Math 1A: introduction to functions and calculus   Oliver Knill, 2011

Lecture 25: Related rates

Before we continue with integration, we include a short flash-back on differentiation. This allows us to solidify the **chain rule**

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

which will be very useful for the integration technique called "substitution". Since the chain rule is often perceived as a difficult concept in calculus, it is good to come back to it again. We take the opportunity also to review a bit our differentiation skills and to take some fresh breath before launching into more advanced integration techniques.

1 Assume we inflate a **balloon** and pump 5 volume units per unit time into it. If the balloon has radius 7, what is the rate of change of the radius? **Solution.** Let $V(r)$ be the volume and $r(t)$ the radius at time t. Since $\boxed{V(r(t)) = 4\pi r(t)^3/3}$, we have by the chain rule

$$5 = d/dt V(r(t)) = 4\pi r(t)^2 r'(t) .$$

This relation allows us to compute $r'(t) = 5/(4\pi r^2) = 5/(4\pi 7^2)$.

2 Hydrophilic **water gel spheres** made from **polyacrylamide polymer** can expand 300 times their original size as you see in class. Assume they have initially a diameter of 1 (cm) and that they expand in 10 hours to its 300 fold volume. Find the rate of change of the radius in time when they have a volume of 100 (cm^3). **Solution.** We have the same rule $\boxed{V = 4\pi r^3/3}$. The problem gives us $d/dt V(r(t)) = 300/10 = 30$. The rest is now the same as in the previous problem: $30 = 4\pi r^2 r'$. Since $r = 100$ we get $r' = 30/(4\pi 100^2)$.


3 The upper part of a **wine glass** has a shape $y = x^2$ with $0 \le y \le 2$. We assume the glass is half full, meaning that the wine level is at $y = 1$. We taste the wine with 1 ml/sec using a straw, ignoring any political and behavioral correctness. How fast does the wine level sink at that moment?
Solution: The area of the wine layer at height y is $A(y) = x^2\pi = y\pi$. The volume is $\boxed{V(y) = \int_0^y y\pi \, dy = y^2\pi/2}$. We know

$$-1 = d/dt V(y(t)) = V'(y)y'(t) = \pi y y'(t)$$

so that $y'(t) = -1/(\pi y)$ and for $y = 1$ this is $-1/\pi$.

4 A person of height 6 feet is located at $x = 6$ and walks with constant speed 1. A lamp at $x = 0$ is at height 10 feet. With what speed does the **shadow** of the person proceed on the floor? **Solution:** If the person is at position x, the shadow's length L satisfies $\boxed{L/6 = (L + x)/10}$ which is $L = 9$. The relation $L/6 = (L + x)/10$ means $L = 3x/2$ so that $L' = 3x'/2 = 3/2$.
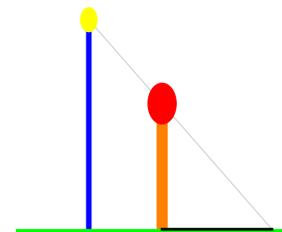
5 **Romeo and Juliet** have meet secretly at position $(0, 0)$ and rush home. Romeo runs with speed 4 meters/seconds to the east. Assume their distance satisfies $l(t) = t^3$. After 10 seconds, they wave back to each other. With what speed does Juliet run at this time? **Solution.** What do we know? $x(t) = 4t$ is the position of Romeo and $l(t) = t^3$. If $y(t)$ is the y position of Juliet, the law we use is Pythagoras $\boxed{l^2 = x^2 + y^2}$ so that $y(t) = \sqrt{l^2 - x^2}$ and $y(10) = \sqrt{1000 - 100} = \sqrt{900} = 30$. Now differentiate the law to get $2ll' = 2xx' + 2yy'$. We know all quantities at time $t = 10$: we know $l = 1000, l' = 300, x = 40, x' = 4, y = 30$ and compute $y' = (2000 * 300 - 80 * 4)/60 = 29984/3$.

We have seen the ladder example twice already:

6 A **ladder** has length 1. Assume slips on the ground away with constant speed 2 in the x-direction. What is the speed of the top part of the ladder sliding down the wall at the time when $x = y$? **Solution** We know $x'(t) = 2$ and that $x(t), y(t)$ are related by $\boxed{x^2(t) + y^2(t) = 1}$. Differentiation gives $2x(t)x'(t) + 2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

7 A **kid slides** down a slide of the shape $\boxed{y = 2/x}$. Assume at height $y = 2$ we have $dy/dt = -7$. What is dx/dt? **Solution:** differentiate the relation to get $y' = -2x'/x^2$. At $y = 2$ we have $x = 1$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.

Image source: http://www.dmfco.com

8 A **canister of oil** releases oil at a constant rate 5. With what rate does the radius of the oil spill increase, when the radius is 1? **Solution.** We have $\boxed{A(r) = r^2\pi}$ and so $5 = A'(r) = 2rr'\pi$. Solving for r' gives $r' = 5/(2r\pi)$ which is $5/(2\pi)$.

Related rates problems link quantities by a $\boxed{\text{rule}}$. These quantities can depend on time. To solve a related rates problem, differentiate the $\boxed{\text{rule}}$ with respect to time and solve for the unknown quantity.

Related rates problems are not so easy. The difficulty comes from the fact that they are often "word problems" which first have to be parsed. We have to find the $\boxed{\text{rule}}$ and differentiate it. In all the problems on this handout, the $\boxed{\text{rule}}$ is boxed. It is important to understand which variables depend on time. If a term x^3 appears for example and x depends on time, then $d/dt\,x^3 = 3x^2 x'$.

Homework

1 The **ideal gas law** $\boxed{pV = T}$ relates pressure p and volume V and temperature T. Assume the temperature $T = 50$ is fixed and the volume is at $V = 2$ and decreased by $V' = -3$. Find the rate p' with which the pressure increases.


2 Assume the **total production rate** P of a new tablet computer product for kids is constant 100 and given by the famous **Cobb-Douglas formula** $\boxed{P = L^{1/3}K^{2/3}}$ where $L = 64$ is the labor and $K = 125$ is the cost. Assume labor is increased at a rate $L' = 2$. What is the cost change K'?

3 You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\phi' = 5$ degree per second (which is $5\pi/180$ radiants per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\boxed{\tan(\phi) = x/h}$ and make a picture to figure out what ϕ is.

4 An **isosceles triangle** with base $2a$ and height h has fixed area $\boxed{A = ah = 1}$. Assume the height is decreased by a rate $h' = -2$. With what rate does a increase if $h = 1/2$?
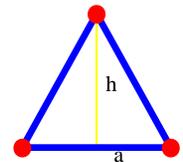
5 There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $\boxed{V = \pi^2 r^4/2}$ increases at a rate $d/dt\,V(r(t)) = 100\pi^2 r^2$. What is r' if the current radius is $r = 47$ (billion light years).