the mass in distance z.

Work from power

If $P(t)$ is the amount of power produced at time t, then $\int_0^T P(t)\, dt$ is the **work**=energy produced in the time interval $[0,T]$.

Energy is the anti-derivative of power.

7 Assume a power plant produces power $P(t) = 1000 + \exp(-t) + t^2 - t$. What is the energy produced from $t=1$ to $t=10$? **Solution.**

Wouldn't be nice to have one of those bikes with interactive training environments in the gym, allowing to ride in the Peruvian or Swiss Mountains, the California coast or in the Italian Tuscany?

Additionally, there should be some computer game features, racing other riders through beaches, deserts or Texan highways (could be on google earth). Training would be so much more entertaining. Business opportunities everywhere. The first offering such training equipment will make a fortune. Until then we are stuck with TV programs which really suck.

Homework

1 The probability distribution which describes the time you have to wait for your next email is $f(x) = 3e^{-3x}$. What is the probability that you get your next email in the next 2 hours, that is between $x=0$ and $x=2$?

2 Assume the probability distribution for the waiting time to the next warm day is $f(x) = (1/4)e^{-x/4}$, where x has days as unit. What is the probability to get a warm day between tomorrow and after tomorrow that is between $x=1$ and $x=2$?

3 A rod modeled over the interval $[0,4]$ has temperature $f(x) = 5 + x^2 - 3x$ at position x. Find the average temperature.

4 A CD Rom has radius 6. If we would place the material at radius x onto one point, we get a density of $f(x) = 2\pi x$. Find the moment of inertia I of the disc. If we spin it with an angular velocity of $w = 20$ rounds per second. Find the energy $E = Iw^2/2$.

Without credit: Explode a CD: http://www.powerlabs.org/cdexplode.htm. Careful!

5 a) You are on a stationary bike in the Hemenway gym and pedal with power

$$P(t) = 200 + 100\sin(10\pi t) - \frac{t}{300} + \frac{t^2}{19440}$$

(in Watts=W). The periodic fluctuations come from a hilly route. The linear term is the "tiring effect" and the quadratic term is due to **endorphins** kicking in eventually. What energy (Joules J=W s) have you produced in the time $t \in [0,1800]$ (s=seconds)?
b) Since we do math not physics, we usually ignore all the units but this one is just too much fun. If you divide the result by 4184, you get **kilo calories** = **food calories**. Eating an apple gives you about 80 food calories. How many apples can you eat after your half hour workout, just to get even?