

Lecture 23: Improper integrals

In this lecture, we look at integrals on infinite intervals or integrals, where the function can get infinite at some point. These integrals are called **improper integrals**. The area under the curve can remain finite or become infinite.

- 1 What is the integral

$$\int_1^{\infty} \frac{1}{x^2} dx ?$$

Since the anti-derivative is $-1/x$, we have

$$\frac{-1}{x} \Big|_1^{\infty} = -1/\infty + 1 = 1.$$

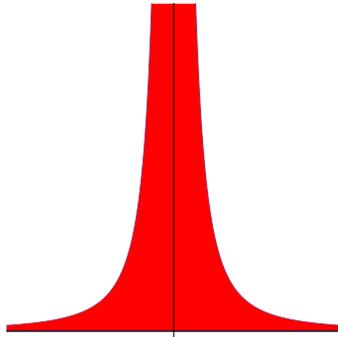
To justify this, compute the integral $\int_1^b 1/x^2 dx = 1 - 1/b$ and see that in the limit $b \rightarrow \infty$, the value 1 is achieved.

In a previous lecture, we have seen a shocking example similar to the following one:

- 2

$$\int_{-1}^1 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{-1}^1 = -1 - 1 = -2.$$

This does not make any sense because the function is positive so that the integral should be a positive area. The problem is this time not at the boundary $-1, 1$. The sore point is $x = 0$ over which we have carelessly integrated over.



The next example illustrates the problem with the previous example better:

- 3 The computation

$$\int_0^1 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_0^1 = -1 + \infty.$$

indicates that the integral does not exist. We can justify by looking at integrals

$$\int_a^1 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_a^1 = -1 + \frac{1}{a}$$

which are fine for every $a > 0$. But this does not converge for $a \rightarrow 0$.

Do we always have a problem if the function goes to infinity at some point?

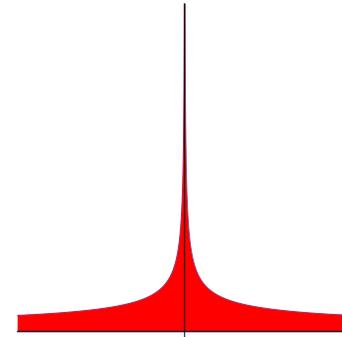
- 4 Find the following integral

$$\int_0^1 \frac{1}{\sqrt{x}} dx.$$

Solution: Since the point $x = 0$ is problematic, we integrate from a to 1 with positive a and then take the limit $a \rightarrow 0$. Since $x^{-1/2}$ has the antiderivative $x^{1/2}/(1/2) = 2\sqrt{x}$, we have

$$\int_a^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x} \Big|_a^1 = 2\sqrt{1} - 2\sqrt{a} = 2(1 - \sqrt{a}).$$

There is no problem with taking the limit $a \rightarrow 0$. The answer is 2. Even so the region is infinite its area is finite. This is an interesting example. Imagining this to be a container for paint. We can fill the container with a finite amount of paint but the wall of the region has infinite length.



- 5 Evaluate the integral $\int_0^1 1/\sqrt{1-x^2} dx$. **Solution:** The antiderivative is $\arcsin(x)$. In this case, it is not the point $x = 0$ which produces the difficulty. It is the point $x = 1$. Take $a > 0$ and evaluate

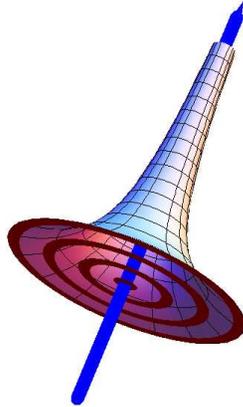
$$\int_0^{1-a} \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) \Big|_0^{1-a} = \arcsin(1-a) - \arcsin(0).$$

Now $\arcsin(1-a)$ has no problem at limit $a \rightarrow 0$. Since $\arcsin(1) = \pi/2$ exists. We get therefore the answer $\arcsin(1) = \pi/2$.

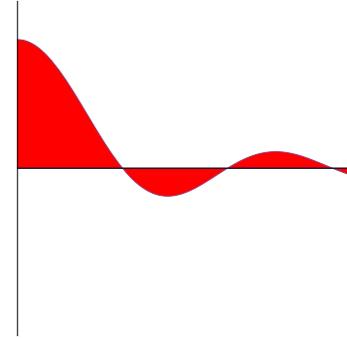
- 6 Rotate the graph of $f(x) = 1/x$ around the x -axes and compute the volume of the solid between 1 and ∞ . The cross section area is π/x^2 . If we look at the integral from 1 to a fixed R , we get

$$\int_1^R \frac{\pi}{x^2} dx = -\frac{\pi}{x} \Big|_1^R = -\pi/R + \pi.$$

This converges for $R \rightarrow \infty$. The volume is π . This famous solid is called **Gabriels trumpet**. This solid is so prominent because if you look at the surface area of the small slice, then it is larger than $dx2\pi/x$. The total surface area of the trumpet from 1 to R is therefore larger than $\int_1^R 2\pi/x dx = 2\pi(\log(R) - \log(1))$, which goes to infinity. We can fill the trumpet with a finite amount of paint but we can not **paint** its surface.



second semester course like Math 1b. The integral can be written as an alternating series, which converges and there are many ways to compute it: ¹

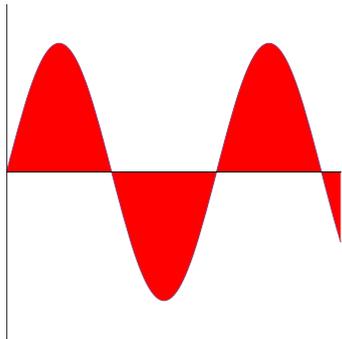


Lets summarize the two cases of improper integrals: infinitely long intervals and a point where the function becomes infinite.

- 1) To investigate the improper integral $\int_a^\infty f(x) dx$ we look at the limit $\int_a^b f(x) dx$ for $b \rightarrow \infty$.
- 1) To investigate improper integral $\int_0^b f(x) dx$ where $f(x)$ is not continuous at 0, we take the limit $\int_a^b f(x) dx$ for $a \rightarrow 0$.

Finally, lets look at the following example

- 7 Evaluate the integral $\int_0^\infty \sin(x) dx$. **Solution.** There is no problem at the boundary 0 nor at any other point. We have to investigate however, what happens at ∞ . Therefore, we look at the integral $\int_0^b \sin(x) dx = -\cos(x)|_0^b = 1 - \cos(b)$. We see that the limit $b \rightarrow \infty$ does not exist. The integral fluctuates between 0 and 2.



The next example leads to a topic in a follow-up course. It is not covered here, but could make you curious:

- 8 What about the integral

$$I = \int_0^\infty \frac{\sin(x)}{x} dx ?$$

Solution. The anti derivative is the Sine integral $Si(x)$ so that we can write $\int_0^b \sin(x)/x dx = Si(b)$. It turns out that the limit $b \rightarrow \infty$ exists and is equal to $\pi/2$ but this is a topic for a

Homework

- 1 Evaluate the integral $\int_1^2 \frac{5}{\sqrt{x-1}} + \pi \cos(\pi x) dx$.
- 2 Evaluate the following integrals
 - a) $\int_0^1 2x/\sqrt{1-x^2} dx$.
 - b) $\int_0^1 2/\sqrt{1-x^2} dx$.
 Hint: For a) think about the chain rule $d/dx f(g(x)) = f'(g(x))g'(x)$
- 3 Evaluate the integral $\int_{-3}^4 (x^2)^{1/3} dx$. To make sure that the integral is fine, check whether \int_{-3}^0 and \int_0^4 work.
- 4 The integral $\int_{-2}^1 1/x dx$ does not exist. We can however take a positive $b > 0$ and look at

$$\int_{-2}^{-b} 1/x dx + \int_b^1 1/x dx = \log |b| - \log |-2| + (\log |1| - \log |b|) = \log(2) .$$

This value is called the **Cauchy principal value** of the integral. Find the principal value of

$$\int_{-4}^5 3/x^3 dx$$

using the same process as before, by cutting out $[-a, a]$ and then taking the limit $a \rightarrow 0$.

- 5 a) Evaluate $\int_{-1}^1 \frac{1}{x^4} dx$ blindly, without worrying that the function is infinite.
- b) Can we give a principal value integral value to $\int_{-1}^1 \frac{1}{x^4} dx$? If yes, find the value. If not, tell why not.

¹Hardy, Mathematical Gazette, 5, 98-103, 1909.