4/5/2011: Second midterm practice II

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) **T** F The formula $\int_0^x f''(x)\,dx = f'(x) - f'(0)$ holds.

Solution:
Apply the fundamental theorem to the derivative.

2) T **F** The area of the lower half disc is the integral $\int_{-1}^1 -\sqrt{1-x^2}\,dx$

Solution:
The area is positive. The integral given is negative.

3) **T** F If the graph of the function $f(x) = x^2$ is rotated around the interval $[0,1]$ we obtain a solid with volume $\int_0^1 \pi x^4\,dx$.

Solution:
Indeed the area is $A(x) = \pi x^4$.

4) **T** F The identity $d/dx \int_0^x f''(t)\,dt = f'(x)$ holds.

Solution:
The result is $f''(x)$.

5) **T** F There is a point in $[0,1]$, where $f'(x) = 0$ if $f(x) = x^3 - x^2 + 1$.

Solution:
Since $f(0) = f(1) = 1$, Rolle's theorem assures this.

6) **T** F The fundamental theorem of calculus assures that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Solution:
Yes, this is one of the important reformulations.

7) **T** F If f is differentiable on $[a,b]$, then $\int_a^b f(x)\,dx$ exists.

Solution:
Yes, we have seen the proof in class.

8) T F The integral $\int_0^{\pi/2} \sin(\sin(x))\, dx$ is positive.

Solution:
$\sin(\sin(x)) > 0$ there so that the integral is positive

9) T F The anti-derivative of an anti-derivative of f is equal to the derivative of f.

Solution:
This is just total nonsense. We would have to differentiate three times the anti derivative of the anti derivative to get to the derivative of f.

10) T F If a function is positive everywhere, then $\int_a^b f(x)\, dx$ is positive too.

Solution:
Yes, the integral has the meaning of an area under the curve.

11) T F If a differentiable function is odd, then $\int_{-1}^1 f(x)\, dx = 0$.

Solution:
We have a cancellation to the left and to the right.

12) T F If $f_c(x)$ is a function with a local minimum at 0 for all $c < 0$ and no local minimum in $[-1, 1]$ for $c > 0$, then $c = 0$ is called a catastrophe.

Solution:
Yes this is a pretty precise definition of a catastrophe.

13) T F The term "improper integral" is a synonym for "indefinite integral".

Solution:
Improper means that we either integrate a function which has a discontinuity or that we integrate over an infinite interval.

14) T F The function $F(x) = x\sin(x)$ is an antiderivative of $\sin(x)$.

Solution:
Too good to be true. Just differentiate and you see that the derivative of F is not f.

15) T F The mean value theorem holds for every continuous function.

Solution:
No, only for differentiable functions.

16) T F Newton and Leibniz were best buddies all their life. Leibniz even gave once the following famous speech: "You guys might not know this, but I consider myself a bit of a loner. I tend to think of myself as a one-man wolf pack. But when my sister brought Isaac home, I knew he was one of my own. And my wolf pack... it grew by one.

Solution:
This line is from the movie "hangover". No, Newton and Leibniz had a dispute about who discovered calculus.

17) T F Any function $f(x)$ satisfying $f(x) > 0$ is a probability density function.

Solution:
What is missing is that the integral over the real line is equal to 1.

18) T F The moment of inertia integral I can be used to compute energy with the relation $E = \omega^2 I/2$ where ω is the angular velocity.

Solution:
Yes, this is why the moment of inertia is an important quantity in physics.

19) T F If $0 \leq f(x) \leq g(x)$ then $0 \leq \int_0^1 f(x)\, dx \leq \int_0^1 g(x)\, dx$.

Solution:
Yes, just look at the areas.

20) T F The improper integral $\int_0^\infty 1/(x^4+1)\,dx$ is finite.

Solution:
It works well at 0 because the function does not have a pole there. The integral is smaller than $\int 1/x^4\,dx$ which has the anti-derivative $-1/(3x^3)$. The improper integral exists.

Problem 2) Matching problem (10 points) No justifications are needed.

From the following functions there are two for which no elementary integral is found. Find them. You can find them by spotting the complement set of functions which you can integrate.

Function	Antiderivative is not elementary	Function	Antiderivative is not elementary
e^{-x^2}		$1/\log(x)$	
$\sin(3x)$		$\tan(3x)$	
$1/x$		$\arctan(3x)$	

Solution:
The $1/\log(x)$ and e^{-x^2} have no elementary integral.

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following problems are related rates problems? Several answers can apply.

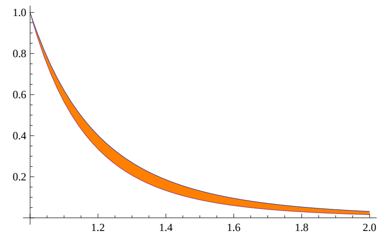
Problem	Related rates?
Find the volume of a sphere in relation to the radius.	
Relate the area under a curve with value of the curve.	
If $x^3+y^3=5$ and $x'=3$ at $x=1$, find y'.	
Find the rate of change of the function $f(x)=\sin(x)$ at $x=1$	
Find r' for a sphere of volume V satisfying $d/dt V(r(t))=15$.	
Find the inflection points of $f(x)=x^3+3x+4$.	
Find the global maxima of $f(x)=x^4+x^3-x$.	

Solution:
The third and fifth are related rates problems.

Problem 4) Area computation (10 points)

a) (5 points) Find the area of the region enclosed by the curves $3-x^4$ and $3x^2-1$.

b) (5 points) Find the area of the region between $1/x^6$ and $1/x^5$ from $x=1$ to $x=\infty$.

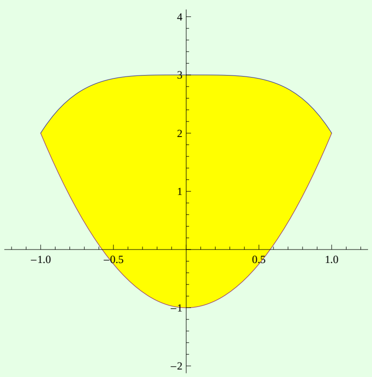


Solution:
a) The two curves intersect at $x = -1$ and $x = 1$ so that the area is

$$\int_{-1}^{1} (3 - x^4) - (3x^2 - 1)\ dx = 3x - x^5/5 - x^3 + x|_{-1}^{1} = 28/5\ .$$

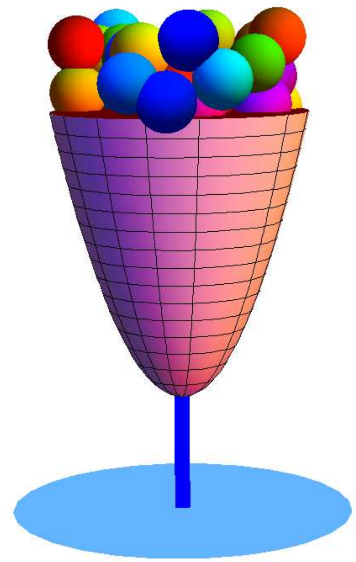
b) The graph of $1/x^5$ is above $1/x^6$. We have

$$\int_1^\infty x^{-5} - x^{-6}\ dx = -x^{-4}/4 + x^{-5}/5|_1^\infty = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}\ .$$



Problem 5) Volume computation (10 points)

Julian eats some magic "Bertie Botts Every Flavor Beans" from a cup which is a rotationally symmetric solid, for which the radius at position x is $\sqrt{x}$ and $0 \leq x \leq 4$. Find the volume of Julian's candy cup.



Solution:
The area at height x is $\sqrt{x}^2\pi = x\pi$ so that the volume is

$$\int_0^4 x\pi\ dx = \pi x^2/2|_0^4 = 8\pi\ .$$

Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x + \tan(x) + \sin(x) + \cos(x) + \log(x)\ dx$.

b) (5 points) $\int_1^3 (x+1)^3\ dx$

Solution:
a) The anti-derivative is $x^2/2 - \log(\cos(x)) - \cos(x) + \sin(x) + (x\log(x) - x)$. (The $\log(x)$ integral is a bit out of line at this stage, but we have seen it at some point).
b) $(x+1)^4/4|_1^3 = (4^4 - 2^4)/4 = 60$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

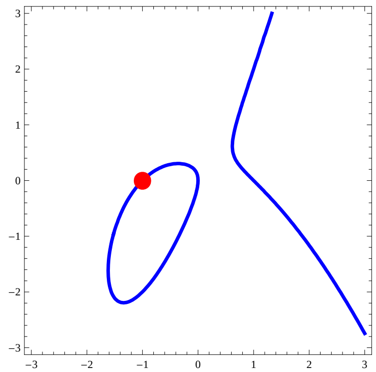
a) (5 points) $\int \sqrt{x^3}\ dx$

b) (5 points) $\int 4/\sqrt{x^5}\ dx$

Solution:
a) The function is $x^{3/2}$. Integration gives $x^{5/2}(2/5)$.
b) The function is $4x^{-5/2}$. Integration gives $4x^{-3/2}(-2/3)$.

Problem 8) Implicit differentiation (10 points)

The curve $y^2 = x^3 + 2xy - x$ is an example of an **elliptic curve**. Find dy/dx at the point $(-1, 0)$ without solving for y first.



Solution:
Differentiate, $2yy' = 3x^2 + 2y + 2xy' - 1$ and solve for $y' = (3x^2 + 2y - 1)/(2y - 2x)$. At the point $(-1, 0)$ this is 1.

Problem 9) Applications (10 points)

The probability density of the exponential distribution is given by $f(x) = (1/2)e^{-x/2}$. The probability to wait for for time x (hours) to get an idea for a good calculus exam problem is $\int_0^x f(x)\ dx$. What is the probability to get a good idea if we wait for $T = 10$ (hours)?

Solution:
$\int_0^{10}(1/2)e^{-x/2}\ dx = (1 - e^{-5})$ which is almost certain. Indeed, we did not have to wait so long to get this great problem.

Problem 10) Applications (10 points)

What is the **average value** of the function

$$f(x) = 4 + 1/(1 + x^2)$$

on the interval $[-1, 1]$?

Solution:
It is $\int_{-1}^{1} f(x)\ dx/2 = (8 - 2\arctan(1))/2 = 4 + \arctan(1) = 4 + \pi/4$.