

Lecture 30: Numerical integration

After repeating and before looking at other integration techniques, we briefly look at some numerical techniques. There are variations of Riemann sums which speed up the computation.

Riemann sum with nonuniform spacing

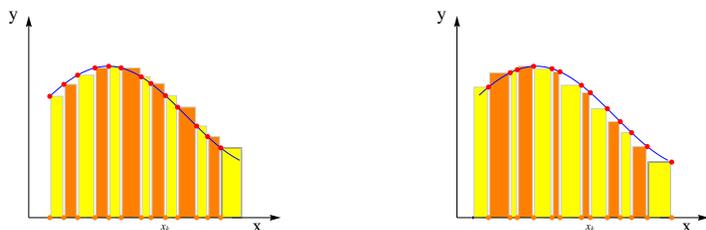
A more general Riemann sum is obtained by choosing n points in $[a, b]$ and defining

$$S_n = \sum_{y_j} f(y_j)(x_{j+1} - x_j) = \sum_{y_j} f(y_j)\Delta x_j$$

where y_j is in (x_j, x_{j+1}) .

This is how Riemann sums are usually introduced in calculus books. The generalization allows to use a small mesh size where the function fluctuates a lot. For theoretical purposes it is mostly equivalent.

The sum $\sum f(x_j)\Delta x_j$ is called the **left Riemann sum**, the sum $\sum f(x_{j+1})\Delta x_j$ the **right Riemann sum**.

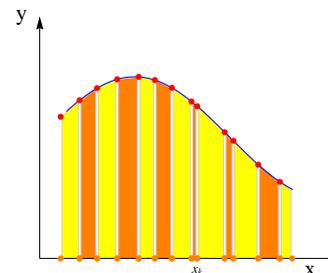


If $x_0 = a, x_n = b$ and $\max_j \Delta x_j \rightarrow 0$ for $n \rightarrow \infty$ then S_n converges to $\int_a^b f(x) dx$.

- 1 If $x_j - x_k = 1/n$ and $z_j = x_j$, then we have the Riemann sum as we defined it earlier.
- 2 You numerically integrate $\sin(x)$ on $[0, \pi/2]$ with a Riemann sum. What is better, the left Riemann sum or the right Riemann sum? Look also at the interval $[\pi/2, \pi]$? **Solution:** you see that in the first case, the left Riemann sum is smaller than the actual integral. In the second case, the left Riemann sum is larger than the actual integral.

Trapezoid rule

The average between the left and right hand Riemann sum is called the **Trapezoid rule**. Geometrically, it sums up areas of trapezoids instead of rectangles.



The **Trapezoid rule** computes

$$\frac{1}{2n} \sum_{k=1}^n [f(x_k) + f(x_{k+1})].$$

Remark. The Trapezoid rule does not change things much in general. In the case of equal spacing $x_k = a + (b - a)k/n$.

$$\frac{1}{2n}[f(x_0) + f(x_n)] + \frac{1}{n} \sum_{k=1}^{n-1} f(x_k).$$

Simpson rule

Much better is the Simpson method:

The **Simpson rule** computes the sum

$$S_n = \frac{1}{6n} \sum_{k=1}^n [f(x_k) + 4f(y_k) + f(x_{k+1})],$$

where y_k are the midpoints between x_k and x_{k+1} .

The Simpson rule is good because it is exact for quadratic functions: you can check for $f(x) = ax^2 + bx + c$ that the formula

$$\frac{1}{v-u} \int_u^v f(x) dx = [f(u) + 4f((u+v)/2) + f(v)]/6$$

holds. To prove it just run the following two lines in Mathematica: (== means "is equal")

```
f[x_] := a x^2 + b x + c;
Simplify[(f[u]+f[v]+4f[(u+v)/2])/6==Integrate[f[x],{x,u,v}]/(v-u)]
```

This actually will imply (as you might see in a course like Math 1b) that the numerical integration for functions which are 4 times differentiable gives numerical results which are n^{-4} close to the actual integral. For 100 division points, this can give accuracy to 10^{-8} already. We see this in a demonstration.

There are other variants which are a bit better but need more function values. If x_k, y_k, z_k, x_{k+1} are equally spaced, then

The **Simpson 3/8 rule** computes

$$\frac{1}{8n} \sum_{k=1}^n [f(x_k) + 3f(y_k) + 3f(z_k) + f(x_{k+1})].$$

This formula is again exact for quadratic functions: for $f(x) = ax^2 + bx + c$, the formula

$$\frac{1}{v-u} \int_u^v f(x) dx = [f(u) + 3f((2u+v)/3) + 3f((u+2v)/3) + f(v)]/6$$

holds. If you are interested, run the two Mathematica lines:

```
f[x_] := a x^2 + b x + c; L=Integrate[f[x],{x,u,v}]/(v-u);
Simplify[(f[u]+f[v]+3f[(2u+v)/3]+3f[(u+2v)/3])/8==L]
```

This 3/8 method can be slightly better than the first Simpson rule.

Monte Carlo Method

A powerful integration method is to choose n random points x_k in $[a, b]$ and look at the sum divided by n . Because it uses randomness, it is called **Monte Carlo method**.

The **Monte Carlo** integral is the limit S_n to infinity

$$S_n = \frac{1}{n} \sum_{k=1}^n f(x_k),$$

where x_k are n random values in $[a, b]$.

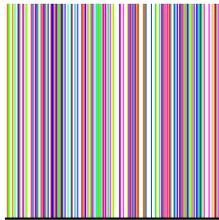
The law of large numbers in probability shows that the **Monte Carlo integral** is equivalent to the **Lebesgue integral** which is more powerful than the Riemann integral. Monte Carlo integration is interesting especially if the function is complicated.

3 The **salt and pepper** function is defined as

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

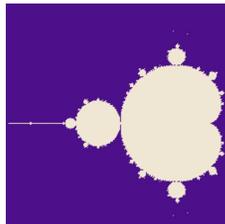
The Riemann integral with equal spacing k/n is equal to 1 for every n . But this is only because we have evaluated the function at rational points, where it is 1.

The Monte Carlo integral gives zero because if we chose a random number in $[0, 1]$ we hit an irrational number with probability 1.



The Salt and Pepper function and the Boston Salt and Pepper bridge (Anne Heywood).

The following two lines evaluate the **area of the Mandelbrot fractal** using Monte Carlo integration. The function F is equal to 1, if the parameter value c of the quadratic map $z \rightarrow z^2 + c$ is in the Mandelbrot set and 0 else. It shoots 100'000 random points and counts what fraction of the square of area 9 is covered by the set. Numerical experiments give values close to the actual value around 1.51.... One could use more points to get more accurate estimates.



```
F[c_] := Block[{z=c, u=1}, Do[z=N[z^2+c]; If[Abs[z]>3, u=0; z=3], {99}]; u;
M=10^5; Sum[F[-2.5+3 Random[]+I(-1.5+3 Random[])], {M}]* (9.0/M)
```

Homework

- 1 Generate $n = 10$ random numbers x_k in $[0, 1]$, then sum up the square x_k^2 of these numbers and divide by $n = 10$. Compare your result with $\int_0^1 x^2 dx$. **Remark.** If using a program, increase the value of n as large as you can. Here is a Mathematica code:

```
n=20; Sum[Random[]^2, {n}]/n
```

Here is an implementation in Perl. Its still possible to cram the code into one line:

```
#!/usr/bin/perl
$n=20;$s=0;for($i=0;$i<$n;$i++){ $f=rand(); $s+=$f*$f;} print $s/$n;
```

- 2 a) Use Simpson's rule to compute $\int_0^\pi \sin(x) dx$ using $n = 2$ intervals $[0, \pi/2]$ and $[\pi/2, \pi]$. On each of these intervals $[a, b]$ compute the Simpson sum $[f(a) + 4f((a+b)/2) + f(b)]/6$ with $f(x) = \sin(x)$. Compare with the actual integral.
b) Now use the 3/8 Simpson rule to estimate $\int_0^\pi f(x) dx$ using $n = 1$ intervals $[0, \pi]$. Again compare with the actual integral.

Instead of adding more numerical methods exercises, we want to practice a bit more integration. The challenge in the following problems is to find out which integration method is best suited. This is good preparation for the final, where we will not reveal which integration method is the best.

- 3 Integrate $\tan(x)/\cos(x)$ from 0 to $\pi/6$.
- 4 Find the antiderivative of $x \sin(x) \exp(x)$.
- 5 Find the antiderivative of $x/\sin(x)^2$.