Lecture 35: Calculus and Economics

In this lecture we look more at applications of calculus and focus mostly on **economics**. This is an opportunity to review extrema problems.

Marginal and total cost

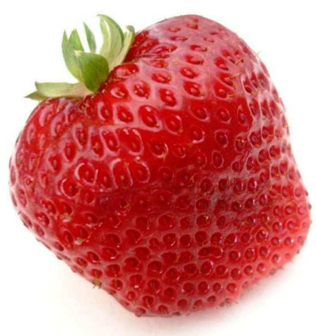
Recall that the **marginal cost** was defined as the derivative of the **total cost**. Both, the marginal cost and total cost are functions of the quantity of goods produced.

1 Assume the total cost function is $C(x) = 10x + 0.01x^2$. Find the marginal cost and the place where the total cost is maximal. **Solution**. Differentiate.

2 You sell spring water. The marginal cost to produce depends on the season and given by $f(x) = 10 - 10\sin(2x)$. For which x is the total cost maximal?

3 The following example is adapted from the book "Dominik Heckner and Tobias Kretschmer: Don't worry about Micro, 2008", where the following strawberry story appears: (verbatim citation in italics):

Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

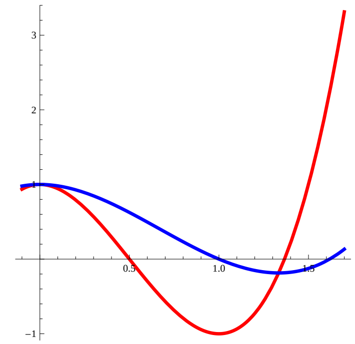


Lets recall that if $F(x)$ is the **total cost function** in dependence of the quantity x, then $F' = f$ is called the **marginal cost**.

The function $g(x) = F(x)/x$ is called the **average cost**.

A point where $f = g$ is called a **break even point**.

4 If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, lets call it the "strawberry theorem":

Strawberry theorem: We have $g'(x) = 0$ if an only if $f = g$.

Proof.

$$g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g) .$$

Volume extremization

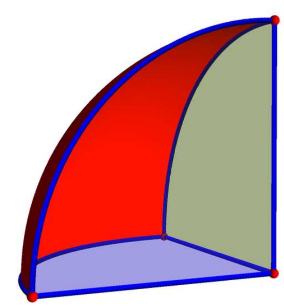
1 Assume the cost to heat a room is $V(x) + A(x) - \pi L(x)$ where V is its volume, A is the surface area and $L(x) = \pi x$ is proportional to length x. A conference center hall is is eighth of a sphere. Its volume, surface area and length are

$$V(x) = \frac{4\pi x^3}{3}\frac{1}{8}, A(x) = (\frac{4\pi}{8} + \frac{3\pi}{4})x^2, L(x) = \pi x .$$

The costs are $\pi/6x^3 + (3\pi/4 + 4\pi/8)x^2 - \pi x$. To extremize the cost, we can minimize

$$f(x) = x^3/6 + 5x^2/4 - x .$$

The minimum is achieved at $x = (-5 + \sqrt{3})/2$.

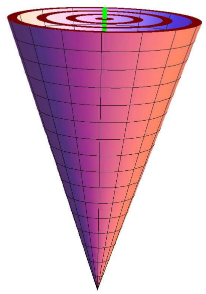


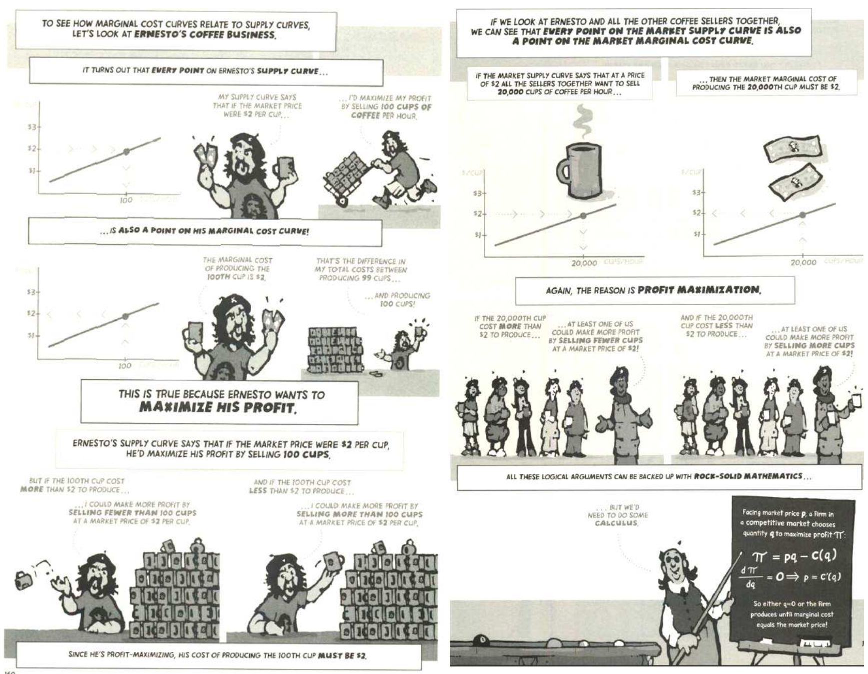
2 A cone shaped solar loudspeaker has to be a cone of volume π. For optimal charging features, the sum of vertical and horizontal shadow areas $hr + \pi r^2$ need to be extremized. Can you get a minimum or maximum? **Solution**. Lets first compute the volume of a cone

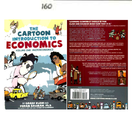
with maximal radius r and height h. At height z, the radius is rz/h. At z the surface area is $A(z) = \pi(hz/r)^2$ so that the volume is

$$V = \int_0^h \pi(r^2z^2/h^2)dz = \pi r^2 h/3 = \pi \ .$$

This means $h = 3/r^2$ and $hr = 3/r$. The cross section is $f(r) = \pi r^2 + 3/r$. Setting $f'(r) = 0$, we get the critical point $(3/(2\pi))^{1/3}$.







Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang.

Homework

1 Verify the Strawberry theorem in the case $f(x) = \cos(x)$.

2 The **production function** in an office gives the production $Q(L)$ in dependence of labor L. Assume $Q(L) = 500L^3 - 3L^5$. Find L which gives the maximal production.

This can be typical: For smaller groups, production usually increases when adding more workforce. After some point, bottle necks occur, not all resources can be used at the same time, management and bureaucracy is added, each individuum has less impact and feels less responsible, meetings slow down production etc. In this range, adding more people will decrease the productivity.

3 **Marginal revenue** f is the rate of change in **total revenue** F. As total and marginal cost, these are functions of the **cost** x. Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the point, where the total revenue has a local maximum.

4 To find the line $y = mx$ through the points $(3, 4), (6, 3), (2, 5)$. We have to minimize the function

$$f(m) = (3m - 4)^2 + (6m - 3)^2 + (2m - 5)^2 \ .$$

5 For any a we look at the solid obtained by rotating the graph of the function $f(r) = a\sin(r/a)$ around the axes over the interval $[0, \pi/a]$. For which a is the volume locally maximal?

P.S. You can see the graph of the volume $V(a)$ in dependence of a below. There are many local maxima. The problem is to find them.