Lecture 38: Review since second midterm

Related rates

Implicit differentiation and related rates are manifestations of the chain rule.

A) related rates: we have an equation $\boxed{F(x,y)=c}$ relating two variables x,y which depend on time t. differentiate the equation with respect to t using the chain rule and solve for y'.
B) implicit differentiation: we have an equation $\boxed{F(x,y(x))=c}$ relating y with x. Differentiate the equation with respect to x using the chain rule and solve for y'.

Examples:
A) $x^3+y^3=1$, $x(t)=\sin(t)$, then $3x^2x'+3y^2y'=0$ so that $y'=-x^2x'/y^2=-\sin^2(t)\cos(t)/(1-\sin^3(t))^{1/3}$.
B) Same example but $x(t)=x$: $y'=-x^2/y^2=-\sin^2(t)/(1-\sin^3(t))^{1/3}$.

Substitution

Substitution replaces $\int f(x)\,dx$ with $\int g(u)\,du$ with $u=u(x), du=u'(x)dx$. Special cases:

A) The antiderivative of $f(x)=g(u(x))u'(x)$, is $G(u(x))$ where G is the anti derivative of g.
B) $\int f(ax+b)\,dx=F(ax+b)/a$ where F is the anti derivative of f.

Examples:
A) $\int \sin(x^5)x^4\,dx=\int \sin(u)\,du/5=-\cos(u)/5+C=-\cos(x^5)/5+C$.
B) $\int \log(5x+7)\,dx=\int \log(u)\,du/5=(u\log(u)-u)/5+C=(5x+7)\log(5x+7)-(5x+7)+C$.

Integration by parts

A) Direct:

$$\int x\,\sin(x)\,dx = x\,(-\cos(x)) - \int 1\,(-\cos(x))\,dx = -x\cos(x)+\sin(x)+C\,dx\,.$$

B) Tic-Tac-Toe: To integrate $x^2\sin(x)$

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The anti-derivative is

$$-x^2\cos(x)+2x\sin(x)+2\cos(x)+C\,.$$

C) Merry go round: Example $I=\int \sin(x)e^x\,dx$. Use parts twice and solve for I.

Partial fractions

A) Make a common denominator on the right hand side $\frac{1}{(x-a)(x-b)}=\frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$. and compare coefficients $1=Ax-Ab+Bx-Ba$ to get $A+B=0, Ab-Ba=1$ and solve for A,B.
B) If $f(x)=p(x)/(x-a)(x-b)$ with different a,b, the coefficients A,B in $\frac{p(x)}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$ can be obtained from

$$A=\lim_{x\to a}(x-a)f(x)=p(a)/(a-b),\ B=\lim_{x\to b}(x-b)f(x)=p(b)/(b-a)\,.$$

Examples:
A) $\int \frac{1}{(x+1)(x+2)}\,dx=\int \frac{A}{x+1}\,dx+\int \frac{B}{x+2}\,dx$. Find A,B by multiplying out and comparing coefficients in the nominator.
B) Directly write down $A=1$ and $B=-1$, by plugging in $x=-2$ after multiplying with $x-2$. or plugging in $x=-1$ after multiplying with $x-1$.

Improper integrals

A) Integrate over infinite domain.
B) Integrate over singularity.

Examples:
A) $\int_0^\infty 1/(1+x^2)=\arctan(\infty)-\arctan(0)=\pi/2-0=\pi/2$.
B) $\int_0^1 1/x^{2/3}\,dx=(3/1)x^{1/3}|_0^1=3$.

Trig substitutions

A) In places like $\sqrt{1-x^2}$, replace x by $\cos(u)$.
B) Use $u=\tan(x/2)$, $dx=\frac{2du}{(1+u^2)}$, $\sin(x)=\frac{2u}{1+u^2}$, $\cos(x)=\frac{1-u^2}{1+u^2}$ to replace trig functions by polynomials.

Examples:
A) $\int_{-1}^1 \sqrt{1-x^2}\,dx=\int_{-\pi/2}^{\pi/2}\cos(u)\cos(u)du=\int_{-\pi/2}^{\pi/2}(1+\cos(2u))/2=\frac{\pi}{2}$.
B) $\int \frac{1}{\sin(x)}\,dx=\int \frac{1+u^2}{2u}\frac{2du}{1+u^2}\ =\int \frac{1}{u}\,du=\log(u)+C=\log(\tan(\frac{x}{2}))+C$.

Applications, keywords to know

Music: hull function, piano function
Economics: average cost, marginal cost and total cost. Strawberry theorem, fit points
Computer science: curvature and Chaikin steps
Statistics: probability density function, cumulative distribution function, expectation, variance.
Geometry: area between two curves, volume of solid
Numerical methods: trapezoid rule, Simpson rule, Newton Method
Psychology: critical points and Catastrophes.
Physics: position, velocity and acceleration.
Gastronomy: turn table to prevent wobbling, bottle calibration.