

Lecture 21: Area computation

If $f(x) \geq 0$, then $\int_a^b f(x) dx$ is the **area under the graph** of $f(x)$ and above the interval $[a, b]$ on the x axes.

If the function is negative, then $\int_a^b f(x) dx$ is negative too and this integral is minus the area below the curve:

Therefore, $\int_a^b f(x) dx$ as the difference of the area above the graph minus the area below the graph.

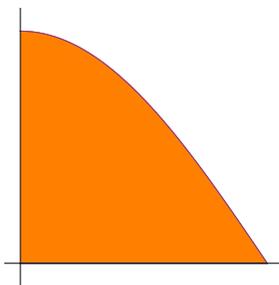
More generally we can also look at areas sandwiched between two graphs f and g .

The area of a region G enclosed by two graphs $f \leq g$ and bound by $a \leq x \leq b$ is

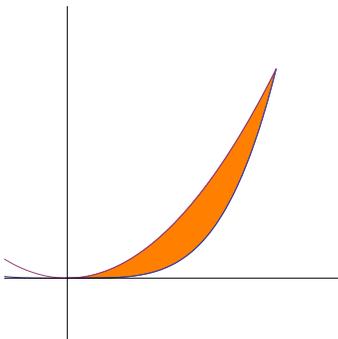
$$\int_a^b g(x) - f(x) dx$$

Make sure that if you have to compute such an integral that $g \geq f$ before giving it the interpretation of an area.

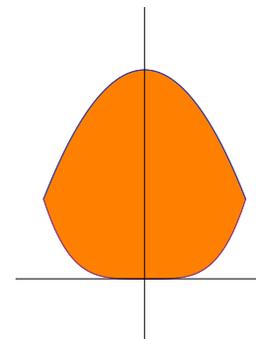
- 1 Find the area of the region enclosed by the x -axes, the y -axes and the graph of the cos function. **Solution:** $\int_0^{\pi/2} \cos(x) dx = 1$.



- 2 Find the area of the region enclosed by the graphs $f(x) = x^2$ and $f(x) = x^4$.



- 3 Find the area of the region enclosed by the graphs $f(x) = 1 - x^2$ and $g(x) = x^4$.

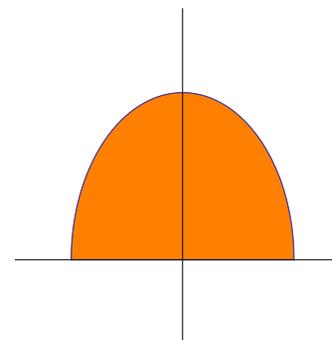


- 4 Find the area of the region enclosed by a half circle of radius 1. **Solution:** The half circle is the graph of the function $f(x) = \sqrt{1 - x^2}$. The area under the graph is

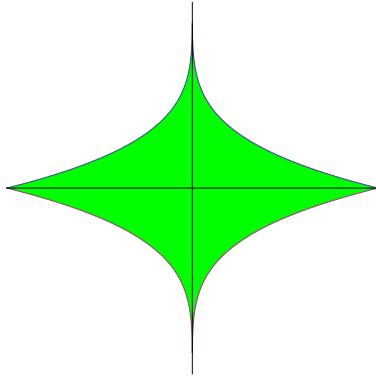
$$\int_{-1}^1 \sqrt{1 - x^2} dx.$$

Finding the anti-derivative is not so easy. We will find techniques to do so, for now we pop it together: we know that $\arcsin(x)$ has the derivative $1/\sqrt{1 - x^2}$ and $x\sqrt{1 - x^2}$ has the derivative $\sqrt{1 - x^2} - x^2/\sqrt{1 - x^2}$. The sum of these two functions has the derivative $\sqrt{1 - x^2} - (1 - x^2)/\sqrt{1 - x^2} = 2\sqrt{1 - x^2}$. We find the anti derivative to be $(x\sqrt{1 - x^2} + \arcsin(x))/2$. The area is therefore

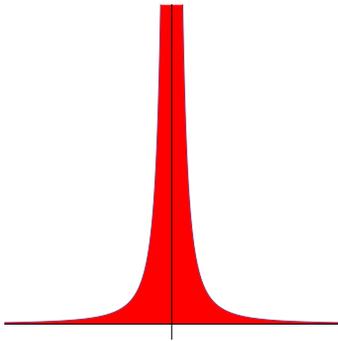
$$\frac{x\sqrt{1 - x^2} + \arcsin(x)}{2} \Big|_{-1}^1 = \frac{\pi}{2}.$$



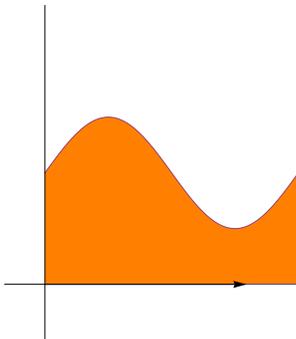
- 5 Find the area of the region between the graphs of $f(x) = 1 - |x|^{1/4}$ and $g(x) = -1 + |x|^{1/4}$.



- 6 Find the area under the curve of $f(x) = 1/x^2$ between -6 and 6 . Solution. $\int_{-6}^6 x^{-2} dx = -x^{-1}|_{-6}^6 = -1/6 - 1/6 = -1/3$. There is something fishy with this computation because $f(x)$ is nonnegative so that the area should be positive. But we obtained a negative answer. What is going on?



- 7 Find the area between the curves $x = 0$ and $x = 2 + \sin(y)$, $y = 2\pi$ and $y = 0$. **Solution:** We turn the picture by 90 degrees so that we compute the area under the curve $y = 0$, $y = 2 + \sin(x)$ and $x = 2\pi$ and $x = 0$.



- 8 **The grass problem.** Find the area between the curves $|x|^{1/3}$ and $|x|^{1/2}$. **Solution.** This example illustrates how important it is to have a picture. This is good advice for any "word problem" in mathematics.

Use a picture of the situation while doing the computation.

Homework

- Find the area of the bounded region enclosed by the graphs $f(x) = x^5 - 12x$ and $g(x) = 2x^2$.
- Find the area of the region enclosed by the four lines $y = x$, $y = 3 - 2x$, $y = -2x$, $y = 3x - 1$.
- Find the area of the region enclosed by the curves $x = 0$, $x = \pi/2$, $y = 4 + \sin(3x)$, $y = 2 + \sin^2(2x)$.
- Write down an integral which gives the area of the region $4x^4 + y^2 \leq 4$ by first writing it as a region between two graphs. Evaluate the integral numerically using Wolfram alpha, Mathematica or any other software.
- The graphs $1 + \sin(x)$ and $2 \cos(x) - 1$ intersect at $x = 0, 2\pi$ and a point between. They define a **humming bird** region, consisting of a larger region and a tail region. Find the area of each part and assume the bird has its eye closed.

