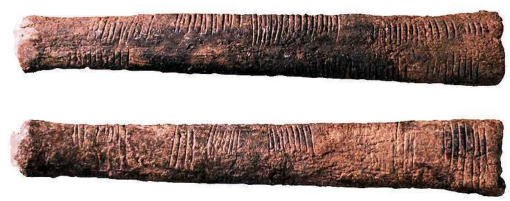
Lecture 1: What is Calculus?

Calculus formalizes the process of **taking differences** and **taking sums**. Differences measure **change**, sums explore how things **accumulate**. The process of taking differences has a limit called **derivative**. The process of taking sums will lead to the **integral**. These two processes are related in an intimate way. In this first lecture, we look at these two processes in the simplest possible setup, where functions are evaluated only on integers and where we do not take any limits. About 25'000 years ago, numbers were represented by units like



$$1, 1, 1, 1, 1, 1, \dots$$

for example carved in the fibula bone of a baboon[1]. It took thousands of years until numbers were represented with symbols like today

$$0, 1, 2, 3, 4, \dots \ .$$

Using the modern concept of function, we can say $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$ and mean that the **function** f assigns to an input like 1001 an output like $f(1001) = 1001$. Lets call $Df(n) = f(n+1) - f(n)$ the **difference** between two function values. We see that the function f satisfies $Df(n) = 1$ for all n. We can also formalize the summation process. If $g(n) = 1$ is the function which is constant 1, then $Sg(n) = g(0) + g(1) + \dots + g(n-1) = 1 + 1 + ... + 1 = n$. We see that $Df = g$ and $Sg = f$. Now lets start with $f(n) = n$ and apply **summation** on that function:

$$Sf(n) = f(0) + f(1) + f(2) + \dots + f(n-1) \ .$$

In our case, we get the following values:

$$0, 1, 3, 6, 10, 15, 21, \dots \ .$$

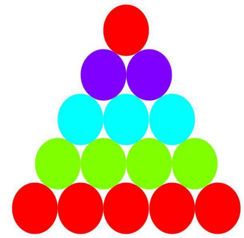
The new function g satisfies $g(1) = 1, g(2) = 3, g(2) = 6$, etc. These numbers are called **triangular numbers**. From the function g we can get f back by taking difference:

$$Dg(n) = g(n+1) - g(n) = f(n) \ .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$ which indeed is $f(5)$. Finding a formula for the sum $Sf(n)$ is not so easy. Can you do it? When **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, Mr. Büttner gave him the task to sum up the first 100 numbers $1+2+\cdots+100$. Gauss found the answer immediately by pairing things up: to add up $1+2+3+\ldots+100$ he would write this as $(1+100)+(2+99)+\cdots+(50+51)$ leading to 50 terms of 101 to get for $n = 101$ the value $g(n) = n(n-1)/2 = 5050$. Taking differences again is easier $Dg(n) = n(n+1)/2 - n(n-1)/2 = n = f(n)$.

[1] This is the **Ishango bone**. The lengths of the marks could also have some astronomical significance but this is not relevant here

Lets add up the new sequence again and compute $h = Sg$. We get the sequence



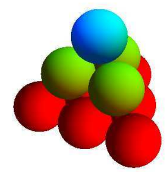
$$0, 1, 4, 10, 20, 35, ...$$

These numbers are called the **tetrahedral numbers** because one use $h(n)$ balls to build a tetrahedron of side length n. For example, we need $h(4) = 20$ golf balls for example to build a tetrahedron of side length 4. The formula which holds for h is $\boxed{h(n) = n(n-1)(n-2)/6}$. We see (see worksheet) that summing the differences gives the function in the same way as differencing the sum:

$$\boxed{SDf(n) = f(n) - f(0), \qquad DSf(n) = f(n)}$$

This is an arithmetic version of the **fundamental theorem of calculus**. The process of adding up numbers will lead to the **integral** $\boxed{\int_0^x f(x)\, dx}$.

The process of taking differences will lead to the **derivative** $\boxed{\frac{d}{dx} f(x)}$. Later in this course, we will encounter the **fundamental theorem of calculus**



$$\boxed{\int_0^x \frac{d}{dt} f(t)\, dt = f(x) - f(0), \qquad \frac{d}{dx} \int_0^x f(t)\, dt = f(x)}$$

and realize why it is such a fantastic result. You see how well it fits the result for difference and sum. A major goal of this course will be to understand this fundamental theorem result and how we apply it. We have just seen it once and packed the essence of the theorem in the above version with S and D. We will revisit the theorem at least twice again. [2] The above version will lead us. Note that if we define $[n]^0 = 1, [n]^1 = n, [n]^2 = n(n-1)/2, [n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1], D[n]^2 = 2[n], D[n]^3 = 3[n]^2$ and in general

$$\boxed{\frac{d}{dx}[x]^n = n[x]^{n-1}}$$

We will generalize this from $h = 1$ to general $h > 0$ and then see that it also holds in the limit $h \to 0$ where it is the familiar formula $(d/dx)x^n = nx^{n-1}$ you might have seen in high school. The calculus you have just seen, contains the essence of single variable calculus. This core idea will become more powerful and natural if we use it together with the concept of limit.

1

Problem: The sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ satisfies the rule $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$? **Solution:** We take the difference between successive numbers and get the sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, ...$$

which is the same sequenc again. We can deduce from this recursion that f has the property that $\boxed{Df(x) = f(x-1)}$. It is called the **Fibonnacci sequence**, a sequence of great fame.

2 **Problem:** Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, ...$ What sequence is that?

Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ is subtracted. It has been relatively easy to find the sum, because we knew what the difference operation did. This example shows:

[2] Many textbooks need hundreds of pages until the fundamental theorem is reached.

We can study differences to understand sums.

The next problem illustrates this too:

3 **Problem:** Find the next term in the sequence
2 6 12 20 30 42 56 72 90 110 132 . **Solution:** Take differences

2	6	12	20	30	42	56	72	90	110	132
2	4	6	8	10	12	14	16	18	20	22
2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0

Now we can add an additional number, starting from the bottom and working us up.

2	6	12	20	30	42	56	72	90	110	132	156
2	4	6	8	10	12	14	16	18	20	22	24
2	2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0	0

4 **Problem:** The function $f(n) = 2^n$ is called the **exponential function**. We have for example $f(0) = 1, f(1) = 2, f(2) = 4, \dots$. It leads to the sequence of numbers

n=	0	1	2	3	4	5	6	7	8	...
f(n)=	1	2	4	8	16	32	64	128	256	...

We can verify that f satisfies the equation $\boxed{Df(x) = f(x)}$. because $Df(x) = 2^{x+1} - 2^x = (2-1)2^x = 2^x$.
This is an important special case of the fact that

The derivative of the exponential function is the exponential function itself.

The function 2^x is a special case of the exponential function when the Planck constant is equal to 1. We will see that the relation will hold for any $h > 0$ and also in the limit $h \to 0$, where it becomes the classical exponential function e^x which plays an important role in science.

5 **Problem:** Look at the function $f(n)$ which gives the n'th prime number. Lets look at the derivatives $D^k f$ but take the absolute value $|D^k(f)|$. In other words, we study $T(f)(n) = |f(n+1) - f(n)|$. We know for example that $f(n) = 2^n$ satisfies $Tf = f$. Lets look at the prime function and apply this differences:

n=	1	2	3	4	5	6	7	8	9	...
$f(n) =$	2	3	5	7	11	13	17	23	29	...
$Tf(n) =$	1	2	2	4	2	4	2	4	6	...
$T^2f(n) =$	1	0	2	2	2	2	2	2	4	...
$T^3f(n) =$	1	2	0	0	0	0	0	2	0	...

The **Gilbreath conjecture** of 1959 claims that the first entry remains 1 for ever when applying this absolute differentiation process. The problem is still open.

Homework

1 Look at the odd numbers $f(n) = 2n+1$. The sequence starts with

$$1, 3, 5, 7, 9, 11, 13, \dots$$

Find a formula for $Sf(n) = f(0) + f(1) + f(2) + \dots + f(n-1)$. For example $g(3) = f(0) + f(1) + f(2) = 1 + 3 + 5 = 9$.

2 We have defined $Sf(n) = f(0) + f(2) + ... + f(n-1)$ and $Df(n) = f(n+1) - f(n)$ and seen

$$\begin{array}{lll} f(n) = [n]^0 = 1 & \text{we have} & g(n) = Sf(n) = n \ . \\ f(n) = [n]^1 = n & \text{we have} & g(n) = Sf(n) = n(n-1)/2. \\ f(n) = [n]^2 = n(n-1)/2 & \text{we have} & g(n) = Sf(n) = n(n-1)(n-2)/6. \end{array}$$

Verify that if $f(n) = n(n-1)(n-2)/6$ then $Sf(n) = n(n-1)(n-2)(n-3)/24$ satisfies $Dg(n) = f(n)$. Can you see a pattern?

3 Find the next term in the sequence $3, 12, 33, 72, 135, 228, 357, 528, 747, 1020, 1353....$ To do so, compute successive derivatives $g = Df$ of f, then $h = Dg$ until you see a pattern.

4 Find $F(n) = Sf(n)$ for the function $f(n) = n^2$. This means we want to find a formula such that $F(1) = 1, F(2) = 5, F(3) = 14$ leading to the sequence of numbers

$$0, 1, 5, 14, 30, 55, 91, 140, 204, 285, \dots \ .$$

We have already have computed Sg for $g(n) = n(n-1)/2$ as well as Sh for $h(n) = n$. Try to write f as a combination of g and h and use the **addition rule** $D(h+h) = Dg + Dh$.

5 Find a formula $g(n) = Sf(n)$ for the function $f(n) = 7^n$. First compute the "derivative" Df of f and go from there.

General remarks about homework

- Make sure to think about the problem yourself first before discussing it with others.
- The time you spend on homework is valuable. Especially the exploration time before you know how to solve it.
- If you do not know how to get started, don't hesitate to ask.

Lecture 2: Functions

A **function** is a rule which assigns to a real number a new real number. The function $f(x) = x^2 - 2x$ for example assigns to the number $x = 4$ the value $4^2 - 8 = 8$. A function is given with a **domain** A, the points where f is defined and a **codomain** B a set of numbers which f can reach. The function $f(x) = x^2 - 2x$ is defined on the entire real axes. While $f(x) \geq -1$, the codomain is the set of real numbers.

Typically, the codomain agrees with the set of real numbers and the domain to be all the numbers, where the function is defined. The function $f(x) = 1/x$ for example is not defined at $x = 0$ so that we chose the domain $A = R \setminus \{0\}$, all numbers except 0. The function $f(x) = 1/x$ takes values in the codomain R. If we choose $A = B$, then $f(x) = 1/x$ reaches every point in B and is invertible. It is its own inverse. Here are a few examples of functions. We will look at them in more detail during the lecture, especially the polynomials, trigonometric functions and exponential function.

identity	$f(x) = x$	power	$f(x) = 2^x$
constant	$f(x) = 1$	exponential	$f(x) = e^x = \exp(x)$
linear	$f(x) = 3x + 1$	logarithm	$f(x) = \log(x) = \exp^{-1}(x)$
quadratic	$f(x) = x^2$	absolute value	$f(x) = \|x\|$
cosine	$f(x) = \cos(x)$	devil comb	$f(x) = \sin(1/x)$
sine	$f(x) = \sin(x)$	bell function	$f(x) = e^{-x^2}$
exponentials	$f(x) = \exp_h(x) = (1 + h)^{x/h}$	witch of Agnesi	$f(x) = \frac{1}{1+x^2}$
logarithms	$f(x) = \log_h(x) = \exp_h^{-1}$	sinc	$\sin(x)/x$

We can build new functions by:

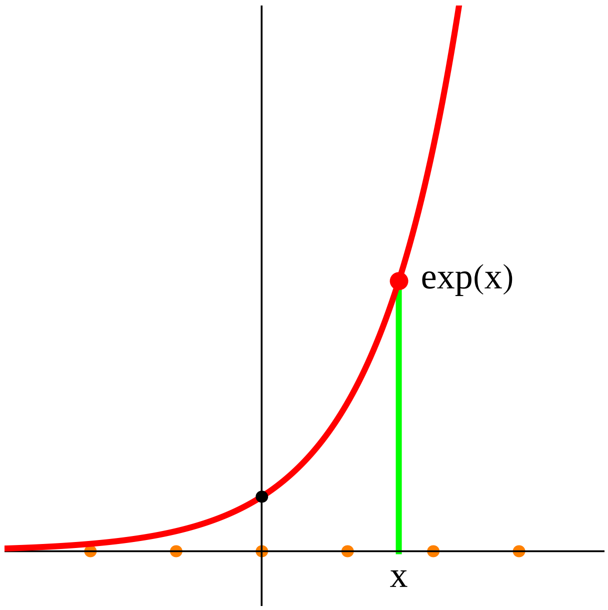
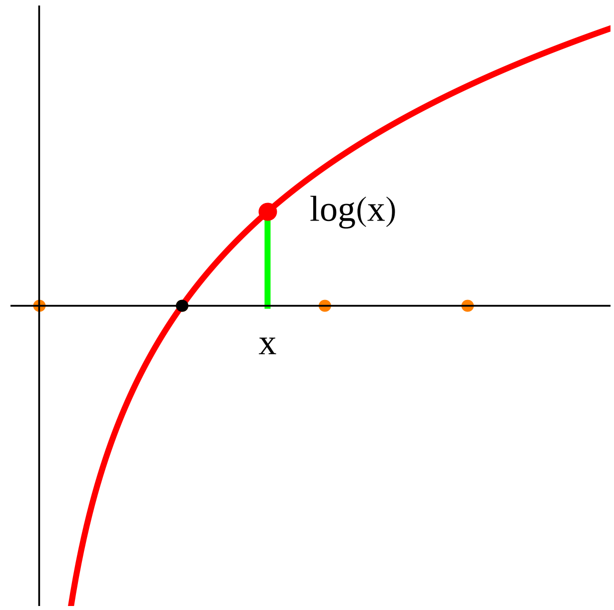
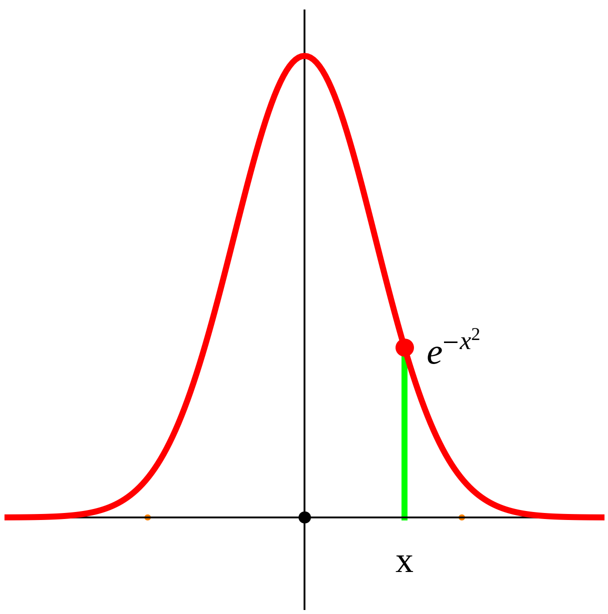
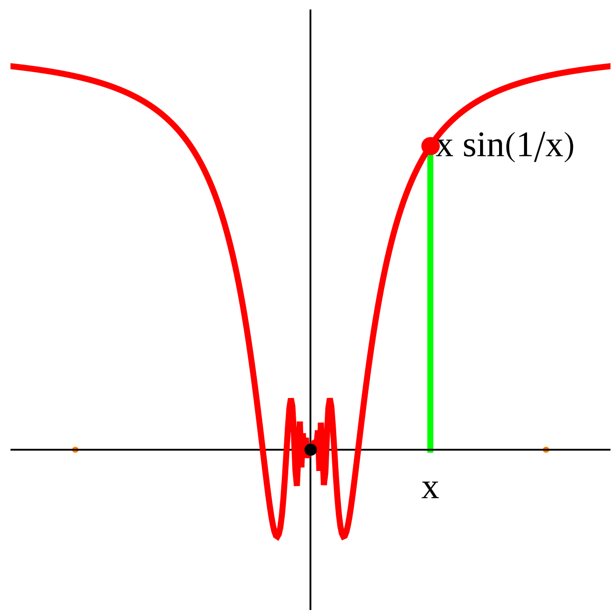
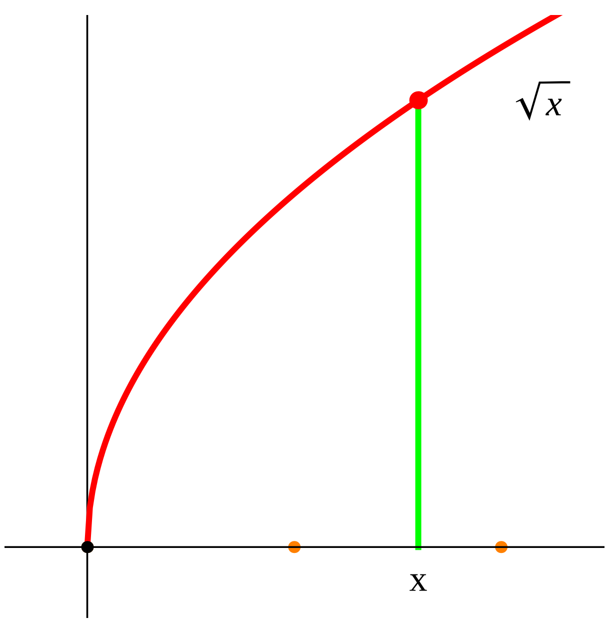
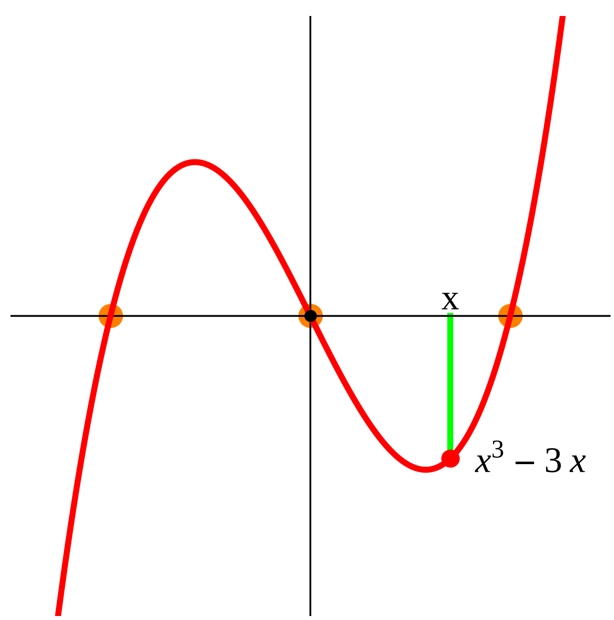
addition	$f(x) + g(x)$
scaling	$2f(x)$
translate	$f(x + 1)$
compose	$f(g(x))$
invert	$f^{-1}(x)$
difference	$f(x + 1) - f(x)$
sum up	$f(x) + f(x + 1) + \ldots$

Here are important functions:

polynomials	$x^2 + 3x + 5$
rational functions	$(x + 1)/(x^4 + 1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arcsin^{-1}(x), \arctan(x)$.
roots	$\sqrt{x}, x^{1/3}$

We will look at these functions **a lot** during this course. The logarithm, exponential and trigonometric functions are especially important.
For some functions, we need to restrict the domain, where the function is defined. For the square root function $\sqrt{x}$ or the logarithm $\log(x)$ for example, we have to assume that the number is positive. We write that the domain is $(0, \infty) = \mathbf{R}^+$. For the function $f(x) = 1/x$, we have to assume that x is different from zero. Keep these three examples in mind.

The **graph** of a function is the set of points $\{(x, y) = (x, f(x))\}$ in the plane, where x runs over the domain A of f. Graphs allow us to **visualize** functions. We can "see them", when we draw the graph.


exp(x)
x

log(x)
x

e^{-x^2}
x

x sin(1/x)
x

$\sqrt{x}$
x

x
$x^3 - 3\,x$

Homework

1 Draw the function $f(x) = x\sin(x)$. Its graph goes through the origin $(0,0)$.
a) A function is called **odd** if $f(-x) = -f(x)$. Is f odd?
b) A function is called **even** if $f(x) = f(-x)$. Is f even?
c) A function is called **monotone increasing** if $f(y) > f(x)$ if $y > x$. Is f monotone increasing? No need to decide this yet analytically. Just draw(*) and decide.

2 A function $f : A \to B$ is called **invertible** or **one to one** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$. For example, the function $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0,\infty)$ to $B = [0,\infty)$. Determine from the following functions whether they are invertible. If they are invertible, find the inverse.
a) $f(x) = \cos(x)$ from $A = [0,\pi/2]$ to $B = [0,1]$
b) $f(x) = x^5$ from $A = \mathbf{R}$ to $B = \mathbf{R}$
c) $f(x) = x^4$ from $A = \mathbf{R}$ to $B = \mathbf{R}$
d) $f(x) = \exp(2x)$ from $A = \mathbf{R}$ to $B = R^+ = (0,\infty)$.
e) $f(x) = 1/(1+x^2)$ from $A = [0,\infty)$ to $B = [0,\infty)$.

3 Look at the function $f_1(x) = 7x(1-x)/2, f_2(x) = f_1(f_1(x)), f_3(x) = f_2(f_2(x))$.
a) Draw the graphs of the functions f_1, f_2, f_3 on the interval $[0,1]$.
b) Can you imagine what $f_{100}(x)$ looks like? You might want to make more experiments here to see the answer. Of course you are allowed to plot the functions with a calculator with Wolfram alpha or with Mathematica.

```
Plot[NestList[(7/2) (# (1 - #)) &, x, 3], {x, 0, 1}]
```

4 Lets call a function $f(x)$ a **composition square root** of a function g if $f(f(x)) = g(x)$. For example, the function $f(x) = x^2+1$ is the composition square root of $g(x) = x^4+2x^2+2$ because $f(f(x)) = (x^2+1)^2+1 = g(x)$. Find the composition square roots of the following functions:
a) $f(x) = \cos(\cos(x))$.
b) $f(x) = x^4$
c) $f(x) = x$
d) $f(x) = x^4 + 2x^2 + 2$
e) $f(x) = e^{e^x}$.

Note that it can be difficult in general to find the square root function in general. Already for basic functions like $\exp(x)$ or $\sin(x)$, we are speechless.

5 A function $f(x)$ has a **root** at $x = a$ if $f(a) = 0$. Roots are places, where the function is zero. Find one root for each of the following functions or state that there is none.
a) $f(x) = \cos(x)$
b) $f(x) = \exp(-x^2)$
c) $f(x) = x^3 - x$
d) $f(x) = \sin(x) - 1$
e) $f(x) = \csc(x) = 1/\sin(x)$

(*) Here is how you can use the Web to plot a function. The example given is $\sin(x)$.

```
http://www.wolframalpha.com/input/?i=Plot+sin(x)
```

Lecture 3: Limits

We have seen that functions like $1/x$ are not defined everywhere. Sometimes, however, functions do not make sense at first but can nevertheless be saved. A silly example is $f(x) = x/x$ which is apriori not defined at $x = 0$ because we divide by x but can be "saved" by noticing that $f(x) = 1$ for all x different from 0. Functions often can be continued to "non-allowed" places if we write the function differently. This often involves dividing out a common factor. Lets look at some examples:

1 The function $f(x) = (x^3 - 1)/(x - 1)$ is at first not defined at $x = 1$. However, for x close to 1, nothing really bad happens. We can evaluate the function at points closer and closer to 1 and get closer and closer to 3. We will say $\lim_{x\to 1} f(x) = 3$. Indeed, as you might have noticed already, we have $f(x) = x^2 + x + 1$ by factoring out the term $(x - 1)$. While the function was initially not defined at $x = 1$, we can assign a natural value 3 at the point $x = 1$ so that the graph continues nicely through that point.

Definition. We write $x \to a$ to say that the number x approaches a from either side. A function $f(x)$ has a **limit** at a point a if there exists b such that $f(x) \to b$ for $x \to a$. We write $\lim_{x\to a} f(x) = b$. It should not matter, whether we approach a from the left or from the right. In both cases, we should get the same limiting value b.

2 The function $f(x) = \sin(x)/x$ is called $\mathrm{sinc}(x)$. It converges to 1 as $x \to 0$. We can see this geometrically by comparing the side $a = \sin(x)$ of a right angle triangle with a small angle $\alpha = x$ and hypotenuse 1 with the length of the arc between B, C of the unit circle centered at A. The arc has length x which is close to $\sin(x)$ for small x. Keep this example in mind. It is a crucial one. The fact that the limit of $f(x)$ exists for $x \to 0$ is some important that it is sometimes called the **fundamental theorem of trigonometry**.

3 The quadratic function $f(x) = x^2$ has the property that $f(x)$ approaches 4 if x approaches 2. To evaluate functions at a point, we do not have to take a limit. The function is already defined there. This is important: most points are "healthy". We do not have to worry about limits there. In most cases we see in real applications we only have to worry about limits when the function involves division by 0. For example $f(x) = (x^4 + x^2 + 1)/x$ needs to be investigated carefully at $x = 0$. You see for example that for $x = 1/1000$, the function is slightly larger than 1000, for $x = 1/1000000$ it is larger than one million. There is no rescue here. The limit does not exist at 0.

4 More generally, for all polynomials, the limit $\lim_{x\to a} f(x) = f(a)$ is defined. We do not have to worry about limits, if we deal with polynomials.

5 For all trigonometric polynomials involving sin and cos, the limit $\lim_{x\to a} f(x) = f(a)$ is defined. We do not have to worry about limits if we deal with trigonometric polynomials like $\sin(3x) + \cos(5x)$. The function $\tan(x)$ however has no limit at $x = \pi/2$. No finite value b can be found so that $\tan(\pi/2 + h) \to b$ for $h \to 0$. This is due to the fact that $\cos(x)$ is zero at $\pi/2$. We have $\tan(x)$ goes to $+\infty$ "plus infinity" for $x \searrow \pi/2$ and $\tan(x)$ goes to $-\infty$ for $x \nearrow \pi/2$. In the first case, we approach $\pi/2$ from the right and in the second case from the left.

6 The **cube root** function $f(x) = x^{1/3}$ converges to 0 as $x \to 0$. For $x = 1/1000$ for example, we have $f(x) = 1/10$ for $x = 1/n^3$ the value $f(x)$ is $1/n$. The cube root function is defined everywhere on the real line, like $f(-8) = -2$ and is continuous everywhere.

Why do we worry about limits at all? One of the main reasons will is that we will define the derivative and integral using limits. But we will also use limits to get numbers like $\pi = 3.1415926, ...$ In the next lecture, we will look at the important concept of continuity, which involves limits too.

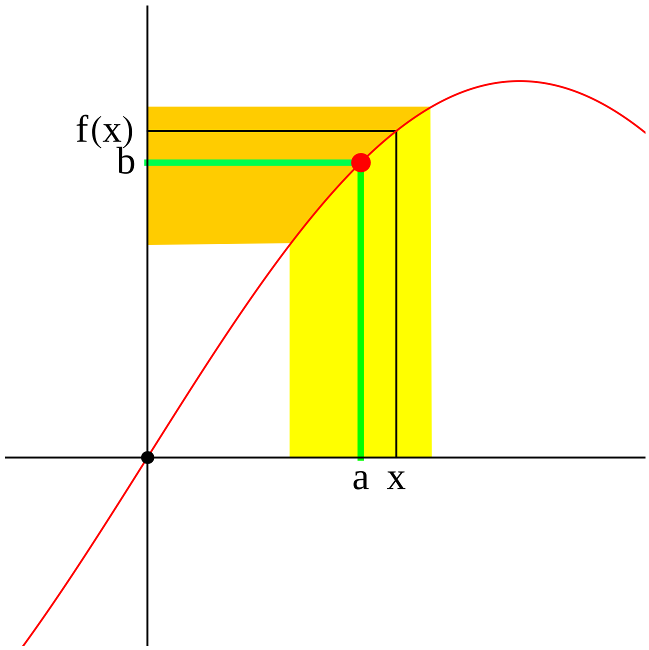


Figure: We can test whether a function has the limit b at a point a if for every vertical interval I containing b there exists a horizontal interval J containing a such that if x is in J, then $f(x)$ is in I. If the function stays bounded, does not oscillate at the point like $\sin(1/x)$ or jump, then the limit exists.

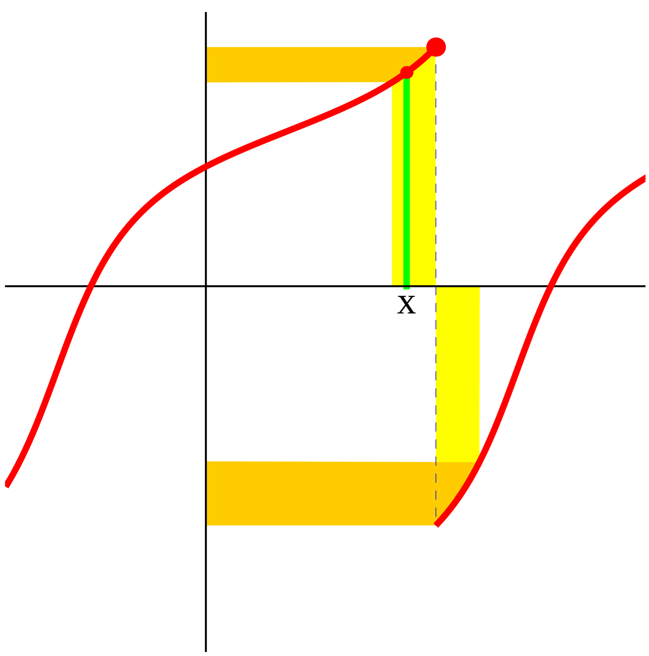


Figure: We see here the function $f(x) = \arctan(\tan(x) + 1)$, where *arctan* is the inverse of *tan* giving the angle from the slope. In this case, the limit does not exist for $a = \pi/2$. If we approach this point a from the right, we are always far below the limiting value. The limit exists from the left if we postulate $f(\pi/2) = \pi/2$. Note that f has a priori no value at $x = \pi/2$ because $\tan(x)$ becomes infinite there.

7 **Problem:** Determine from the following functions whether the limits $\lim_{x\to 0} f(x)$ exist. If the limit exists, find it.
a) $f(x) = \cos(x)/\cos(2x)$
b) $f(x) = \tan(x)/x$
c) $f(x) = (x^2 - x)/(x - 1)$
d) $f(x) = (x^4 - 1)/(x^2 - 1)$
e) $f(x) = (x + 1)/(x - 1)$
f) $f(x) = x/\sin(x)$
g) $f(x) = 5x/\sin(6x)$
h) $f(x) = \sin(x)/x^2$
i) $f(x) = \sin(x)/\sin(2x)$
j) $f(x) = \exp(x)/x$

Solutions:

a) There is no problem at all at $x = 0$. Both, the nominator and denominator converge to 1. The limit is $\boxed{1}$
b) This is $\text{sinc}(x)/\cos(x)$. There is no problem at $x = 0$ for sinc nor for $1/\cos(x)$. The limit is $\boxed{1}$.
c) We can heal this function. It is the same as $x + 1$ everywhere except at $x = 1$ where it is not defined. But we can continue the simplified function $x + 1$ through $x = 1$. The limit is $\boxed{2}$.
d) We can heal this function. It is the same as $x^2 + 1$. The limit is $\boxed{2}$.
e) There is no problem at $x = 0$. There is mischief at $x = 1$ although but that is far, far away. At $x = 0$, we get $\boxed{1}$.
f) This is the prototype, the fundamental theorem of trig! We know that the limit is $\boxed{1}$.
g) This can be written as $f(x) = (5/6)6x/\sin(6x) = (5/6)\text{sinc}(6x)$. The function $6x/\sin(6x)$ converges to 1 by the fundamental theorem of trigonometry. Therefore the limit is $\boxed{5/6}$.
h) This limit does not exist. It can be written as $\text{sinc}(x)/x$. Because $\text{sinc}(x)$ converges to 1. we are in trouble when dividing again by x. $\boxed{\text{There is no limit.}}$
i) We know $\sin(x)/x \to 1$ so that also $\sin(2x)/(2x)$ has the limit 1. If we divide them, see $\sin(x)/\sin(2x) \to 1/2$. The result is $\boxed{1/2}$.
j) The limit does not exist because $\exp(x) \to 1$ but $1/x$ goes to infinity.

Homework

1 Find the limits of each of the following functions at the point $x \to 0$. You can use the fact that $\sin(x)/x$ has the limit 1 as $x \to 0$.
a) $f(x) = (x^2 - 1)/(x - 1)$
b) $f(x) = \sin(3x)/x$
c) $f(x) = \sin^2(5x)/x^2$
d) $f(x) = \sin(3x)/\sin(5x)$

2 a) Graph of the function

$$f(x) = \frac{(1 - \cos(x))}{x^2} .$$

b) Where is the function f defined? Can you find the limit at the places, where it is not defined?

3 a) Can you see the limit of $g(h) = [f(x+h) - f(x)]/h$ as a function of h at the point $x = 0$ for the function $f(x) = \sin(x)$?
b) Verify that the function $f(x) = \exp_h(x) = (1+h)^{x/h}$ satisfies $[f(x+h) - f(x)]/h = f(x)$.
Remark. The exponential function can be defined as $e^x = \exp(x) = \lim_{h\to 0} \exp_h(x)$.

4 Find the limits for $x \to 0$:
a) $f(x) = (x^2 - 2x + 1)/(x - 1)$.
b) $f(x) = 2^x$.
c) $f(x) = 2^{2^x}$.
d) $f(x) = \sin(\sin(x))/\sin(x)$.

5 We explore in this problem the limit of the function $f(x) = x^x$ if $x \to 0$. Can we find a limit? Take a calculator or use Wolfram α and experiment. What do you see when $x \to 0$? Only optional: can you find a explanation for your experiments?

Lecture 4: Continuity

A **function** f is called **continuous** at a point p if a value $f(p)$ can be found such that $f(x) \to f(p)$ for $x \to p$. A function f is called **continuous on** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$.

In the interior (a, b), the limit needs to exist both from the right and from the left. At the boundary a only the right limit needs to exist and at b only the left limit. Intuitively, a function is continuous if you can **draw the graph of the function without lifting the pencil**. Continuity means that small changes in x results in small changes of $f(x)$.

1 Any polynomial as well as $\cos(x), \sin(x), \exp(x)$ are continuous everywhere. Also the sum and product of such functions is continuous. For example

$$\sin(x^3 + x) - \cos(x^5 + x^3)$$

is continuous everywhere.

2 The function $f(x) = 1/x$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to a **pole**. The **division by zero** kills continuity. Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere.

3 The function $f(x) = \log|x|$ is continuous for x different from 0. It is not continuous at 0 because $f(x) \to -\infty$ for $|x| \to 0$. Keep the two examples, $1/x$ and $\log|x|$ in mind.

4 The function $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi, x = 2\pi$ and any multiple of π. It has poles there because $\sin(x)$ is zero there and because we would divide by zero at such points.

5 The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just chose $x_n = 2/(4k + 1)$ and $z_n = 2/(4k - 1)$.

6 The **signum function** $f(x) = \text{sign}(x) = \left[\begin{array}{ll} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{array} \right.$ is not continuous at 0. It is a prototype of a function which has a **jump** discontinuity at 0.

We can refine the notion of continuity and say that a function is **continuous from the right**, if there exists a limit from the right $\lim_{x \downarrow a} f(x) = b$. Similarly a function f can be continuous from the left only. Most of the time we mean with "continuous"= "continuous on the real line".

Rules:
a) If f and g are continuous, then $f + g$ is continuous.
b) If f and g are continuous, then $f * g$ is continuous.
c) If f and g are continuous and if $g > 0$ then f/g is continuous.
d) If f and g are continuous, then $f \circ g$ is continuous.

7 $\sqrt{x^2+1}$ is continuous everywhere on the real line.

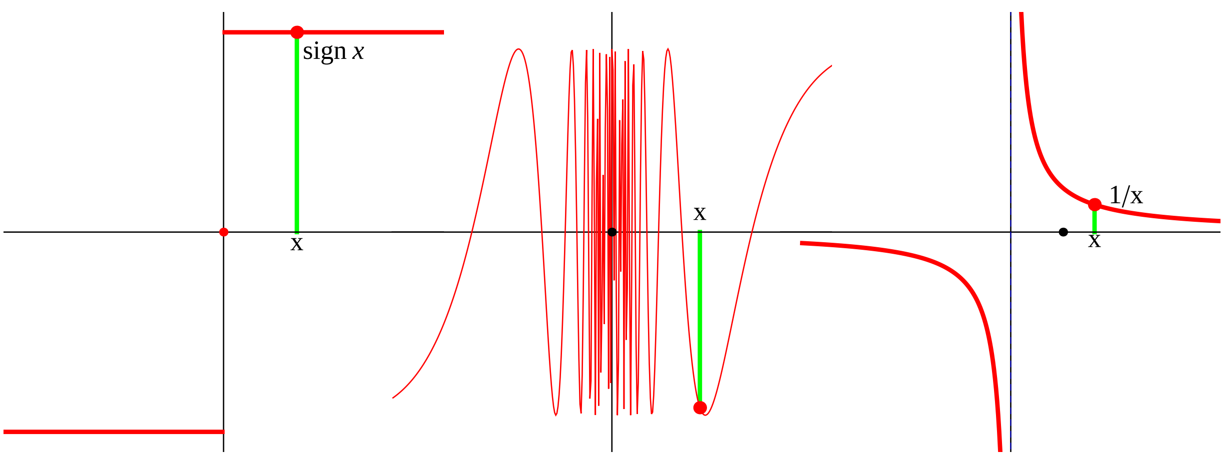
8 $\cos(x) + \sin(x)$ is continuous everywhere.

9 The function $f(x) = \log(|x|)$ is continuous everywhere except at 0. Indeed since for every integer n, we have $f(e^{-n}) = -n$, this can become arbitrarily large for $n \to \infty$ even so e^{-n} converges to 0 for n running to infinity.

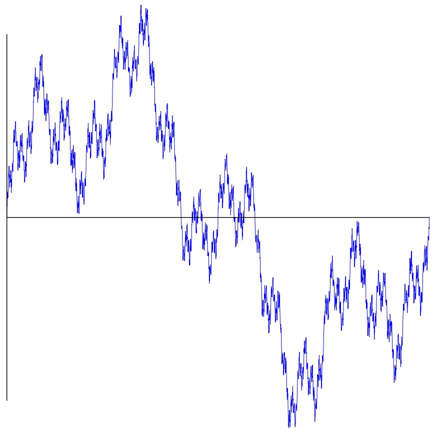
10 While $\log(|x|)$ is not continuous at $x = 0$, the function $1/\log|x|$ is continuous at $x = 0$. Is it continuous everywhere?

11 The function $f(x) = [\sin(x+h) - \sin(x)]/h$ is continuous for every $h > 0$. We will see next week that nothing bad happens when h becomes smaller and smaller and that the continuity will not deteriorate. Indeed, we will see that we get closer and closer to the cos function.

There are three major reasons, why a function is not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity. Here are the prototype examples. We will look at more during the lecture.



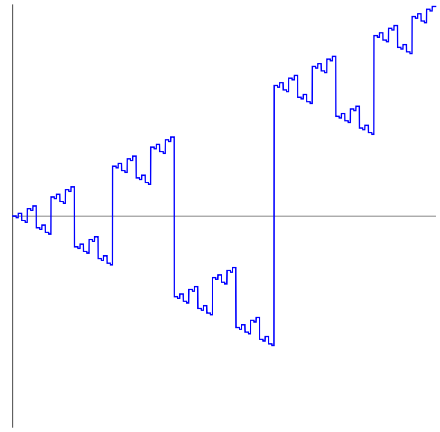
Why do we like continuity? We will see many reasons during this course but for now lets just say that:



A wild continuous function. This Weierstrass function is believed to be a fractal.

"Continuity tames a function.

It can be pretty wild, but not too crazy."



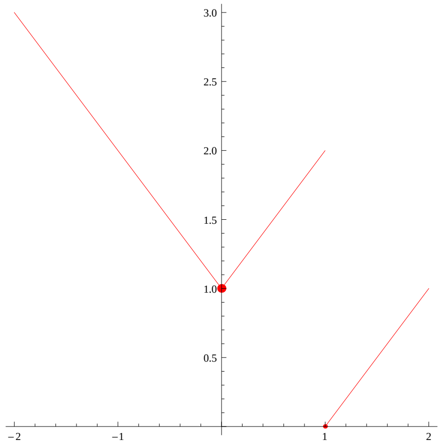
A crazy discontinuous function. It is discontinuous at every point and known to be a fractal.

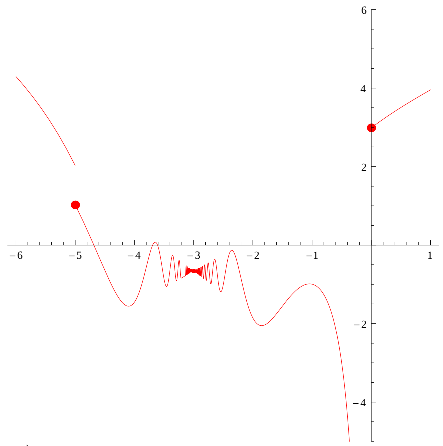
Continuity will be useful later for extremization. A continuous function on an interval $[a, b]$ has a maximum and minimum. And if a continuous function is negative at some place and positive at an other, there is a point between, where it is zero. These are all useful properties to have and they do not hold if a function is not continuous.

12 **Problem** Determine for each of the following functions, where discontinuities appear:
a) $f(x) = \log(|x^2 - 1|)$
b) $f(x) = \sin(\cos(\pi/x))$
c) $f(x) = \cot(x) + \tan(x) + x^4$
d) $f(x) = x^4 + 5x^2 - 3x + 4$
e) $f(x) = \frac{x^2-4x}{x}$
Solution.
a) $\log(|x|)$ is continuous everywhere except at $x = 0$. Since $x^2 - 1 = 0$ for $x = 1$ or $x = -1$, the function $f(x)$ is continuous everywhere except at $x = 1$ and $x = -1$.
b) The function π/x is continuous everywhere except at $x = 0$. Therefore $\cos(\cos(\pi/x)$ is continuous everywhere except possibly at $x = 0$. We have still to investigate the point $x = 0$ but there, the function $\cos(\pi/x)$ takes values between -1 and 1 for points arbitrarily close to $x = 0$. The function $f(x)$ takes values between $\sin(-1)$ and $\sin(1)$ arbitrarily close to $x = 0$. It is not continuous there.
c) The function x^4 is continuous everywhere. The function $\tan(x)$ is continuous everywhere except at the points $k\pi$. The function $\cot(x)$ is continuous everywhere except at points $\pi/2 + k\pi$. The function f is therefore continuous everywhere except at the point $x = k\pi/2$, multiples of $\pi/2$.
d) The function is a polynomial. We know that polynomials are continuous everywhere.
e) The function is continuous everywhere except at $x = 0$, where we have to look at the function more closely. But we can heal the function by dividing nominator and denominator by x which is possible for x different from 0. The healed function is $f(x) = x - 4$.

Homework

1 On which intervals is the following function continuous?


a)


b)

2 For the following functions, determine the points, where f is not continuous.
a) $f(x) = \cot(1-x)$
b) $x^3 \cos(1/x)$
c) $\text{sign}(x)/x$
d) $\textbf{sinc}(x) + \sin(x^2) + x^{22} + \log|x|$
e) $\frac{x^2+5x+x^4}{x-1}$
State which kind of discontinuity appears.

3 Construct a function which has a jump discontinuity, an oscillatory one as well as an escape to infinity. Can you construct an example where two of these flaws happen at the same point? Can you even construct an example, where all three happen at the same point?

4 Heal the following functions:
a) $(x^4 - 16)/(x-2)$
b) $x^5 + x^3/(x^2+1)$
c) $((\sin(x))^3 - \sin(x))/\sin(x)$.
d) $(x^3 + 3x^2 + 3x + 1)/(x^2 + 2x + 1)$
e) $(x^{1000} - 1)/(x^{100} - 1)$

5 Are the following function continuous? Break the functions up into simpler functions and analyze each. If you are not sure, also experiment by plotting the functions.
a) $\sin(\frac{1}{2+\sin(x)}) + |\cos(x)| + \frac{\sin(x)}{x} + x^5 + x^3 + 1 + \frac{1}{\exp(x)}$.
b) $\frac{1}{\log|x|} + x^5 - \sin(\sin(\sin(x))) - \exp(\exp(\exp(x)))$

Lecture 5: Intermediate Value Theorem

If $f(a) = 0$, then the value a is called a **root** of f. The function $f(x) = \cos(x)$ for example has the root $x = \pi/2$ or $x = 3\pi/2$.

1 $f(x) = 4x + 6$. Find the roots of f. **Answer**: set the function equal to 0 and solve for x. We get $4x + 6 = 0$

2 $f(x) = x^2 + 2x + 1$ Find the roots of f. **Answer**: we can write $f(x) = (x+1)^2$. The function has the root $x = -1$.

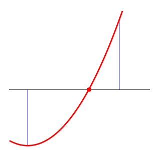
3 $f(x) = (x - 2)(x + 6)(x + 3)$. Find the roots of f.

4 $f(x) = 12 + x - 13x^2 - x^3 + x^4$. Find the roots of f. We do not have a formula for this, but we can try. Indeed, we see that for $x = 1, x = -3, x = 4, x = -1$ we have roots.

5 $f(x) = \exp(x)$. This function does not have any root.

6 $f(x) = 2^x - 16$ has the root $x = 2$.

Intermediate value theorem of Bolzano. If f is continuous on $[a, b]$ and $f(a), f(b)$ have different signs, there is a root of f in (a, b).



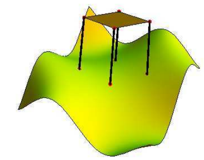
Proof. We can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at the point $c = (a + b)/2$. If $f(c) < 0$, then look take $[c, b]$ as your new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of size $2^{-n}(b - a)$. Continuity assures that $f(u_n) - f(v_n) \to 0$ and $f(u_n), f(v_n)$ have different signs. Both u_n, v_n converge to a root of f.

7 The function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root. Solution. The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. The intermediate value theorem assures there is a point where $f(x) = 0$.

8 There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem.

9 There exists a point on the earth, where the temperature is the same as the temperature on its antipode. Solution: Lets draw a meridian through the north and south pole and let $f(x)$ be the temperature on that circle. Define $g(x) = f(x) - f(x + \pi)$. If this function is zero on the north pole, we have found our point. If not, $g(x)$ different signs on the north and south pole. There exists therefore a point, where the temperature is the same.

10 **Wobbly Table Theorem.** On an arbitrary floor, a square table can be turned so that it does not wobble any more.



Why? The 4 legs ABCD are on a square. Let x be the angle of the line AC with with some coordinate axes if we look from above. Given the angle x, we can position the table **uniquely** as follows: the center of ABCD is on the z-axes, the legs ABC are on the floor and AC points in the direction x. Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. (If it is negative, the argument is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h. This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.

Define $Df(x) = (f(x+h) - f(x))/h$. Lets call it the **derivative** of f for the constant h. We will study it more in the next lecture. But you have verified for example $D\exp_h(x) = \exp_h(x)$ in a homework.

Lets call a point p, where $Df(x) = 0$ a **critical point** for h. Lets call a point a a **local maximum** if $f(a) \geq f(x)$ in an open interval containing a. Define similarly a **local minimum** as a point where $f(a) \leq f(x)$.

11 The function $f(x) = x(x - h)(x - 2h)$ has the derivative $Df(x) = 3x(x - h)$ as you have verified in the case $h = 1$ in the first lecture of this course in a worksheet. We will write $[x]^3 = x(x - h)(x - 2h)$ and $[x]^2 = x(x - h)$. The computation just done tells that $D[x]^3 = 3[x]^2$. Since $[x]^2$ has exactly two roots $0, h$, the function $[x]^3$ has exactly 2 critical points.

12 More generally for $[x]^{n+1} = x(x - h)(x - 2h)...(x - nh)$ we have $D[x]^{n+1} = (n + 1)D[x]^n$. Because $[x]^n$ has exactly n roots, the function $[x]^{n+1}$ has exactly n critical points. Keep the formula

$$\boxed{D[x]^n = n[x]^{n-1}}$$

in mind!

13 The function $\exp_h(x) = (1 + h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. Because this function has no roots and the derivative is the function itself, the function has no critical points. Indeed, this function is monotone.

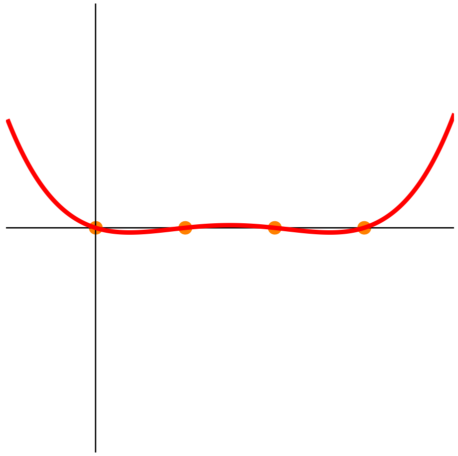


Figure: We see the function $[x]^4 = x(x-h)(x-2h)(x-3h)$ with $h = 0.5$. This function has 3 critical points because $D[x]^4 = 4[x]^3$ and $[x]^3$ has roots at $0, h, 2h$. There are three local maxima or minima according to the theorem.

Later in the course, we will look at the derivative Df in the limit when $h \to 0$. And then the critical points are places where the tangent is horizontal. In our case now, a critical point is a point so that if we walk by a step h to the right, the function does not change. For now, just remember the formula $D[x]^n = n[x]^{n-1}$. It will be the same formula later on when we go to the limit $h \to 0$.

Critical points lead to extrema as we will see later in the course. In our discrete setting we can say:

Fermat's maximum theorem If f is continuous and has a critical point a for h, then f has either a local maximum or local minimum inside the open interval $(a, a+h)$.

Look at the range of the function f restricted to $[a, a+h]$. It is a bounded interval $[c, d]$ by the intermediate value theorem. There exists especially a point u for which $f(u) = c$ and a point v for which $f(v) = d$. These points are different if f is not constant on $[a, a+h]$. There is therefore one point, where the value is different than $f(a)$. If it is larger, we have a local maximum. If it is smaller we have a local minimum.

14 **Problem.** Verify that a cubic polynomial has maximally 2 critical points. **Solution** $f(x) = ax^3 + bx^2 + cx + d$. Because the x^3 terms cancel in $f(x+h) - f(x)$, this is a quadratic polynomial. It has maximally 2 roots.

What we have called "critical point" here will in the limit $h \to 0$ be called "critical point" later in this course. While the h-critical point notion makes sense for any continuous function, we will need more regularity to take the limit $h \to 0$. This limit $h \to 0$ will be one of the major features.

Homework

1 Find the roots for $f(x) = -18 + 9x + 20x^2 - 10x^3 - 2x^4 + x^5$. You are told that all roots are integers.

2 Use the intermediate value theorem to verify that $f(x) = x^5 - 6x^4 + 8$ has at least two roots on $[-2, 2]$.

3 Madonnas height is 161 cm. Lady Gagas height is 155 cm. Gaga was born March 28, 1986, Madonna was born August 16, 1958. Gaga owns probably 0.5 billions, Madonna owns probably 1.3 billion.
a) Can you argue that there was a moment when Gaga's height was exactly half of Madonnas height?
b) Can you argue that there was a moment when Gaga's fortune was exactly a third of Madonnas fortune?
c) Can you argue that if you drive the 190 miles from here to New York in 4 hours then there are at least two moments of time when you drive with exactly 40 miles per hours. The trip is not part of a larger trip. Your start is in Boston and your Destination is New York.

4 Argue why there is a solution to
a) $1 + \cos(x) = x$.
b) $\exp(2x) = x$.
c) $\mathrm{sinc}(x) = x^4$.
d) Why does the following argument not work:
The function $f(x) = 1/\cos(x)$ satisfies $f(0) = 1$ and $f(\pi) = -1$. There exists therefore a point x where $f(x) = 0$.
e) Does the function $f(x) = x + \log|\log|\log|x|||$ have a root?

5 a) Draw the graph of $f(x) = x^3 - x$.
b) Locate the local maxima and minima.
c) Find the critical points of f to the constant $h = 1$. That means, find the places, where $f(x+1) - f(x) = 0$.
d) For every point a you have found in c), verify that there is a local maximum or minimum in $[a, a+1]$.

Lecture 6: Some examples

Here are some worked out examples, similar to what we expect you to do for the homework of lecture 6: The homework should be straightforward, except when finding $Sf(x)$, we want to add a constant such that $Sf(0) = 0$. In general, you will not need to evaluate functions and can leave terms like $\sin(5x)$ as they are. If you have seen calculus already, then you could do this exercice by writing

$$\frac{d}{dx} f(x)$$

instead of $Df(x)$ and by writing

$$\int_0^x f(x)\, dx$$

instead of $Sf(x)$. We did not introduce the derivative df/dx nor the integral $\int_0^x$ yet. For now, just use the Differentiation rules and integrations rules in the box to the right to solve the problem.

1 **Problem:** Find the derivative $Df(x)$ of the function $f(x) = \sin(5 \cdot x) + x^7 + 3$.
Answer: From the differentiation rules, we know $\boxed{Df(x) = 5\cos(5 \cdot x) + 7x^6}$.

2 **Problem:** Find the derivative $Df(0)$ of the same function $f(x) = \sin(5 \cdot x) + 5x^7 + 3$.
Answer: We know $Df(x) = 5\cos(5 \cdot x) + 35x^6$. Plugging in $x = 0$ gives $\boxed{5}$.

3 **Problem:** Find the integral $Sf(x)$ of the function $f(x) = \sin(5 \cdot x) + 5x^7 + 3$.
Answer: From the integration rules, we know $\boxed{Sf(x) = -\cos(5 \cdot x)/5 + 5x^8/8 + 3x}$.

4 **Problem:** Find the integral $Sf(1)$ of the function $f(x) = x^2 + 1$.
Answer: From the integration rules, we know $Sf(x) = x^3/3 + x$. Plugging in $x = 1$ gives $\boxed{1/3 + 1}$ if we use the functions in the limit $h \to 0$. For positive h, we have to evaluate $x(x-h)(x-2h)/3 + x$ for $x = 1$ which is $(1-h)(1-2h)/3 + 1$

5 **Problem:** Find the integral $Sf(1)$ of the function $f(x) = \exp(4 \cdot x)$.
Answer: From the integration rules, we know $Sf(x) = \exp(4 \cdot x)/4 - 1/4$. We have added a constant such that $Sf(0) = 0$. Plugging in $x = 1$ gives $\boxed{\exp(4)/4 - 1/4}$.

6 **Problem:** Assume $h = 1/1000$. Determine the value of

$$\frac{1}{1000}[f(\frac{0}{1000}) + f(\frac{1}{1000}) + ... + f(\frac{999}{1000})]$$

for the function $f(x) = -\sin(7x) + \exp(3x)$.
Answer: The problem asks for $Sf(1)$. We first compute $Sf(x)$ taking care that $Sf(0) = 0$.

$$Sf(x) = \cos(7x)/7 + \exp(3x)/3 - (1/7 + 1/3)\ .$$

Now plug in $x = 1$ to get $\boxed{\cos(7)/7 + \exp(3)/3 - (1/7 + 1/3)}$.

Lecture 6: Fundamental theorem

Calculus is the theory of **differentiation** and **integration**. We explore this still in a discrete setup and practice differentiation and integration. We fix a positive constant h and take differences and sums. Without taking limits, we prove a fundamental theorem of calculus. You can so differentiate and integrate polynomials, exponentials and trigonometric functions. Later we will do the same with real derivatives and integrals. But now, we can work with arbitrary continuous functions.

Given a function $f(x)$, define the **differential quotient**

$$\boxed{Df(x) = (f(x+h) - f(x))\,\frac{1}{h}}$$

If f is continuous then Df is a continuous. We call it also "derivative".

1 Lets take the constant function $f(x) = 5$. We get $Df(x) = (f(x+h)-f(x))/h = (5-5)/h = 0$ everywhere. You can see that in general, if f is a constant function, then $Df(x) = 0$.

2 $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$. You see in general that if f is a linear function $f(x) = ax + b$, then $Df(x) = a$ is constant.

3 If $f(x) = ax + b$, then $Df(x) = \boxed{a}$.

For constant functions, the derivative is zero. For linear functions, the derivative is the slope.

4 For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h$ which is $\boxed{2x + h}$.

Given a function f, define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \leq jh < x$. That is, if k is such that $(k-1)h$ is the largest below x, then

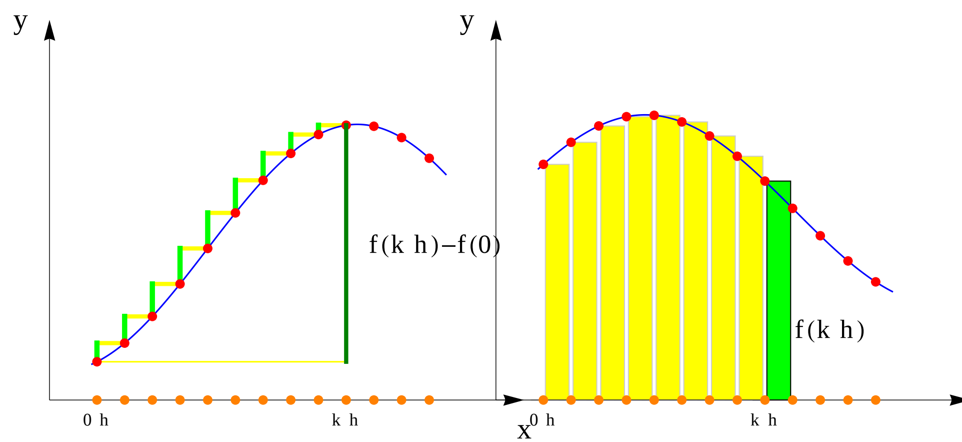
$$\boxed{Sf(x) = h[\ f(0)+f(h)+f(2h)+\cdots+f((k-1)h)\ ]}$$

We call Sf also the "integral" or "antiderivative" of f.

5 Compute $Sf(x)$ for $f(x) = 1$. **Solution**. We have $Sf(x) = 0$ for $x \leq h$, and $Sf(x) = h$ for $h \leq x < 2h$ and $Sf(x) = 2h$ for $2h \leq x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but grows in quantized steps.

The difference	$Df(x)$	will become the	**derivative**	$f'(x)$	.
The sum	$Sf(x)$	will become the	**integral**	$\int_0^x f(t)\,dt$	.

Df	means	rise over run	and is close to	the	slope	of the	graph of f.
Sf	means	areas of rectangles	and is close to	the	area	under the	graph of f.



Theorem: Sum the differences and get $SDf(kh) = f(kh) - f(0)$

Theorem: Difference the sum and get $DSf(kh) = f(kh)$

6 For $f(x) = [x]_h^m = x(x-h)(x-2h)...(x-mh+h)$ we have

$$f(x+h) - f(x) = (x(x-h)(x-2h)...(x-kh+2h))\,((x+h) - (x-mh+h)) = [x]^{m-1}hm$$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. Lets leave the h away to get the important formula $\boxed{D[x]^m = m[x]^{m-1}}$

We can establish from this differentiation formulas for **polynomials**.

7 If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$.

The fundamental theorem allows us to integrate and get the right values at the points k/n:

8 Find Sf for the same function. The answer is $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero. As a rescaled exponential, it is continuous and monotone.

9 You have already computed the derivative in a homework. Lets do it again. The function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. **Solution**: $\exp_h(x+h) = (1+h)\exp(x)$ shows that. $\boxed{D\exp_h(x) = \exp_h(x)}$

10 Define $\exp(a \cdot x) = (1+ah)^{x/h}$. It satisfies $\boxed{D\exp_h(a \cdot x) = a\exp_h(a \cdot x)}$ We write a dot because $\exp_h(ax)$ is not equal to $\exp_h(a \cdot x)$. What is important to us is only the differentiation rule for this function.

11 If we allow a to become complex, we get $\exp(1+ia)(1+aih)^{x/h}$. We still have $D\exp_h^{ai}(x) = ai\exp_h^{ai}(x)$. Taking real and imaginary parts define new functions $\exp_h^{ai}(x) = \cos_h(a\cdot x) + i\sin_h(a\cdot x)$. Despite the fact that we have for a moment escaped to the complex, these functions exist and morph into the familiar cos and sin functions for $h \to 0$. But in general, for any $h > 0$ and any a, we have $\boxed{D\cos_h(a\cdot x) = -a\sin_h(a\cdot x) \text{ and } D\sin_h(a\cdot x) = a\cos_h(a\cdot x).}$ If h is the size of the Planck constant $h = 1.616\cdot 10^{-35}m$, we would notice a difference between the cos and $\cos_h$ only if an x-ray traveling for 13 billion years. It would appear as a gamma ray burst.

Homework

We leave the h away in this homework. To have more fun, also define $\log_h$ as the inverse of $\exp_h$ and define $1/[x]_h = D\log_h(x)$ for $x > 0$. If we start integrating from 1 instead of 0 as usual we write $S_1 f$ and get $S_1 1/[x]_h = \log_h(x)$. We also write here x^n for $[x]_h^n$ and write $\exp(a\cdot x) = e^{a\cdot x}$ instead of $\exp_h^a(x)$ and $\log(x)$ instead of $\log_h(x)$ because we are among friends. Use the differentiation and integration rules on the right to find derivatives and integrals of the following functions:

1 Find the derivatives $Df(x)$ of the following functions:

a) $f(x) = x^6 + 6x^4 + x$
a) $f(x) = x^2 + 3\log(x)$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Df(0)$?

2 Find the integrals $Sf(x)$ of the following functions:

a) $f(x) = x^4$.
b) $f(x) = x^2 + 6x^7 + x$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Sf(1)$?

3 Find the derivatives $Df(x)$ of the following functions

a) $f(x) = \exp(3\cdot x) + x^6$
b) $f(x) = 4\exp(-3\cdot x) + 9x^6$
c) $f(x) = -\exp(5\cdot x) + x^6$

4 Find the integrals $Sf(x)$ of the following functions

a) $f(x) = \exp(6\cdot x) - 3x^6$
b) $f(x) = \exp(8\cdot x) + x^6$
c) $f(x) = -\exp(5\cdot x) + x^6$

5 Define $f(x) = \sin(4\cdot x) - \exp(2\cdot x) + x^4$ and assume $h = 1/100$ in part c).
a) Find $Df(x)$
b) Find $Sf(x)$
c) Determine the value of

$$\frac{1}{100}[f(\frac{0}{100}) + f(\frac{1}{100}) + \cdots + f(\frac{99}{100})]\ .$$

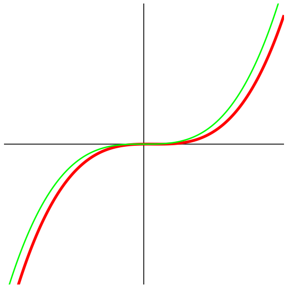
All calculus on 1/3 page

Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

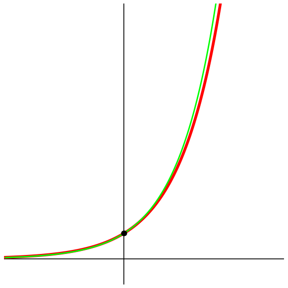
Differentiation rules	**Integration rules (for $x = kh$)**
$Dx^n = nx^{n-1}$	$Sx^n = x^{n+1}/(n+1)$
$De^{a\cdot x} = ae^{a\cdot x}$	$Se^{a\cdot x} = e^{a\cdot x}/a$
$D\cos(a\cdot x) = -a\sin(a\cdot x)$	$S\cos(a\cdot x) = \sin(a\cdot x)/a$
$D\sin(a\cdot x) = a\cos(a\cdot x)$	$S\sin(a\cdot x) = -\cos(a\cdot x)/a$
$D\log(x) = 1/x$	$S\frac{1}{x} = \log(x)$

Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x+h)$.

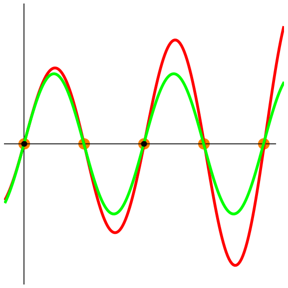
Pictures



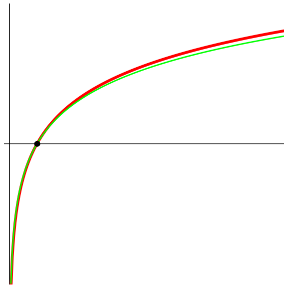
$[x]_h^3$ for $h = 0.1$



$\exp_h(x)$ for $h = 0.1$



$\sin_h(x)$ for $h = 0.1$



$\log_h(x)$ for $h = 0.1$

Lecture 7: Rate of change

A function f leads to a new function

$$Df(x) = \frac{f(x+h) - f(x)}{h} .$$

It is the **rate of change** of the function with step size h. When changing x to $x+h$ and get a response change $f(x)$ to $f(x+h)$. In this lecture, we take the limit $h \to 0$ and derive the important formulas $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}\exp(x) = \exp(x)$, $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$ which we have seen already in a discrete setting.

1 You walk up a snow hill of height $f(x) = 30 - x^2$ meters. You walk with a step size of $h = 0.5$ meters. You are at position $x = 3$. How much do you climb or descend when making an other step? We have $f(3) = 21$ and $f(3.5) = 17.75$. We have walked down 3.25 meters. How steep was the snow hill at this point? We have to divide the height difference by the walking distance: $-3.25/0.5 = -7.5$.

Today, we take the limit $h \to 0$:

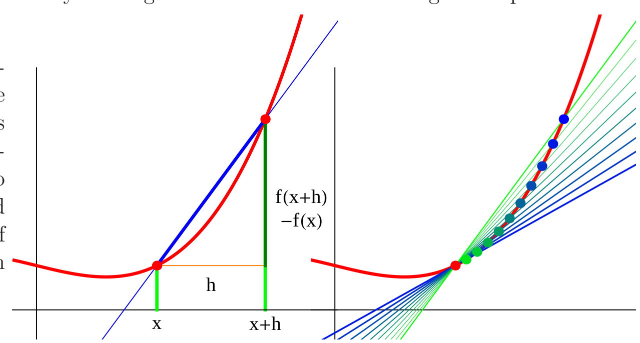
If the limit $\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exist, we say f is **differentiable** at the point x. The value is called the **derivative** or **instantaneous rate of change** of the function f at x. We denote the limit also with $f'(x)$.

2 In the previous problem, $f(x) = 30 - x^2$ we have

$$f(x+h) - f(x) = [30 - (x+h)^2] - [30 - x^2] = -2xh - h^2$$

Dividing this by h gives $-2x - h$. The limit $h \to 0$ gives $-2x$. We have just seen that for $f(x) = x^2$, we get $f'(x) = -2x$. For $x = 3$, this is -6. The actual slope of the snow hill is a bit smaller than the estimate done by walking. The reason is that the hill gets steeper.

The derivative $f'(x)$ has a geometric meaning. It is the slope of the tangent at x. This is an important geometric interpretation. It is useful to think about x as "time" and the derivative as the rate of change of the quantity $f(x)$ in time.

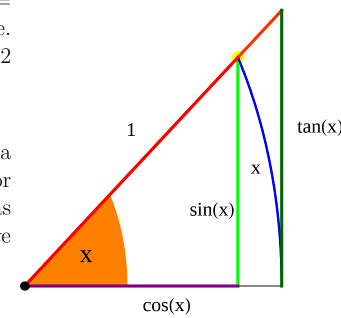


For $f(x) = x^n$, we have $f'(x) = nx^{n-1}$.

Proof: $f(x+h) - f(x) = (x+h)^n = (x^n + nx^{n-1}h + a_2h^2 + ... + h^n) - x^n = nx^{n-1}h + a_2h^2 + ... + h^n)$. If we divide by h, we get $nx^{n-1} + h(a_2 + \ldots + h^{n-2})$ for which the limit $h \to 0$ exists: it is nx^{n-1}. This is an important result because most functions can be approximated very well with polynomials.

For $f(x) = \sin(x)$ we have $f'(0) = 1$ because the differential quotient is $[f(0+h) - f(0)]/h = \sin(h)/h = \text{sinc}(h)$. We have already seen that the limit is 1 before. Lets look at it again geometrically. For all $0 < x < \pi/2$ we have

$$\sin(x) \leq x \leq \tan(x) .$$



3 [dividing by 2 squeezes the area of the sector by the area of triangles.] Because $\tan(x)/\sin(x) = 1/\cos(x) \to 1$ for $x \to 0$, the value of $\text{sinc}(x) = \sin(x)/x$ must go to 1 as $x \to 0$. Renaming the variable x with the variable h, we see the **fundamental theorem of trigonometry**

$$\boxed{\lim_{h \to 0} \frac{\sin(h)}{h} = 1}$$

4 For $f(x) = \cos(x)$ we have $f'(x) = 0$. To see this, look at $f(0+h) - f(0) = \cos(h) - 1$. Geometrically, we can use Pythagoras $\sin^2(h) + (1-\cos(h))^2 \leq h^2$ to see that $2 - 2\cos(h) \leq h^2$ or $1 - \cos(h) \leq h^2/2$ so that $(1 - \cos(h))/h \leq h/2$ and this goes to 0 for $h \to 0$. We have just nailed down an other important identity

$$\boxed{\lim_{h \to 0} \frac{(1-\cos(h))}{h} = 0}.$$

The interpretation is that the tangent is **horizontal** for the cos function at $x = 0$. We will call this a critical point later on.

5 From the previous two examples, we get

$$\cos(x+h) - \cos(x) = \cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x) = \cos(x)(\cos(h) - 1) - \sin(x)\sin(h)$$

because $(\cos(h) - 1)/h \to 0$ and $\sin(h)/h \to 1$, we see that $[\cos(x+h) - \cos(x)]/h \to -\sin(x)$.

For $f(x) = \cos(ax)$ we have $f'(x) = -a\sin(ax)$.

6 Similarly,

$$\sin(x+h) - \sin(x) = \cos(x)\sin(h) + \sin(x)\cos(h) - \sin(x) = \sin(x)(\cos(h) - 1) + \cos(x)\sin(h)$$

because $(\cos(h)-1)/h \to 0$ and $\sin(h)/h \to 1$, we see that $[\sin(x+h) - \sin(x)]/h \to \cos(x)$.

for $f(x) = \sin(ax)$, we have $f'(x) = a\cos(ax)$.

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

Like π, the Euler number e is irrational. Here are the first digits: 2.7182818284590452354. If you want to find an approximation, just pick a large n, like $n = 100$ and compute $(1 + 1/n)^n$. For $n = 100$ for example, we see $101^{100}/100^{100}$. We only need 101^{100} and then put a comma after the first digit to get an approximation. Interested why the limit exists: verify hat the fractions $A_n = (1 + 1/n)^n$ increase and $B_n = (1 + 1/n)^{(n+1)}$ decrease. Since $B_n/A_n = (1 + 1/n)$ which goes to 1 for $n \to \infty$, the limit exists. The same argument shows that $(1+1/n)^{xn} = \exp_{1/n}(x)$ increases and $\exp_{1/n}(x)(1+1/n)$ decreases. The limiting function $\exp(x) = e^x$ is called the **exponential function**. Remember that if we write $h = 1/n$, then $(1 + 1/n)^{nx} = \exp_h(x)$ considered earlier in the course. We can sandwich the exponential function between $\exp_h(x)$ and $(1 + h)\exp_h(x)$:

$$\exp_h(x) \le \exp(x) \le \exp_h(x)(1 + h), \quad x \ge 0 \; .$$

For $x < 0$, the inequalities are reversed.

7 Lets compute the derivative of $f(x) = e^x$ at $x = 0$. **Answer**. We have for $x \le 1$

$$1 \le\le (e^x - 1)/x \le 1 + x \; .$$

Therefore $f'(0) = 1$. The exponential function has a graph which has slope 1 at $x = 0$.

8 Now, we can get the general case. It follows from $e^{x+h} - e^x = e^x(e^h - 1)$ that the derivative of $\exp(x)$ is $\exp(x)$.

For $f(x) = \exp(ax)$, we have $f'(x) = a\exp(ax)$.

It follows from the properties of taking limits that $(f(x) + g(x))' = f'(x) + g'(x)$. We also have $(af(x))' = af'(x)$. From this, we can now compute many derivatives

9 Find the slope of the tangent of $f(x) = \sin(3x) + 5\cos(10x) + e^{5x}$ at the point $x = 0$. **Solution**: $f'(x) = 3\cos(3x) - 50\sin(10x) + 5e^{5x}$. Now evaluate it at $x = 0$ which is $3 + 0 + 5 = 8$.

Finally, lets mention an example of a function which is not everywhere differentiable.

10 The function $f(x) = |x|$ has the properties that $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x =< 1$. The derivative does not exist at $x = 0$ evenso the function is continuous there. You see that the slope of the graph jumps discontinuously at the point $x = 0$.

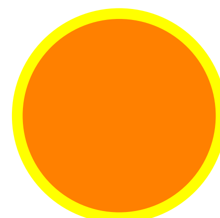
For a function which is discontinuous at some point, we don't even attempt to differentiate it there. For example, we would not even try to differentiate $\sin(4/x)$ at $x = 0$ nor $f(x) = 1/x^3$ at $x = 0$ nor $\sin(x)/|x|$ at $x = 0$. Remember these bad guys?

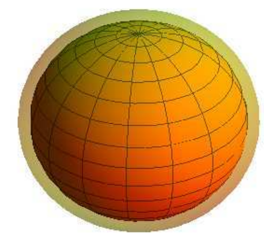
To the end, you might have noticed that in the boxes, more general results have appeared, where x is replaced by ax. We will look at this again but in general, the relation $f'(ax) = af(ax)$ holds ("if you drive twice as fast, you climb twice as fast").

Homework

1 For which of the following functions does the derivative $f'(x)$ exist at every x?
a) $\sin^2(x)$ b) $|\exp(x)|$ c) $\exp(x) + \sin(15x)$
d) $|\cos(x)|$ e) $\sin(1/x)$ f) $|\exp(x)| + |1 + \sin(15x)|$

2 a) A circle of radius x has the area $f(r) = \pi r^2$. Find $\frac{d}{dr}f(r)$. Can you visualize why this is the same than the circumference of the circle.
b) The sphere of radius r has the volume $f(r) = 4\pi r^3/3$. Find $\frac{d}{dr}f(r))$ and compare it with the surface area of the sphere.
c) A **hypersphere** of radius r has the **hyper volume** $f(r) = \pi^2 r^4/2$. Find $\frac{d}{dr}f(r)$, the volume of the boundary sphere.





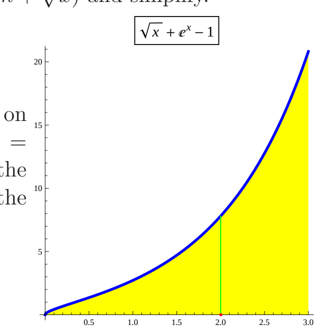
3 Find the derivatives of the following functions at the point $x = 2$.
a) $f(x) = 5\exp(x) + \sin(x) + x + x^2 + x^3 + x^4 + x^5$.
b) $f(x) = (x^5 - 1)/(x - 1) + \cos(2x)$. First heal this function.
c) $f(x) = \frac{1+4x+6x^2+4x^3+x^4}{x^2+2x+1}$. Also here, first heal!

4 In this problem we compute the derivative of $\sqrt{x}$ for $x > 0$. To do so, we have to find the limit

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \; .$$

Hint: multiply the top and the bottom with $(\sqrt{x+h} + \sqrt{x})$ and simplify.

5 A rocket with Mars Rover "Curiosity" lifts off on November 26, 2011. The height at time t is $h(t) = e^t - 1 + \sqrt{t}$, at least for the first few seconds. Find the rate of change of the height at time $t = 1$. Use the previous problem to get the derivative of $\sqrt{t}$.



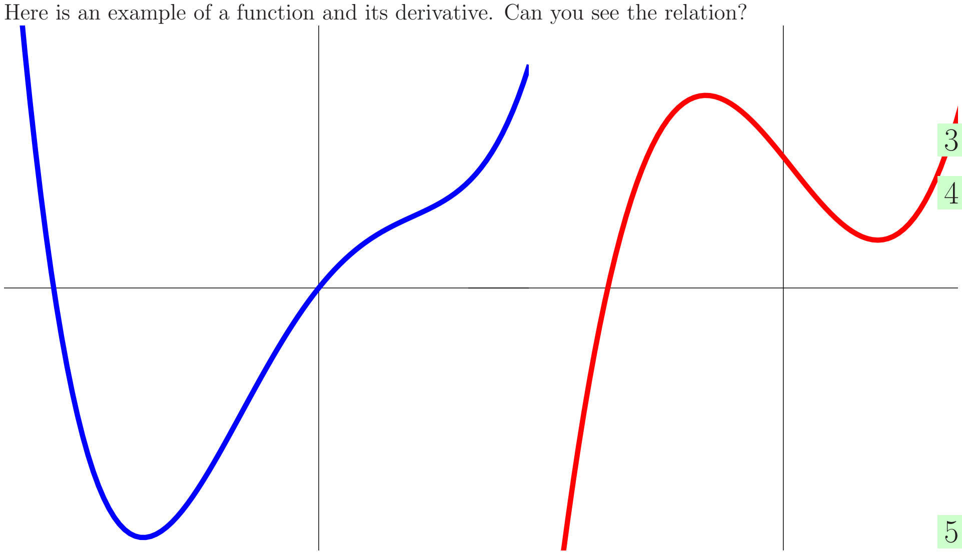
Lecture 8: The derivative function

We have defined the derivative $f'(x) = \frac{d}{dx}f(x)$ as a limit of $Df(x)$ for $h \to 0$. We have seen that $\frac{d}{dx}x^n = nx^{n-1}$ holds for integer n. We also know already that $\sin'(x) = \cos(x), \cos'(x) = -\sin(x)$ and $\exp'(x) = \exp(x)$. We can already differentiate a lot of functions and evaluate the derivative $f'(x)$ at some point x. This is the slope of the curve at x.

1 Find the derivative $f'(x)$ of $f(x) = \sin(\pi x) + \cos(\pi x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$. **Solution**: $f'(x) = \pi\cos(\pi x) - \pi\sin(\pi x) - 1/(2\sqrt{x} - 1/x^2 + 4x^3$. Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

The function which takes the derivative at a given point is called the **derivative function**. For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$. In this lecture, we want to understand the new function and its relation with f. What does it mean if $f'(x) > 0$. What does it mean that $f'(x) < 0$. Do the roots of f tell something about f' or do the roots of f' tell something about f?

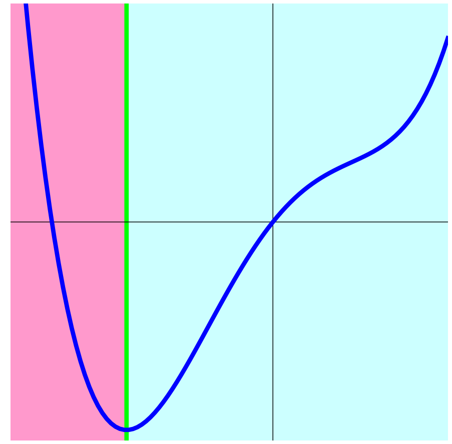
Here is an example of a function and its derivative. Can you see the relation?

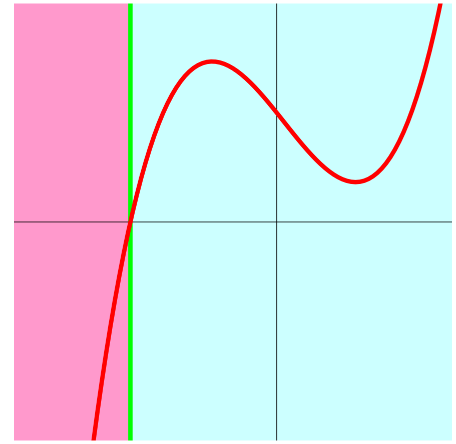


To understand the relation, it is good to distinguish intervals, where $f(x)$ is increasing or decreasing. This are the intervals where $f'(x)$ is positive or negative.

A function is called **monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Lets look at the previous example again.





2 Can you find a function f which is bounded $|f(x)| \leq 1$ and such that $f'(x)$ is unbounded?

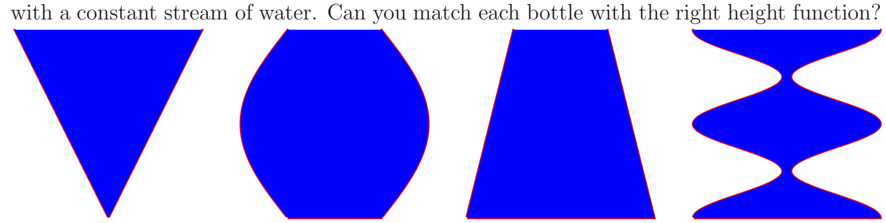
Given the function $f(x)$, we can define $g(x) = f'(x)$ and then take the derivative g' of g. This second derivative $f''(x)$ is called the **acceleration**. It measures the rate of change of the tangent slope. For $f(x) = x^4$, for example we have $f''(x) = 12x^2$. If $f''(x) > 0$ on some interval the function is called **concave up**, if $f''(x) < 0$, it is **concave down**.

3 Find a function f which has the property that its acceleration is constant equal to 10.

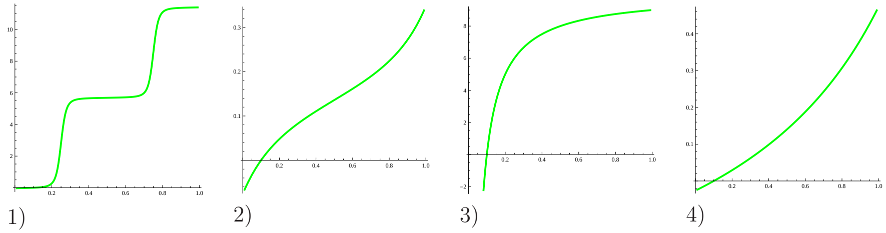
4 Can you find a function f which is bounded $|f(x)| \leq 1$ and such that $f''(x)$ is positive everywhere?

Finally we look at a famous inverse problem called **bottle calibration problem**. We fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time t. Assume the radius of the bottle is $f(z)$ at height z. Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that f is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of g and f. Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions g qualitatively.

5 In each of the bottles, we call g the height of the water level at time t, when filling the bottle with a constant stream of water. Can you match each bottle with the right height function?



a) b) d) c)



The key is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h)-g(t)]$ times the area πf^2 is a constant times the time difference $h = dt$, we have

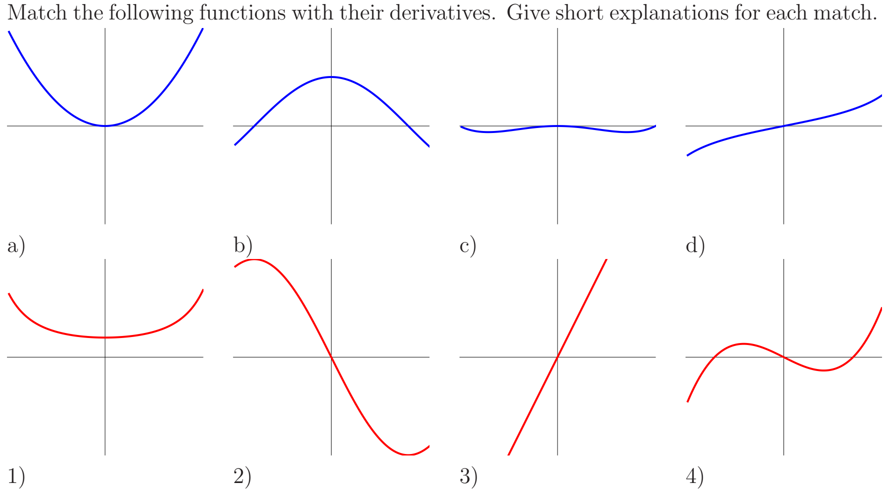
$$g' = \frac{1}{\pi f^2} \; .$$

This formula relates the derivative function of g with the thickness $f(t)$ of the bottle at height g. It tells that if f is large, then g' is small and if f is small, then g' is large.

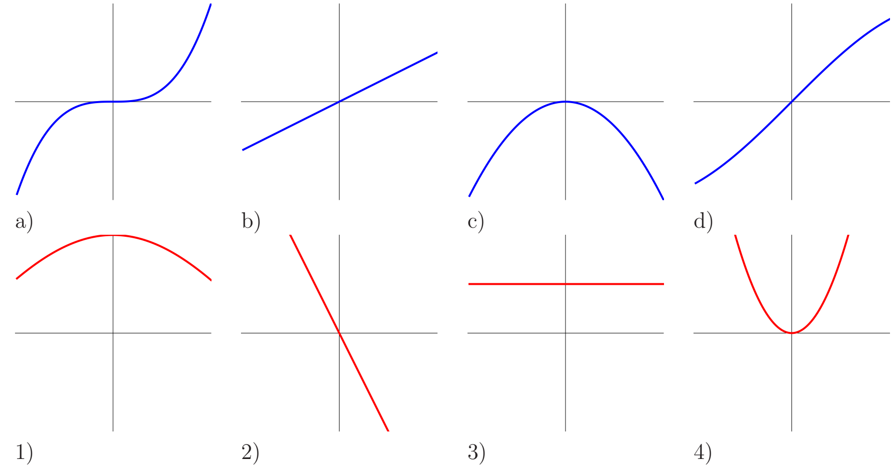
Homework

1 For the following functions, determine on which intervals the function is monotonically increasing or decreasing.
a) $f(x) = x^3 - x$ on $[-2, 2]$
b) $f(x) = \sin(\pi x)$ on $[-\pi, \pi]$
c) $f(x) = x^4 - 2x^2$ on $[-2, 2]$.

2 Match the following functions with their derivatives. Give short explanations for each match.



3 Match also the following functions with their derivatives. Give short explanations documenting your reasoning in each case.



4 Draw for the following functions the graph of the function $f(x)$ as well as the graph of its derivative $f'(x)$. You do not have to compute the derivative analytically as a formula here since we do not have all tools yet to compute the derivatives. The derivative function you draw needs to have the right qualitative shape however.
a) The Gaussian bell curve or the **"To whom the bell tolls"** function

$$f(x) = e^{-x^2}$$

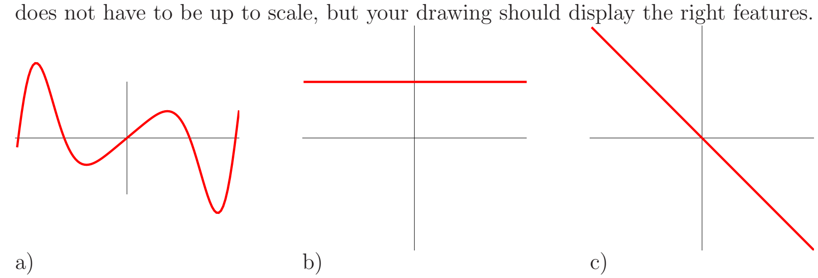
b) The **witch of Maria Agnesi**.

$$f(x) = \frac{1}{1+x^2}$$

c) The **three gorges function**

$$f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} \; .$$

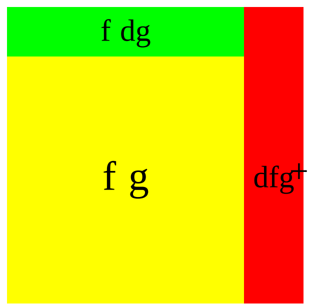
5 Below you the graphs of three derivative functions $f'(x)$. In each case you are told that $f(0) = 1$. Your task is to draw the function $f(x)$ in each of the cases a),b),c). Your picture does not have to be up to scale, but your drawing should display the right features.



Lecture 9: The product rule

In this lecture, we look at the derivative of a product of functions. The product rule is also called **Leibniz rule** named after **Gottfried Leibniz**, who found it in 1684. It is a very important rule because it allows us to differentiate many more functions. If we wanted to compute the derivative of $f(x) = x\sin(x)$ for example, we would have to get under the hood of the function and compute the limit $\lim(f(x+h) - f(x))/h$. We are too lazy for that. Lets start with the identity

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)]$$



which can be written as $D(fg) = Dfg^+ + fDg$ with $g^+(x) = g(x+h)$. This **quantum Leibniz rule** can also be seen geometrically: the rectangle of area $(f+df)(g+dg)$ is the union of rectangles with area $f \cdot g$, $f \cdot dg$ and $df \cdot g^+$. Divide this relation by h to see

$$\frac{[f(x+h) - f(x)]}{h} \cdot g(x+h) \to f'(x) \cdot g(x)$$
$$f(x) \cdot \frac{[g(x+h) - g(x)]}{h} \to f(x) \cdot g'(x) .$$

We get the extraordinarily important **product rule**:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) .$$

Remark: the Quantum Leibniz rule can also be seen in the **Babylonian calculus** developed in the first hour. Take $h = 1$ and compute $Dx^2 = D(xx) = Dx(x+1) + xDx = 1 \cdot (x+1) + x \cdot 1 = 2x+1$. Indeed, we have seen that summing up all odd numbers $2x+1$ gives the squares x^2.

1 Find the derivative function $f'(x)$ for $f(x) = x^3\sin(x)$. **Solution:** We know how to differentiate x^3 and $\sin(x)$ so that $f'(x) = 3x^2\sin(x) + x^3\cos(x)$.

2 While we know

$$\frac{d}{dx}x^5 = 5x^4$$

lets compute this with the Leibniz rule and write $x^5 = x^3 \cdot x^2$. We have

$$\frac{d}{dx}x^3 = 3x^2, \frac{d}{dx}x^2 = 2x .$$

The Leibniz rule gives us $d/dx^5 = 3x^4 + 2x^4 = 5x^4$.

3 Lets look at a few derivatives related to functions where we know the answer already but where we can check things using the product formula:

- $\frac{d}{dx}(x^3 \cdot x^5)$
- $\frac{d}{dx}e^{3x}e^{5x}$
- $\frac{d}{dx}\sqrt{x}/\sqrt{x}$
- $\frac{d}{dx}\sin(x)\cos(x)$

Before we look at the quotient rule which allows to differentiate $f(x)/g(x)$ we can also write the later as $f(x) \cdot 1/g(x)$ and use a rule telling us how to differentiate $1/g(x)$. This is the **reciprocal rule**:

If $g(x) \neq 0$, then
$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2} .$$

In order to see this $h = 1/g$ and differentiate the equation $1 = g(x)h(x)$ on both sides. The product rule gives $0 = g'(x)h(x) + g(x)h'(x)$ so that $h'(x) = -h(x)g'(x)/g(x) = -g'(x)/g^2(x)$.

4 Find the derivative of $f(x) = 1/x^4$. **Solution**: $f'(x) = -4x^3/x^8 = -4/x^5$. The same computation shows that $\frac{d}{dx}x^n = nx^{n-1}$ holds for all integers n.

The formula $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$ holds for all integers n.

The **quotient rule** is obtained by applying the product rule to $f(x) \cdot (1/g(x))$ and using the reciprocal rule:

If $g(x) \neq 0$, then
$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[f'(x)g(x) - f(x)g'(x)]}{g^2(x)} .$$

5 Find the derivative of $f(x) = \tan(x)$. **Solution**: because $\tan(x) = \sin(x)/\cos(x)$ we have

$$\tan'(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} .$$

6 Find the derivative of $f(x) = \frac{2-x}{x^2+x^4+1}$. **Solution**. We apply the quotient rule and get $[(-1)x^2 + x^4 + 1 + (2-x)(2x + 4x^3)]/(x^2 + x^4 + 1)$.

Here are some more problems with solutions:

7 Find the second derivative of $\tan(x)$. **Solution**. We have already computed $\tan'(x) = 1/\cos^2(x)$. Differentiate this again with the quotient rule gives

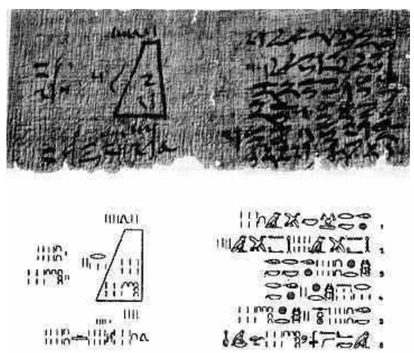
$$\frac{-\frac{d}{dx}\cos^2(x)}{\cos^4(x)} .$$

We still have to find the derivative of $\cos^2(x)$. The product rule gives $\cos(x)\sin(x) + \sin(x)\cos(x) = 2\cos(x)\sin(x)$. Our final result is

$$2\sin(x)/\cos^3(x) .$$

8 A cylinder has volume $V = \pi r^2 h$, where r is the radius and h is the height. Assume the radius grows like $r(t) = 1 + t$ and the height shrinks like $1 - \sin(t)$. Does the volume grow or decrease at $t = 0$?
Solution: The volume $V(t) = \pi(1 + t)^2(1 - \sin(t))$ is a product of two functions $f(t) = \pi(1 + t)^2$ and $g(t) = (1 - \sin(t)$. We have $f(0) = 1, g'(0) = 2, f'(0) = 2, g(0) = 1$. The product rule gives gives $V'(0) = \pi 1 \cdot (-1) + \pi 2 \cdot 1 = \pi$. The volume increases in volume at first.

9 On the **Moscow papyrus** dating back to 1850 BC, the general formula $V = h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h appeared. Assume $h(t) = 1 + \sin(t), a(t) = 1 + t, b(t) = 1 - 2t$. Does the volume of the truncated pyramid grow or decrease at first? **Solution**. We could fill in $a(t), b(t), h(t)$ into the formula for V and compute the derivative using the product rule. A bit faster is to write $f(t) = a^2 + ab + b^2 = (1+t)^2 + (1-3t)^2 + (1+t)(1-3t)$ and note $f(0) = 3, f'(0) = -6$ then get from $h(t) = (1+\sin(t))$ the data $h(0) = 1, h'(0) = 1$. So that $V'(0) = (h'(0)f(0) - h(0)f'(0))/3 = (1\cdot 3 - 1(-6))/3 = -1$. The pyramid shrinks in volume at first.



10 We pump up a balloon and let it fly. Assume that the thrust increases like t and the resistance decreases like $1/\sqrt{1-t}$ since the balloon gets smaller. The distance traveled is $f(t) = t/\sqrt{1-t}$. Find the velocity $f'(t)$ at time $t = 0$.

Homework

1 Find the derivatives of the following functions:
a) $f(x) = \sin(3x)\cos(10x)$.
b) $f(x) = \sin^2(x)/x^2$.
c) $f(x) = x^4 \sin(x)\cos(x)$.
d) $f(x) = 1/\sqrt{x}$.
e) $f(x) = \cot(x) + (1 + x)/(1 + x^2)$.

2 a) Verify that for $f(x) = g(x)h(x)k(x)l(x)$ the formula $f' = g'hkl + gh'kl + ghk'l + ghkl'$ holds.
b) Verify the following formula for derivative of $f(x) = g(x)^4$: $f'(x) = 4g^3(x)g'(x)$. We will derive this later using the chain rule. Don't use that rule yet.

3 If $f(x) = \mathrm{sinc}(x) = \sin(x)/x$, find its derivative $g(x) = f'(x)$ and then the derivative of $g(x)$. Then evaluate it numerically at $x = 0$.

4 Find the derivative of

$$\frac{\sin(x)}{1 + \cos(x) + \frac{x^4}{1+\cos^2(x)}}$$

at $x = 0$.

5 a) We have already computed the derivative of $f(x) = \sqrt{x}$ in the last homework by directly computing the limit. Lets do it using the product rule. Use part a) of this problem to compute the derivative of

$$g(x) = f(x) \cdot f(x)$$

Use the obtained identity $g'(x) = ...$ to get a formula for $f'(x) = \frac{d}{dx}\sqrt{g(x)}$.
b) Use the same method and the above homework problem 2) in this homework set to compute the derivative of the cube root function $f(x) = x^{1/4}$.

Remark: Also this last problem 5) is a preparation for the chain rule, we see next Monday. Avoid using the chain rule already here.

Lecture 10: The chain rule

How do we take the derivative of a composition of functions? It is the chain rule which will allow us to compute derivatives like for $f(x) = \sin(x^7)$ which is a composition of two functions $f(x) = x^7$ and $g(x) = \sin(x)$. The product rule does not work here. The functions are "chained", we evaluate first x^7 then apply sin to it. In order to differentiate, we the derivative of the first function we evaluate x^7 then multiply this with the derivative of the function sin at x^7. The answer is $7x^6 \cos(x^7)$.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) .$$

The chain rule follows from the identity

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h} .$$

Write $H(x) =$ g(x+h)-g(x) in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h} .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(g(x))$ and the second factor has $g'(x)$ as a limit. The chain rule is one reason why classical calculus is so elegant: $D(f(g)) = (D_H f)(g(x))D(g(x))$. The h has changed.

1 Find the derivative of $f(x) = (4x-1)^{17}$. **Solution** The inner function is $g(x) = 4x - 1$. It has the derivative 4. We get therefore $f'(x) = 17(4x-1)^6 \cdot 4 = 28(4x-1)^6$. Remark. We could have expanded out the power $(4x-1)^{17}$ first and avoided the chain rule. Avoiding the **chain rule** is called the **pain rule**.

2 Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution**: applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

3 For linear functions $f(x) = ax + b, g(x) = cx + d$, the chain rule can readily be checked. We have $f(g(x)) = a(cx + d) + b = acx + ad + b$ which has the derivative ac. Indeed this is the definition of f times the derivative of g. You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

4 Find the derivative of the natural logarithm function $\log(x)$ [1] **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x)) \log'(x) = x \log'(x) = 1$. Therefore $\log'(x) = 1/x$.

[1] We always write $\log(x)$ for the natural log. The ln notation is old fashioned and only used in obscure places like calculus books and calculators from the last millenium.

$\frac{d}{dx} \log(x) = 1/x$.

Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0, \pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$.

5 Find the derivative of $\arcsin(x)$. **Solution**. We write $x = \sin(\arcsin(x))$ and differentiate.

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} .$$

6 Find the derivative of $\arccos(x)$. **Solution**. We write $x = \cos(\arccos(x))$ and differentiate.

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}} .$$

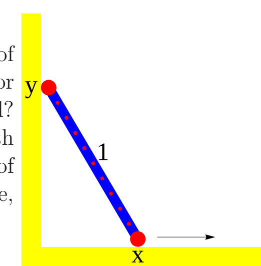
7 $f(x) = \sin(x^2 + 3)$. Then $f'(x) = \cos(x^2 + 3)2x$.

8 $f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x))) \cos(\sin(x)) \cos(x)$.

Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

9 Lets compute the derivative of $\sin(\sqrt{x^5 - 1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5 - 1}$ gives $\cos(\sqrt{x^5 - 1}) \frac{d}{dx} \sqrt{x^5 - 1}$. Apply now the chain rule again for the derivative on the right hand side.

10 Here is the famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0, y)$ on the wall with $(x, 0)$ on the floor. We want to express y as a function of x. We have $y = f(x) = \sqrt{1-x^2}$. Taking the derivative, assuming $x' = 1$ gives $f'(x) = -2x/\sqrt{1-x^2}$.



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

11 For the brave: find the derivative of $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))))$.

Homework

1 Find the derivatives of the following functions
a) $f(x) = \sin(\sqrt{x})$
b) $f(x) = \tan(1/x^7)$
c) $f(x) = \exp(1/(1+x))$
d) $(2+\sin(x))^{-5}$

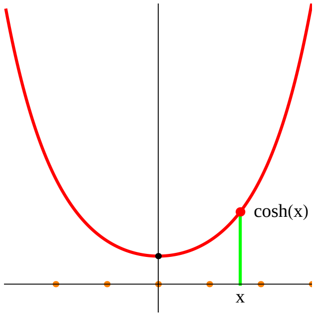
2 Find the derivatives of the following functions at $x = 1$.
a) $f(x) = x^8 \log(x)$. (log is natural log)
b) $\sqrt{x^5+1}$

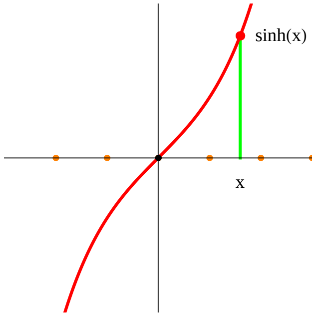
3 a) Find the derivative of $f(x) = 1/x$ by differentiating the identity $xf(x) = 1$.
b) Find the derivative of $f(x) = \text{arccot}(x)$ by differentiating $\cot(\text{arccot}(x)) = x$.

4 a) Find the derivative of $f(x) = \sqrt{x}$ by differentiating the identity $f(x)^2 = x$.
b) Find the derivative of $f(x) = x^{m/n}$ by differentiating the identity $f(x)^n = x^m$.

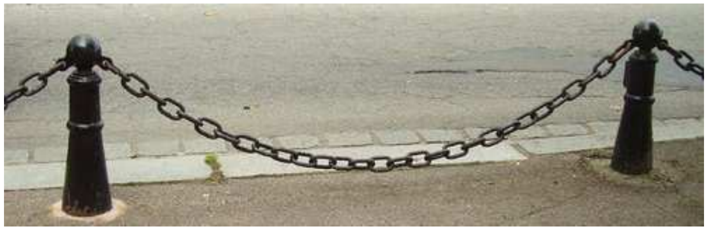
> The function $f(x) = [\exp(x) + \exp(-x)]/2$ is called $\cosh(x)$.
> The function $f(x) = [\exp(x) - \exp(-x)]/2$ is called $\sinh(x)$.
> They are called **hyperbolic cosine** and **hyperbolic sine**. The first is even, the second is odd. You can see directly using $\exp'(x) = \exp(x)$ and $\exp'(-x) = -\exp(-x)$ that $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$. Furthermore $\exp = \cosh + \sinh$ writes exp as a sum of an even and odd function.

5 a) Find the derivative of the inverse $\text{arccosh}(x)$ of $\cosh(x)$.
b) Find the derivative of the inverse $\text{arcsinh}(x)$ of $\sinh(x)$.





The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine.



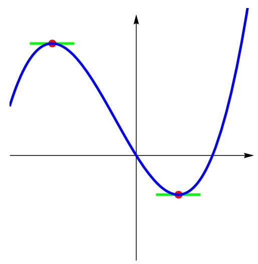
Lecture 11: Local extrema

Today we look at the problem to find extrema. We want to maximize nice quantities and minimize unpleasant ones. Extremizing quantities is also the most important principle nature follows. Important laws in physics like Newtons law, equations describing light, or matter can be based on the principle of extremization. The most important intuitive insight is that at maxima or minima the tangent to the graph needs to be horizontal. This leads to a zero derivative and the notion of critical points:

A point x_0 is a **critical point** of a differentiable function f if $f'(x_0) = 0$.

In some textbooks, critical points include points where f' is not defined. [1] We do here **not** include these points in the list of critical points. They are points outside the domain of definition of f' and will be treated separately.

1 Find the critical points of the function $f(x) = x^3 + 3x^2 - 24x$. **Solution:** we compute the derivative as $f'(x) = 3x^2 + 6x - 24$. The roots of f' are $2, -4$.



A point is called a **local maximum** of f, if there exists an interval $U = (p-a, p+a)$ around p, such that $f(p) \geq f(x)$ for all $x \in U$. A **local minimum** is a local maximum of $-f$. Local maxima and minima together are called **local extrema**.

2 The point $x = 0$ is a local maximum for $f(x) = \cos(x)$. The reason is that $f(0) = 1$ and $f(x) < 1$ nearby.

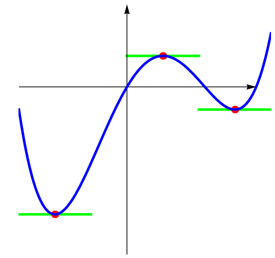
3 The point $x = 1$ is a local minimum for $f(x) = (x-1)^2$. The function is zero at $x = 1$ and positive everywhere else.

Fermat: If f is differentiable and has a local extremum at x, then $f'(x) = 0$.

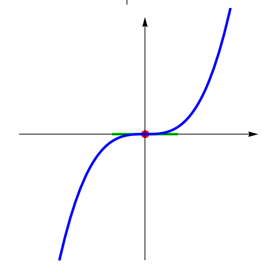
Why? Assume the derivative $f'(x) = c$ is not zero. We can assume $c > 0$ otherwise replace f with $-f$. By the definition of limits, for some large enough h, we have $f(x+h) - f(x)/h \geq c/2$. But this means $f(x+h) \geq f(x) + hc/2$ and x can not be a local maximum. Since also $(f(x) - f(x-h))/h \geq c/2$ for small enough h, we also have $f(x-h) \leq f(x) - hc/2$ and x can not be a local minimum.

[1] Important definitions have to be simple

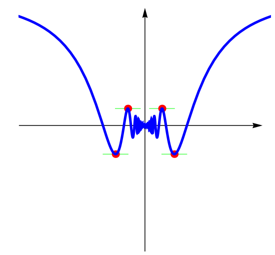
4 The derivative of $f(x) = 72x - 30x^2 - 8x^3 + 3x^4$ is $f'(x) = 72 - 60x - 24x^2 + 12x^3$ By plugging in integers (calculus teachers like integer roots because students like integer roots!) we can guess the roots $x = 1, x = 3, x = -2$ and see $f'(x) = 12(x-1)(x+2)(x-3)$. The critical points are $1, 3, -2$.



5 We have already seen that $f'(x) = 0$ does not assure that x is a local extremum. The function $f(x) = x^3$ is a counter example. It satisfies $f'(0) = 0$ but 0 is not a local extremum. It is an example of an **inflection point**, a point where f'' changes sign.



6 Lets look at one nasty example. The function $f(x) = x\sin(1/x)$ is continuous at 0 but there are infinitely many critical points near 0.



If $f''(x) > 0$, then the graph of the function is concave up. If $f''(x) < 0$ then the graph of the function is concave down.

Second derivative test. If x is a critical point of f and $f''(x) > 0$, then f is a local minimum. If $f''(x) < 0$, then f is a local maximum.

If $f''(x_0) > 0$ then $f'(x)$ is negative for $x < x_0$ and positive for $f'(x) > x_0$. This means that the function decreases left from the critical point and increases right from the critical point. Similarly, if $f''(x_0) < 0$ then $f'(x)$ is positive for $x < x_0$ and $f'(x)$ is positive for $x > x_0$. This means that the function increases left from the critical point and increases right from the critical point.

7 The function $f(x) = x^2$ has one critical point at $x = 0$. Its second derivative is 2 there.

8 Find the local maxima and minima of the function $f(x) = x^3 - 3x$ using the second derivative test. **Solution:** $f'(x) = 3x^2 - 3$ has the roots $1, -1$. The second derivative $f''(x) = 6x$ is negative at $x = -1$ and positive at $x = 1$. The point $x = -1$ is therefore a local maximum and the point $x = 1$ is a local minimum.

9 Find the local maxima and minima of the function $f(x) = \cos(\pi x)$ using the second derivative test.

10 For the function $f(x) = x^5 - x^3$, the second derivative test is inconclusive at $x = 0$. Can you nevertheless see the critical points?

11 Also for the function $f(x) = x^4$, the second derivative test is inconclusive at $x = 0$. The second derivative is zero. Can you nevertheless see whether the critical point 0 is local maximum of local minimum?

Finally, lets look at an example, where we can practice more the chain rule.

12 Find the critical points of $f(x) = 4\arctan(x) + x^2$. **Solution**. The derivative is

$$f'(x) = \frac{4}{1+x^2} + 2x = \frac{2x+2x^3+4}{1+x^2} .$$

We see that $x = -1$ is a critical point. There are no other roots of $2x + 2x^3 + 4 = 0$. How did we get the derivative of arctan again? Differentiate

$$\tan(\arctan(x)) = x$$

and write $u = \arctan(x)$:

$$\frac{1}{\cos^2(u)} \arctan'(x) = 1 .$$

Use the identity $1 + \tan^2(u) = 1/\cos^2(u)$ to write this as

$$(1 + \tan^2(u)) \arctan'(x) = 1 .$$

But $\tan(u) = \tan(\arctan(x)) = x$ so that $\tan^2(u) = x^2$. And we have

$$(1 + x^2) \arctan'(x) = 1 .$$

Now solve for $\arctan'(x)$:

$$\arctan'(x) = \frac{1}{1+x^2} .$$

Homework

1 Find all critical points for the following functions. If there are infinitely many, indicate their structure. For $f(x) = \cos(x)$ for example, the critical points can be written as $\pi/2 + k\pi$, where k is an integer.
a) $f(x) = x^4 - 3x^2$.
b) $f(x) = 3 + \sin(\pi x)$
c) $f(x) = \exp(-x^2)x^2$.
d) $f(x) = \cos(\sin(x))$

2 Find all the maxima and minima using the second derivative test:
a) $f(x) = x\log(x)$, where $x > 0$.
b) $f(x) = 1/(1 + x^2)$
c) $f(x) = x^2 - 2x + 1$.
d) $f(x) = 2x\tan(x)$, where $-\pi/2 < x < \pi/2$

3 Verify that a cubic equation $f(x) = x^3 + ax^2 + bx + c$ always has an inflection point, a point where $f''(x)$ changes sign.
Hint. Remember the wobbling table!

4 Depending on c, the function $f(x) = x^4 - cx^2$ has either one or three critical points. Find these points for a general c and use the second derivative test to see whether they are maxima or minima. The answer will depend on c. Where does the answer change?

5 a) Engineer a function which has exactly 2 local maximum and 1 local minimum.
b) Find a function which has exactly 2 local maxima and no local minimum.

Lecture 12: Global extrema

In this lecture we are interested in the points where a function is maximal overall. These **global extrema** can occur at critical points of f or at the boundary of the domain, where f is defined.

A point p is called a **global maximum** of f if $f(p) \geq f(x)$ for all x. A point p is called a **global minimum** of f if $f(p) \leq f(x)$ for all x.

How do we find global maxima? We just make a list of all local extrema and boundary points, then pick the largest. Global maxima or minima do not need to exist. The function $f(x) = x^2$ has a global minimum at $x = 0$ but no global maximum. The function $f(x) = x^3$ has no global extremum at all. We can however look at global maxima on finite intervals.

1 Find the global maximum of $f(x) = x^2$ on the interval $[-1, 2]$. **Solution**. We look for local extrema at critical points and at the boundary. Then we compare all these extrema to find the maximum or minimum. The critical points are $x = 0$. The boundary points are $-1, 2$. Comparing the values $f(-1) = 1, f(0) = 0$ and $f(4) = 4$ shows that f has a global maximum at 2 and a global minimum at 0.

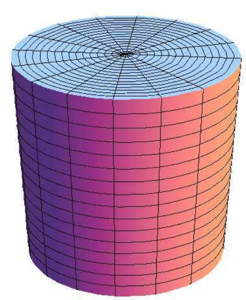
Extreme value theorem A continuous function f on a finite interval $[a, b]$ attains a global maximum and a global minimum.

Here is the argument: Because the function is continuous, the image of the interval $[a, b]$ is a closed interval $[c, d]$. [1] There is a point such that $f(x) = c$, which is a global minimum and a point where $f(x) = d$ which is a global maximum.

Note that the global maximum or minimum can also also on the boundary or points where the derivative dos not exist.

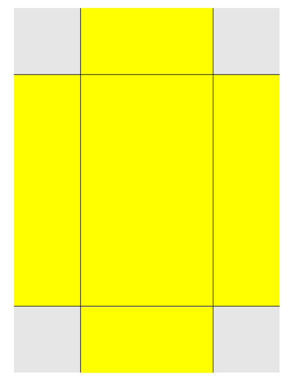
2 Find the global maximum and minimum of the function $f(x) = |x|$. The function has no absolute maximum as it goes to infinity for $x \to \infty$. The function has a global minimum at $x = 0$ but the function is not differentiable there. The point $x = 0$ is a point which does not belong to the domain of f'.

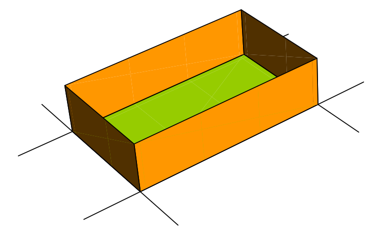
3 A **soda can** is a cylinder of volume $\pi r^2 h$. The surface area $2\pi rh + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is 2π, we can solve the equation for $h = (1 - r^2)/r = 1/r - r$ **Solution:** The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.



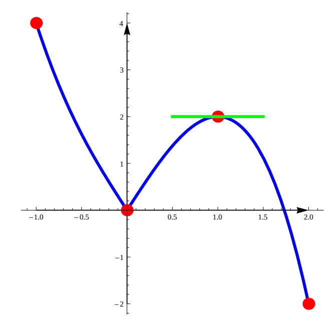
[1] This statement needs more justification but is intuitive enough that we can accept it.

4 Take a card of 2×2 inches. If we cut out 4 squares of equal side length x at the corners, we can fold up the paper to a tray with width $(2 - 2x)$ length $(2 - 2x)$ and height x. For which $x \in [0, 1]$ is the tray volume maximal?
Solution The volume is $f(x) = (2 - 2x)(2 - 2x)x$. To find the maximum, we need to compare the critical points which is at $x = 1/3$ and the boundary points $x = 0$ and $x = 1$.





5 Find the global maxima and minima of the function $f(x) = 3|x| - x^3$ on the interval $[-1, 2]$.
Solution. For $x > 0$ the function is $3x - x^3$ which can be differentiated. The derivative $3 - 3x^2$ is zero at $x = 1$. For $x < 0$ the function is $-3x - x^3$. The derivative is $-3 - x^2$ and has no root. The only critical points are 1. There is also the point $x = 0$ which is not in the domain where we can differentiate the function. We have to deal with this point separately. We also have to look at the boundary points $x = -1$ and $x = 2$. Making a list of function values at $x = -1, x = 0, x = 1, x = 2$ gives the maximum.



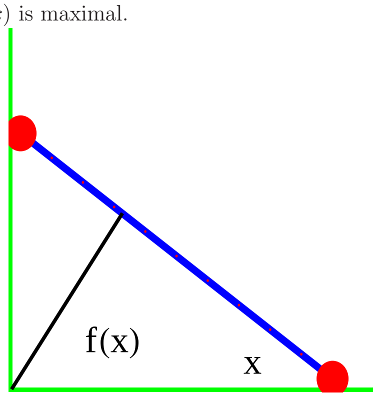
Homework

1 Find the global maxima and minima of the function $f(x) = (x-2)^2$ on the interval $[-1, 4]$.

2 Find the global maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 36x$ on the interval $[-4, 4]$

3 A candy manufacturer builds spherical candies. Its effectiveness is $A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r. Find the radius, where $f(r) = A(r) - V(r)$ has a global maximum for $r \geq 0$.

4 A ladder of length 1 is one side at a wall and on one side at the floor. First verify that the distance from the ladder to the corner is $f(x) = \sin(x)\cos(x)$. Find the angle x for which $f(x)$ is maximal.



5 a) The function $S(x) = -x\log(x)$ is called the **entropy function**. Find the probability $0 < x \leq 1$ which maximizes entropy. important principle in all science is that nature tries to maximize entropy. In some sense we compute here the number of maximal entropy.

b) We can write $1/x^x = e^{-x\log(x)}$. Find the positive value x, where x^{-x} has a local maximum. [2]

Entropy has been introduced by Bolzman. It is important in physics and chemistry.

[2] We have used the identity $a^b = e^{b\log(a)}$

Lecture 13: Hopitals rule

The rule

This Hopital's rule works is a miracle remedy and solves all our remaining worries about limits:

> **Hopital's rule**. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then
> $$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} .$$

Lets see how it works:

1 Lets prove **the fundamental theorem of trigonometry** again:

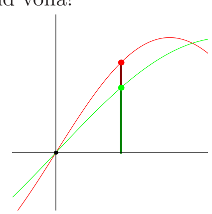
$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 .$$

Why did we work so hard for this? Note that we used the fundamental theorem to derive the derivatives for cos and sin at all points. In order to apply l'Hopital, we had to know the derivative. Our work to establish the limit was not in vain.

The proof of the rule is almost comic in its simplicity if we compare it with how fantastically useful it is:
Since $f(p) = g(p) = 0$ we have $Df(p) = f(p+h)/h$ and $Dg(p) = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$ the **quantum l'Hopital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} .$$

Now take the limit $h \to 0$. And voilà! [1]



Sometimes, we have to administer a medicine twice. To use this, l'Hopital can be improved in that the condition $g'(0) = 0$ can be replaced by the requirement that the limit $\lim_{x \to p} f'(x)/g'(x)$ exists. Instead of having a rule which replaces a limit with an other limit (we cure a disease with a new one!) we formulate it in the way how it is actually used. The second derivative case could easily be generalized for higher derivatives. There is no need to memorize this. Just remember that you can check in several times to a hospital.

> If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f''(x)}{g''(x)}$ if the limit to the right exists.

[1] Some books refer to the intermediate value theorem here. This is not necessary.

2 Find the limit $\lim_{x \to 0}(1 - \cos(x))/x^2$. Remember that this limit had also been pivotal to compute the derivatives of trigonometric functions. **Solution**: differentiation gives

$$\lim_{x \to 0} -\sin(x)/2x .$$

This limit can be obtained with l'Hopital again.

$$\lim_{x \to 0} -\sin(x)/(2x) = \lim_{x \to 0} -\cos(x)/2 = -1/2 .$$

3 Find the limit $f(x) = (\exp(x^2) - 1)/\sin(x^2)$ for $x \to 0$.

4 What do you get if you apply l'Hopital to the limit $[f(x+h) - f(x)]/h$ as $h \to 0$?

5 Find $\lim_{x \to \infty} x \sin(1/x)$. **Solution.** Write $y = 1/x$ then $\sin(y)/y$. Now we have a limit, where the denominator and nominator both go to zero.

The case when both sides converge to infinity can be reduced the other case by looking at $A = f/g = (1/g(x))/(1/f(x))$ which has the limit $g'(x)/g^2(x)/f'(x)/f^2(x) = g'(x)/f'(x)((1/g)/(1/f))^2 = g'/f'(f^2/g^2) = (g'/f')A^2$, so that $A = f'(p)/g'(p)$. We see:

> If $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = \infty$ for $x \to p$ and $g'(p) \neq 0$, then
> $$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)} .$$

2 What is the limit $\lim_{x \to 0} x^x$? This answers the intriguing question: what is **What is 0^0?**
Solution: Because $x^x = e^{x \log(x)}$, it is enough to understand the limit $x \log(x)$ for $x \to 0$.

$$\lim_{x \to 0} \frac{\log(x)}{1/x} .$$

Now the limit can be seen as the limit $(1/x)/(-1/x^2) = -x$ which goes to 0. Therefore $\lim_{x \to 0} x^x = 1$. (We assume $x > 0$ to have real values x^x)

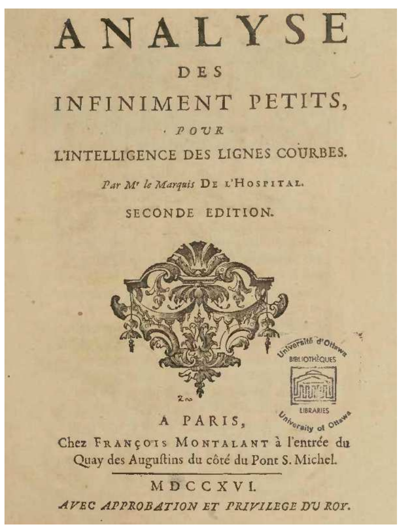
3 Find the limit $\lim_{x \to 2} \frac{x^2 - 4x + 4}{\sin^2(x-2)}$.
Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.
Hopital's rule always works in calculus situations, where functions are differentiable. The rule can fail if differentiability of f or g fails. Here is an other "rare" example:

4 **Deja Vue**: Find $\frac{\sqrt{x^2+1}}{x}$ for $x \to \infty$. L'Hopital gives $x/\sqrt{x^2+1}$ which in terms gives again $\frac{\sqrt{x^2+1}}{x}$. Apply l'Hopital again to get the original function. We got an infinite loop. If the limit is A, then the procedure tells that it is equal to $1/A$. The limit must therefore be 1. This case can be covered easily without l'Hopital: divide both sides by x to get $\sqrt{1 + 1/x^2}$. Now, we can see the limit 1.

5 Given a differentiable function satisfying $g(0) = 0$. Verify that the limit $\lim_{x \to 0} f(g(x))/g(x)$ is $f'(0)$. **Solution:** You check in the homework that the result is $f'(g(0))$.

History

The "first calculus book", the world has known was "Analyse des Infiniment Petits pour l"intelligence des Lignes Courbes". It appeared in 1696 and was written by **Guillaume de l'Hopital**, a text if typeset in a modern font would probably fit onto 50-100 pages.[2] It is now clear that the mathematical content of Hopital's book is mostly due to **Johannes Bernoulli** who became a mathematical "mercenary" for l'Hopital: *Clifford Truesdell write in his article "The New Bernoulli Edition",* [3] *about this "most extraordinary agreement in the history of science": l'Hopital wrote: "I will be happy to give you a retainer of 300 pounds, beginning with the first of January of this year ... I promise shortly to increase this retainer, which I know is very modest, as soon as my affairs are somewhat straightened out ... I am not so unreasonable as to demand in return all of your time, but I will ask you to give me at intervals some hours of your time to work on what I request and also to communicate to me your discoveries, at the same time asking you not to disclose any of them to others. I ask you even not to send here to Mr. Varignon or to others any copies of the writings you have left with me; if they are published, I will not be at all pleased. Answer me regarding all this ..."* Bernoulli's response is lost, but a letter from l'Hopital indicates that it was quickly accepted. From this point on, Bernoulli was a "giant enchained" (Truesdell). Clifford Truesdell also mentions that the book of l'Hopital has remained the standard for Calculus for a century.



ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par Mr le Marquis DE L'HOSPITAL.
SECONDE EDITION.
A PARIS,
Chez FRANÇOIS MONTALANT à l'entrée du
Quay des Auguſtins du côté du Pont S. Michel.
MDCCXVI.
AVEC APPROBATION ET PRIVILEGE DU ROY.

[2]Stewart's book with 1200 pages probably contains about 4 million characters, about 12 times more than l'Hopital's book. It also contains more material of course. The OCRed text of l'Hopital's book of 200 pages has 300'000 characters.

[3]Isis, Vol. 49, No. 1, 1958, pages 54-62

Homework

1 For the following functions, find the limits as $x \to 0$:
a) $5x/\sin(x)$
b) $(\exp(x)-1)/(\exp(3x)-1)$
c) $\sin^2(3x)/\sin^2(5x)$
d) $\frac{\sin(x^2)}{\sin^2(x)}$
e) $\sin(\sin(x))/x$.

2 For the following functions, find the limits as $x \to 1$:
a) $(x^2-x-1)/(\cos(x-1)-1)$
b) $(\exp(x)-e)/(\exp(3x)-e^3)$
c) Find the limit as $x \to \infty$:
$(x^2-x-1)/\sqrt{x^4+1}$. (Find the limit of $(x^2-x-1)^2/(x^4+1)$ first and then take the square root of the limit).
d) Find the limit as $x \to 1$
$(x-4)/(4x+\sin(x)+8)$

3 Use l'Hopital to compute the following limits at $x = 0$:
a) $\log(5x)/\log|x|$.
b) $\lim_{x\to 0} 1/\log|x|$
c) $\text{sinc}'(x) = (\cos(x)x - \sin(x))/x^2$
d) $\log|\log|1+x||/\log|\log|2+x||$.

4 We have seen how to compute limits with healing. Solve the following healing problems with l'Hopital at $x = 1$:
a) $\frac{x^{100}-1}{x^{10}-1}$.
b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

5 a) Assume a function $f(x)$ satisfies $f(0) = 0$ and $f'(0) \neq 0$. Verify the following formula
$$\lim_{x\to 0} f(ax)/f(bx) = a/b \; .$$
b) Given a differentiable function g satisfying $g(0) = 0$ and a differentiable function f. Verify that
$$\lim_{x\to 0} \frac{f(g(x))}{g(x)} = f'(0) \; .$$

Lecture 14: Newton's method

In the intermediate value theorem lecture we computed roots of functions by divide and conquer: start with an interval $[a, b]$ such that $f(a) < 0$ and $f(b) > 0$, then successively half the interval always choosing the side on which the function takes different signs at the boundary. We are then $(b-a)/2^n$ close to the root in n steps. If the function is differentiable we can do much better and use the value of the derivative at a boundary point to get closer. If we draw a tangent at $(x, f(x))$, then

$$f'(x) = \frac{f(x)}{x - T(x)} .$$

because $f'(x)$ is the slope of the tangent and the right hand side is "rise" over "run". If we solve for $T(x)$ we get

The **Newton map** is defined as

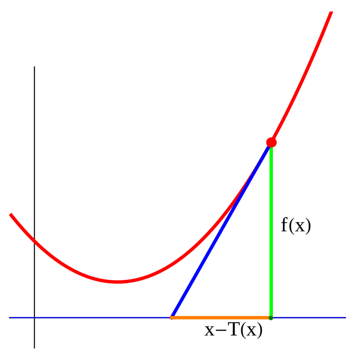
$$T(x) = x - \frac{f(x)}{f'(x)} .$$

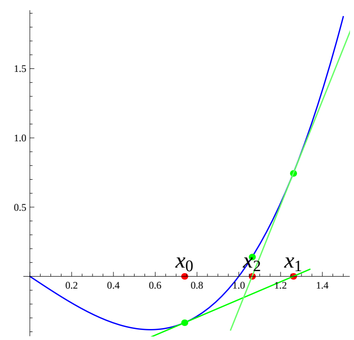
Newton's method is the process to apply this map again and again until we are sufficiently close to the root. It is an extremely fast method to find the root of a function. Start with a point x, then compute a new point $x_1 = T(x)$, where

$$T(x) = x - f(x)/f'(x) .$$

Now iterate this again and again.

If p is a root such that $f'(p) \neq 0$, and x_0 is close to p, then $x_1 = T(x), x_2 = T^2(x)$ converges to the root p.





1 If $f(x) = ax + b$, we reach the root in one step.

2 If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0 but not as fast as the method promises. Indeed, the root is also a critical point which slows us down.

3 The Newton map brings us to infinity if we start at a critical point.

Newton used the method to find the roots of polynomials. The method is so fast that it amazes: Starting 0.1 close to the point, we have after one step 0.01 after 2 steps 0.0001 after 3 steps 0.00000001 and after 4 steps 0.0000000000000001.

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$ if we start sufficiently close to the root.

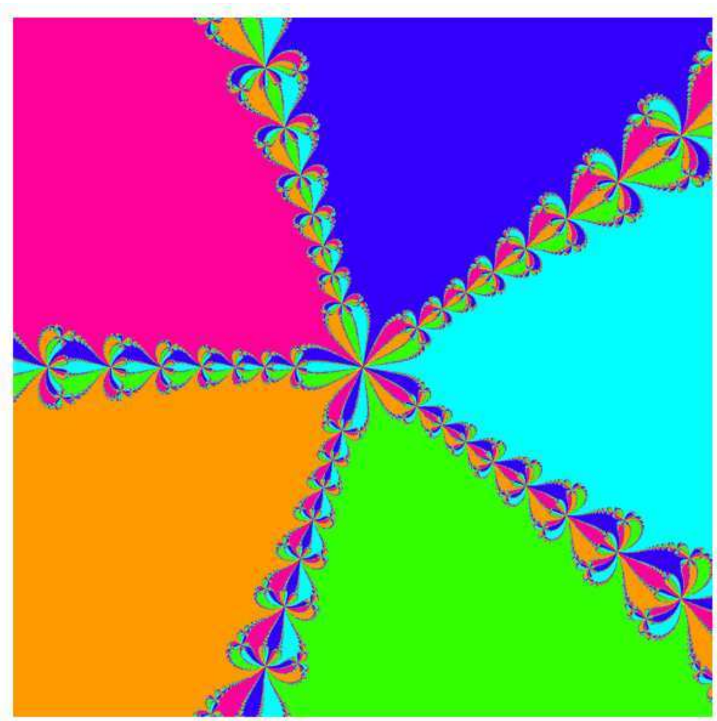
In 10 steps we can get a $2^{10} = 1024$ digits accuracy. Having a fast method to compute roots is useful. For example in computer graphics, where things can not be fast enough. Also in number theory, when working with integers having thousands of digits the Newton method can help. Besides that, there is theoretical use which can explain for example the stability of planetary motion.

4 Verify that the Newton map $T(x)$ in the case $f(x) = (x-1)^3$ has the property that we approach the root $x = 1$. **Solution**. You see that the approach is not that fast: we get $T(x) = x + (1-x)/3 = (1+2x)/3$. It converges exponentially fast, but not super exponential. The reason is that the derivative at $x - 1$ is not zero. That slows us down.

If we have several roots, and we start at some point, to which root will the Newton method converge? Doe it at all converge? This is an interesting question. It is also historically intriguing because it is one of the first cases, where "chaos" can be observed at the end of the 19'th century.

5 Find the Newton map in the case $f(x) = x^5 - 1$. **Solution** $T(x) = x - (x^5 - 1)/(5x^4)$.

If we look for roots in the complex like for $f(x) = x^5 - 1$ which has 5 roots in the complex plane, the basin of attraction of each of the points is a complicated set, a so called **Newton fractal**. Here is the picture:



6 Lets compute $\sqrt{2}$ to 12 digits accuracy - by hand! We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned} T(1) &= 1 - (1-2)/2 = 3/2 \\ T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\ T(17/12) &= 577/408 \\ T(577/408) &= 665857/470832 \end{aligned}$$

This is already $1.6 \cdot 10^{-12}$ close to the real root!

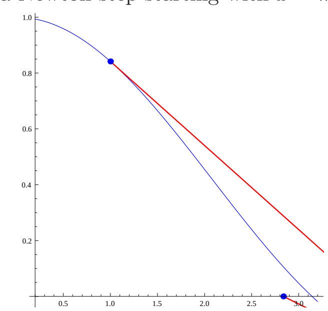
7 To find the cube root of 10 we have to find a root of $f(x) = x^3 - 10$. The Newton map is $T(x) = x - (x^3 - 10)/(3x^2)$. If we start with $x = 2$, we get the following steps $2, 13/6, 3277/1521, 105569067476/49000820427$. After three steps we have a result which is already $2.2 \cdot 10^{-9}$ close to the root.

The Newton method is an incredibly fast algorithm to get roots x_0 of equations. Simply scrumtrulescent.

Homework

1 Find the Newton map $T(x) = x - f(x)/f'(x)$ in the following cases
a) $f(x) = x^4$
b) $f(x) = e^{3x}$
c) $f(x) = e^{-x^2}$
d) $f(x) = 2\tan(x)$.

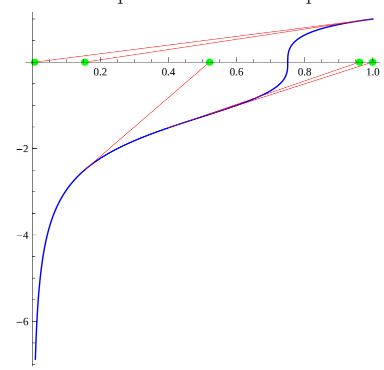
2 a) The sinc function $f(x) = \sin(x)/x$ has a root between 1 and 4. We get closer to the root by doing a Newton step starting with $x = \pi/2$. Do this step graphically



3 The Newton map is handy to compute square roots. Assume we cant to find the square root of 99. We have to solve $\sqrt{99} = x$ or $f(x) = x^2 - 99 = 0$. Perform two Newton steps $T(x) = x - (x^2 - 99)/(2x)$ starting at $x = 10$.

4 a) Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$ and $f(x) = x^6$.
b) Find the Newton step $T(x)$ in general if $f(x) = x^\alpha$, where α is a real number.

5 Verify that the Newton map in the case $f(x) = (4 - 3/x)^{1/3}$ is the quadratic map $T(x) = 4x(1-x)$. This is an example of a Newton map which does not converge.



Lecture 15: Review for first midterm

Major points

A function is **continuous**, if whenever x, y are close, also $f(x), f(y)$ are close. Formally, for every a there exists $b = f(a)$ such that $\lim_{x \to a} f(x) = b$ for every a. The Intermediate value theorem: $f(a) > 0, f(b) < 0$ implies f having a root in (a, b).

If $f'(x) = 0$ and $f''(x) > 0$ then x is a local minimum. If $f'(x) = 0$ and $f''(x) < 0$ then x is a local maximum. To find **global extrema**, compare local extrema and boundary values.

If $f' > 0$ then f is increasing, if $f' < 0$ it is decreasing. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a horizontal tangent.

Hôptial's theorem tells that limits $\lim_{x \to p} f(x)/g(x)$, where $f(p) = g(p) = 0$ or $f(p) = g(p) = \infty$ with $g'(p) \neq 0$ are given by $f'(p)/g'(p)$.

With $Df(x) = (f(x+h) - f(x))/h$ and $S(x) = h(f(h) + f(2h) + ...f(kh))$ we have $SDf(kh) = f(kh) - f(0)$ and $DS(f(kh)) = f(kh)$. This is a preliminary fundamental theorem of calculus.

Roots of $f(x)$ with $f(a) < 0, f(b) > 0$ can be obtained by the dissection method by applying the **Newton map** $T(x) = x - f(x)/f'(x)$ again and again.

Algebra reminders

Healing:	$(a+b)(a-b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)$
Denominator:	$1/a + 1/b = (a+b)/(ab)$
Exponential:	$(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b \log(a)}$
Logarithm:	$\log(ab) = \log(a) + \log(b)$. $\log(a^b) = b \log(a)$
Trig functions:	$\cos^2(x) + \sin^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots:	$a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	$x^3 + 2x^2 + 3x + 1$	Exponential	$5e^{3x}$
Rational functions	$(x+1)/(x^3 + 2x + 1)$	Logarithm	$\log(3x)$
Trig functions	$2\cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$f(x) = x^n$	nx^{n-1}	$f(x) = \sin(ax)$	$a\cos(ax)$
$f(x) = e^{ax}$	ae^{ax}	$f(x) = \tan(x)$	$1/\cos^2(x)$
$f(x) = \cos(ax)$	$-a\sin(ax)$	$f(x) = \log(x)$	$1/x$

Differentation rules

Addition rule	$(f+g)' = f' + g'$.	Quotient rule	$(f/g)' = (f'g - fg')/g^2$.
Scaling rule	$(cf)' = cf'$.	Chain rule	$(f(g(x))' = f'(g(x))g'(x)$.
Product rule	$(fg)' = f'g + fg'$.	Easy rule	simplify before deriving

Extremal problems

1 Build a fence of length $x + 2y = 12$ with dimensions x and y with maximal area $A = xy$.

2 Find the largest area $A = 4xy$ of a rectangle with vertices $(x, y), (-x, y), (-x, -y), (x, -y)$ inscribed in the ellipse $x^2 + 2y^2 = 1$.

3 Which isosceles triangle of height h and base $2x$ and area $xh = 1$ has minimal circumference $2x + 2\sqrt{x^2 + h^2}$?

4 Where is the distance $\sqrt{x^2 + y^2}$ of the parabola $y = x^2 - 2$ to the point $(0, 0)$ minimal?

5 A cone of height $h = 1 + x$ and radius $r = \sqrt{1 - x^2}$ is tightly enclosed by a unit sphere centered at height x. Maximize the volume $r^2 \pi h/3$ of the cone.

6 Maximize $f(x) = \sin(x)$ on $[0, \pi]$.

Limit examples

$\lim_{x \to 0} \sin(x)/x$	l'Hopital 0/0	$\lim_{x \to 1} (x^2 - 1)/(x + 1)$	heal directly
$\lim_{x \to 0} (1 - \cos(x))/x^2$	l'Hopital 0/0 twice	$\lim_{x \to \infty} \exp(x)/(1 + \exp(x))$	l'Hopital
$\lim_{x \to 0} x \log(x)$	l'Hopital ∞/∞	$\lim_{x \to 0} (x + 1)/(x + 5)$	no work necessary

Important things

Summation and taking differences is at the hart of calculus
The 3 major types of discontinuities are jump, oscillation, infinity
Dissection and Newton methods are algorithms to find roots. Disection needs continuity, Newton needs
The fundamental theorem of trigonometry is $\lim_{x \to 0} \sin(x)/x = 1$.
The derivative is the limit $Df(x) = [f(x+h) - f(x)]/h$ as $h \to 0$. It is called rate of change.
The rule $D(1+h)^{x/h} = (1+h)^{x/h}$ leads to $\exp'(x) = \exp(x)$.

More Examples

1 Is $1/\log|x|$ continuous at $x = 0$. Answer: yes

2 Is $\log(1/|x|)$ continuous at $x = 0$. Answer: no

3 Find $\lim_{x \to 1} (x^{1/3} - 1)/(x^{1/4} - 1)$. Answer: 4/3.

4 Find $\lim_{x \to 1} \sin(5x - 5)/(x - 1)$. Answer: 5.

5 Find $\lim_{x \to 2} \frac{3 - \sqrt{7+x}}{x - 2}$. Answer $-1/6$

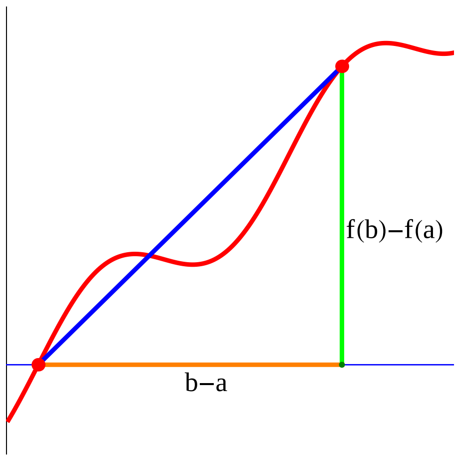
6 Find $\arcsin'(5x^2)$. Answer: $10x(1 - 25x^4)^{-1/2}$

Lecture 16: Mean value theorem

In this lecture, we look at the **mean value theorem** and a special case called **Rolle's theorem**. It is important later when we study the fundamental theorem of calculus. Unlike the intermediate value theorem which applied for continuous functions, the mean value theorem involves derivatives:

Mean value theorem: For a differentiable function f and an interval (a, b), there exists a point p inside the interval, such that

$$f'(p) = \frac{f(b) - f(a)}{b - a} .$$



Here are a few examples which illustrate the theorem:

1 Verify with the mean value theorem that the function $f(x) = x^2 + 4\sin(\pi x) + 5$ has a point where the derivative is 1.
Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1 - 0) = 5$.

2 Verify that the function $f(x) = 4\arctan(x)/\pi - \cos(\pi x)$ has a point where the derivative is 3.
Solution. We have $f(0) = -1$ and $f(1) = 2$. Apply the mean value theorem.

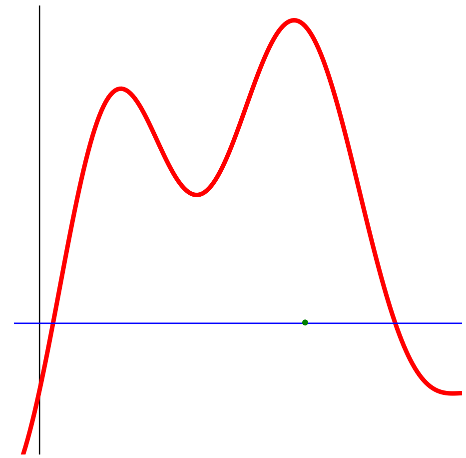
3 A biker drives with velocity $f'(t)$ at position $f(b)$ at time b and at position a at time a. The value $f(b) - f(a)$ is the distance traveled. The fraction $[f(b) - f(a)]/(b - a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

4 The function $f(x) = \sqrt{1 - x^2}$ has a graph on $(-1, 1)$ on which every possible slope is taken.
Solution: We can see this with the intermediate value theorem because $f'(x) = x/\sqrt{1 - x^2}$ gets arbitrary large near $x = -1$ or $x = 1$. The mean value theorem shows this too because we can take intervals $[a, b] = [-1, -1 + c]$ for which $[f(b) - f(a)]/(b - a) = f(-1 + c)/ci \sim \sqrt{c}/c = 1/\sqrt{c}$ gets arbitrary large.

Why is the theorem true? The function $h(x) = f(a) + cx$, where $c = (f(b) - f(a)/(b - a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are

done. By tilting the picture, we have reduced the statement to a special case which is important by itself:

Rolle's theorem: If $f(a) = f(b)$ and f is differentiable, then there exists a critical point p of f in the interval (a, b).



Here is the proof: If it were not true, then $f'(x) \neq 0$ and we would have $f'(x) > 0$ everywhere or $f'(x) < 0$ everywhere. The monotonicity would mean however that $f(b) > f(a)$ or $f(b) < f(a)$.

Here is a second proof: Fermat's theorem assures that there is a local maximum or local minimum of f in (a, b). At this point the derivative is zero. This means $f'(x) = 0$.

We have also seen a related fact that if f is continuous and $f(a) = f(b)$ then there is a local maximum or local minimum in the interval (a, b). This fact is more general and applies to every continuous function. The derivative does not need to exist.

5 There is a point in $[0, 1]$ where $f'(x) = 0$ with $f(x) = x(1 - x^2)(1 - \sin(\pi x))$. **Solution:** We have $f(0) = f(1) = 0$. Use Rolle's theorem.

6 Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$. **Solution:** The function is nonnegative and zero at $0, \pi$.It is also differentiable and so by Rolle's theorem there is a critical point. Remark. We can not use Rolle's theorem to show that there is a local maximum even so the extremal value theorem assures us that this exist.

7 Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g. [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is 1/2 where $g(1/2) = 21/2 > 0$.]

Who was the first to find the **mean value theorem**? It is not so clear. Joseph Louis Lagrange is one candidate. Also Augustin Louis Cauchy (1789-1857) is credited for a modern formulation of the theorem.

Joseph Louis Lagrange, 1736-1813.

Augustin Louis Cauchy, 1789-1857.

What about **Michel Rolle**? He lived from 1652 to 1719, mostly in Paris. No picture of him seems available. Rolle also introduced the n'th root notation like when writing the cube root as $\sqrt[3]{x}$.

Homework

1 The function $f(x) = 1 - |x|$ satisfies $f(-1) = f(1) = 0$ but there is no point where $f'(x) = 0$. Is this a counter example to Rolle's theorem?

2 Use Rolle's theorem and the intermediate value theorem to show that the function $f(x) = x^3 + 3x + 1$ has exactly one root. You do not have to find the root.

3 We look at the function $f(x) = \log|x| + \sin(x)$ on the positive real line
a) Use the **mean value theorem** to assure the there is a p where $f'(p) = 1000$.

b) Use the **intermediate value theorem** applied to the function $f'(x)$ to assure the

same.

4 **Cauchy's mean value theorem** states that for any two differentiable function and any interval (a, b), there exists c for which $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$. We want to prove this here. Define the function $h(x) = (f(b)-f(a))(g(x)-g(a))-(g(b)-g(a))(f(x)-f(a))$.
a) Verify that $h(a) = h(b) = 0$.

b) Compute $h'(x)$.

c) Use Rolle's theorem to verify that there is a c for which $h'(c) = 0$.

5 Given the function $f(x) = x\sin(x)$ and the function $g(x) = \cos(x)$. Verify (using Cauchy's mean value theorem from the previous problem) that there is a point $p \in (0, \pi/2)$ for which $f'(p)/g'(p) = -\pi/2$. You do not have to find the point.

Lecture 17: Catastrophes

In this lecture, we look closer at extrema problems. More precisely, we are interested how extrema change when a parameter changes. Nature, economies, processes favor extrema. Extrema change smoothly with parameters. How come that the outcome is often not smooth? What is the reason that political change can go so fast once a tipping point is reached? One can explain this with mathematical models. We look at a simple example, which explains the general principle:

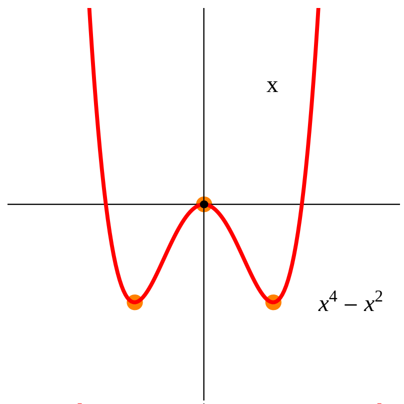
> If a local minimum seizes to become a local minimum, a new stable position is favored. This can be far away from the original situation.

To get started, lets look at an extremal problem

Find all the extrema of the function

$$f(x) = x^4 - x^2$$

1 **Solution:** $f'(x) = 4x^3 - 2x$ is zero for $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$. The second derivative is $12x^2 - 2$. It is negative for $x = 0$ and positive at the other two points. We have two local minima and one local maximum.

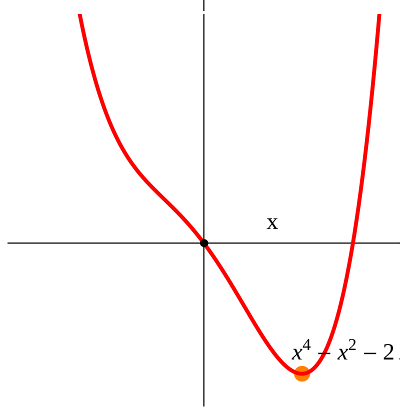


Now find all the extrema of the function

2

$$f(x) = x^4 - x^2 - 2x$$

There is only one critical point. It is $x = 1$.

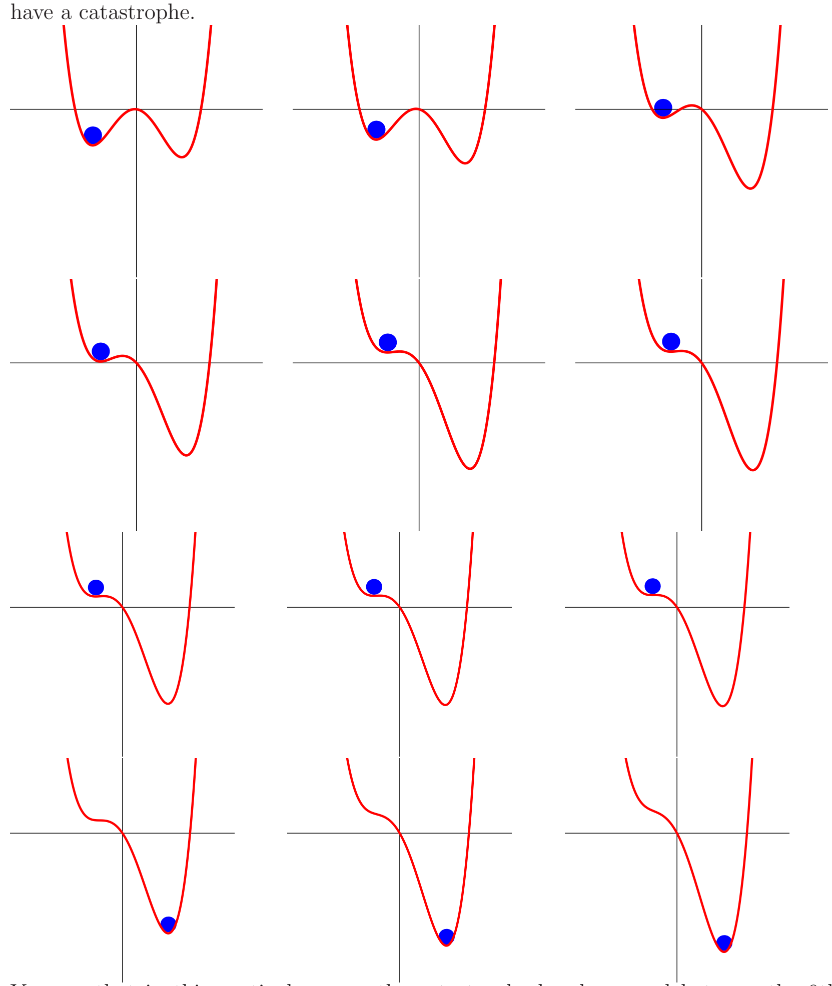


Something has happened when the first example was morphed to the second example. The local minimum to the left has disappeared. Assume the function f measures the prosperity of some kind and c is a parameter. We look at the position of the first critical point of the function. Catastroph theorists call this the **Delay assumption**:

> A **stable equilibrium** is a local minimum of the function. Assume the system depends on a parameter, then the minimum depends on this parameter. It remains a stable equilibrium until it disappears. If that happens, the system settles in a neighboring stable equilibrium.

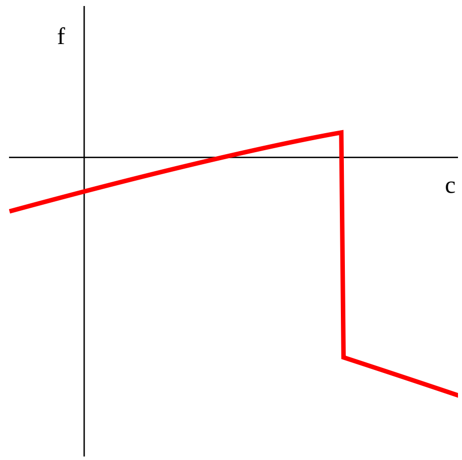
> A parameter value for which a stable minimum disappears, is called a **catastrophe**.

In order to visualize a catastrophe, we draw the graph of the function $f_c(x)$ for various parameters c and look at the local minima. At a parameter value, where a local minimum disappears, we have a catastrophe.

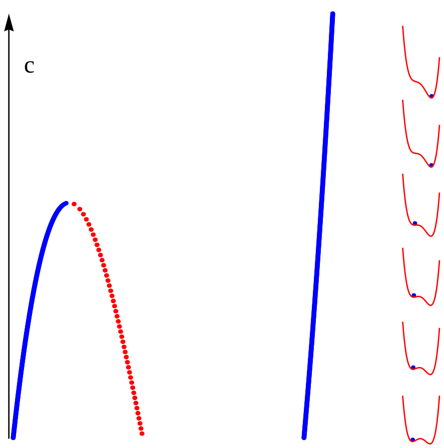


You see that in this particular case, the catastrophe has happened between the 9th and 10th picture.

Here is the position of the equilibrium point plotted in dependence of c.



A parameter value c for which a local minimum disappears is called a **catastrophe**.



A **Bifurcation diagram** displays the equilibrium points as they change in dependence of the parameter c. The vertical axes is the parameter c, the horizontal axes is x. At the bottom for $c = 0$, we have three equilibrium points, two local minima and one local maximum. At the top for $c = 1$ we have only one local minimum.

Catastrophes always go for the worse in the sense that the value decreases. It is not possible to reverse the process and have a catastrophe, where the minimum jumps up.

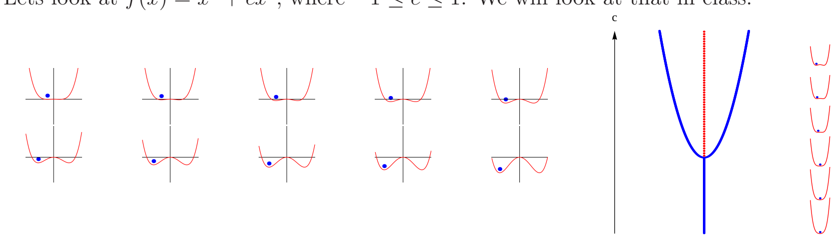
Look again at the above "movie" of graphs. But run it backwards and use the same principle. We do not end up at the position we started with. The new equilibrium stays the equilibrium. Decreasing the food prizes again did not reverse the process of change in Egypt for example.

Catastrophes are in general irreversible.

We know this from daily experience: it is easy to screw up a relationship, get sick, have a ligament torn or lose trust. Building up a relationship, getting healthy or gaining trust usually happens slowly only. Ruining the economy of a country or a company or losing a good reputation of a brand is very easy. It takes a long time to regain it.

Local minima can change discontinuously, when a parameter is changed. This can happen with perfectly smooth functions and smooth parameter changes.

3 Lets look at $f(x) = x^4 + cx^2$, where $-1 \leq c \leq 1$. We will look at that in class.



Homework

We study a catastrophe for the function

$$f(x) = x^6 - x^4 + cx^2 ,$$

where c is a parameter between 0 and 1.

1 a) Find all the critical points in the case $c = 0$ and analyze their stability. b) Find all the critical points in the case $c = 1$ and analyze their stability.

2 Plot the graph of f for at least 10 values of c between 0 and 1. You can use software, a graphing calculator or Wolfram alpha. Mathematica code is below.

3 If you change from $c = -0.3$ to 0.6, pinpoint the value for the catastrophe and show a rough plot of $c \to f(x_c)$, the value at the first local minimum x_c in dependence of c. The text above provides this graph for an other function. It is the graph with a discontinuity.

4 If you change back from $c = 0.6$ to -0.3 pinpoint the value for the catastrophe. It will be different from the one in the previous question.

5 Sketch the bifurcation diagram. That is, if $x_k(c)$ is the k'th equilibrium point, then draw the union of all graphs of $x_k(c)$ as a function of c (the c-axes pointing upwards). As in the two example provided, draw the local maximum with dotted lines.

```
Manipulate[ Plot[x^6 - x^4 + c x^2, {x, -1, 1}], {c, 0, 1}]
```

Lecture 18: Riemann integral

In this lecture, we define the integral $\int_0^x f(t)\,dt$ if f is a differentiable function and compute it for some basic functions.

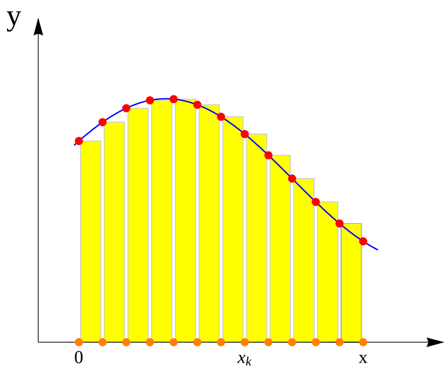
First a reminder. We have defined the **Riemann sums**

$$Sf(x) = h[\,f(0) + f(h) + f(2h) + \cdots + f(kh)]\ ,$$

where k is the largest integer such that $kh < x$. Lets write S_n if we want to stress that the parameter $h = 1/n$ was used in the sum. We define the **integral** as the limit of these sums $S_n f$, when the **mesh size** $h = 1/n$ goes to zero.

Define

$$\int_0^x f(t)\,dt = \lim_{n\to 0} S_n f(x)\ .$$



For any differentiable function, the limit exists

Proof: Lets first assume $f \geq 0$ on $[0, x]$. Let M be such that $f \leq M$ and $f' \leq M$ on $[0, x]$. The Riemann sum $S_n f(x)$ is the total area of k rectangles. Let $Sf(x)$ denote the area under the curve. If M is the maximal slope of f on $[0, x]$, then on each interval $[j/n, (j + 1)/n]$, we have $|f(x) - f(j/n)| \leq M/n$ so that the area error is smaller than M/n^2. Additionally, we have a piece above the interval $[kh, x]$ with area $\leq M/n$. If we add all the $k \leq xn$ "roof area errors" and the "side area" up, we get

$$Sf(x) - S_n f(x) \leq \frac{kM}{n^2} + \frac{M}{n} \leq \frac{xnM}{n^2} + \frac{M}{n} = \frac{xM + M}{n}\ .$$

This converges to 0 for $n \to \infty$. The limit is therefore the area $Sf(x)$. For a general, not necessarily nonnegative function, we write $f = g - h$, where g, h are nonnegative (see homework) and have $\int_0^x f(x)\,dx = \int_0^x g(x)\,dx - \int_0^x h(x)\,dx$.

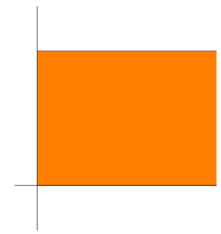
For nonnegative f, the value $\int_0^x f(x)\,dx$ is the **area between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

Remark: the Riemann integral is defined here as the limit $h \sum_{x_k = kh \in [0,x)} f(x_k)$. It converges to the area under the curve for all **continuous** functions but since we work with differentiable functions in calculus we restricted to that. Not **all** bounded functions can be integrated naturally

like this. There are discontinuous functions like the **salt and pepper function** which is defined to be $f(x) = 1$ if x is rational and zero else. Now $Sf(x) = 1$ for rational h and $Sf(x) = 0$ if h is irrational. Therefore, an other integral, the **Lebesgue integral** is used too: it can be defined as the limit $\frac{1}{n}\sum_{k=1}^n f(x_k)$ where x_k are **random points** in $[0, x]$. This **Monte-Carlo integral** definition of the Lebesgue integral gives the integral 0 for the salt and pepper function because rational numbers have zero probability.

Remark: Many calculus books define the Riemann integral with partitions $x_0 < x_1 < ..., x_n$ of points of the interval $[0, x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring x_j goes to zero. The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$, where y_k is arbitrarily chosen inside the interval (x_k, x_{k+1}). For continuous functions, the limiting result is the same the $Sf(x)$ sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated and keep a wide mesh, where the function does not change much. This leads to **numerical analysis** of integrals.

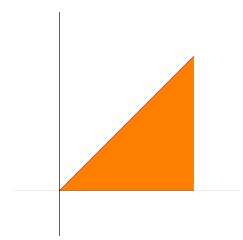
1 Let $f(x) = c$ be constant everywhere. Now $\int_0^x f(t)\,dt = cx$. We can see also that $cnx/n \leq S_n f(x) \leq c(n + 1)x/n$.



2 Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Remark: we could also have added up the Riemann sum but thats more painful: for every $h = 1/n$, let k be the largest integer smaller than $xn = x/h$. Then (remember Gauss's punishment?)

$$S_n f(x) = \frac{1}{n}\sum_{j=1}^k \frac{cj}{n} = \frac{ck(k + 1)/2}{n^2}\ .$$

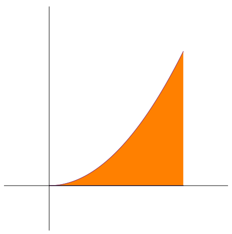
Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\,dt = cx^2/2$.

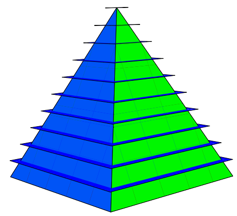


3 Let $f(x) = x^2$. In this case,we can not see the numerical value of the area geometrically. But since we have computed $S[x^2]$ in the first lecture of this course and seen that it is $[x^3]/3$ and since we have defined $S_h f(x) \to \int_0^x f(t)\,dt$ for $h \to 0$ and $[x^k] \to x^k$ for $h \to 0$, we know that

$$\int_0^x t^2\,dt = \frac{x^3}{3}\ .$$

This example actually computes the **volume of a pyramid** which has at distance t from the top an area t^2 cross section. Think about $t^2 dt$ as a slice of the pyramid of area t^2 and height dt. Adding up the volums of all these slices gives the volume.

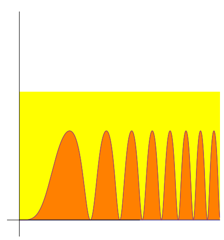




Linearity of the integral (see homework) $\int_0^x f(t)+g(t)\ dt = \int_0^t f(t)\ dt + \int_0^x g(t)\ dt$ and $\int_0^x \lambda f(t)\ dt = \lambda \int_0^x f(t)\ dt$.

Upper bound: If $0 \leq f(x) \leq M$ for all x, then $\int_0^x f(t)\ dt \leq Mx$.

4 $\int_0^x \sin^2(\sin(\sin(t)))/x\ dt \leq x$. **Solution**. The function $f(t)$ inside the interval is nonnegative and smaller or equal to 1 The graph of f is therefore contained in a rectangle of width x and height 1.



We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If f and g satisfy $|f(x) - g(x)| \leq c$, then
$$\int_0^x |f(x) - g(x)|\ dx \leq cx\ .$$

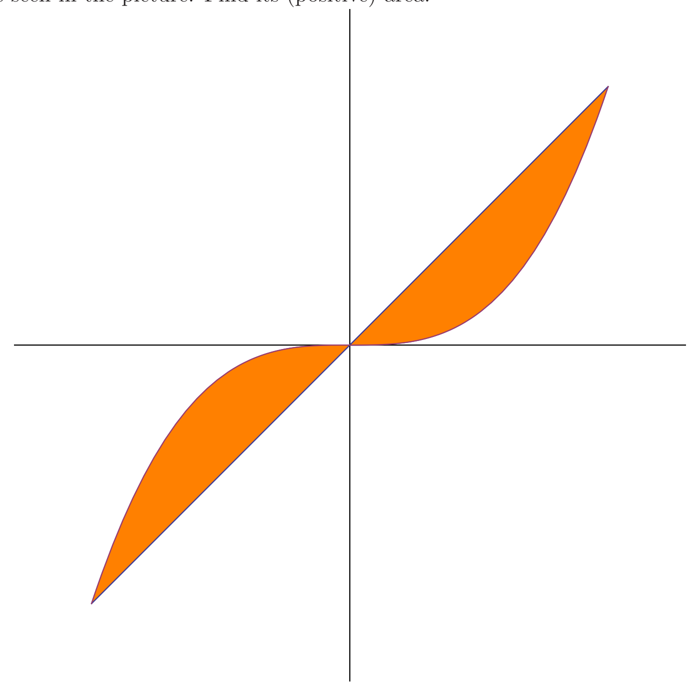
We know identities like $S_n[x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$ already. Since $[x]_h^k - [x]^k \to 0$ we have $S_n[x]_h^k - S_n[x]^k \to 0$ and from $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$.
The other equalities are the same since $\exp_h(x) = \exp(x) \to 0$. This gives us:

$\int_0^x t^n\ dt = \frac{x^{n+1}}{n+1}$	$\int_0^x \cos(t)\ dt = \sin(x)$
$\int_0^x e^t\ dt = e^x - 1$	$\int_0^x \sin(t)\ dt = 1 - \cos(x)$

Homework

1 a) Find the integral $\int_0^x 6t^7 + 4t^3 + e^t\ dt$.
b) Calculate $\int_0^{\log(2)} e^{-t}\ dt$.
c) Find $\int_0^{\pi/2} \cos(t)\ dt$.

2 The region enclosed by the graph of x and the graph of x^3 has a propeller type shape as seen in the picture. Find its (positive) area.



3 Argue geometrically with areas why the following statements hold:

- $\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$.
- $\int_a^b f(x)\ dx + \int_a^b g(x)\ dx = \int_a^b f(x) + g(x)\ dx$.
- $\int_a^b \lambda f(x)\ dx = \lambda \int_a^b f(x)\ dx$.
- $0 \leq m \leq f(x) \leq M$ implies $(b-a)m \leq \int_a^b f(x)\ dx \leq (b-a)M$.

4 a) Verify that f can be written as a difference of two nonnegative functions. To do so, show that $g(x) = \max(f(x), 0)$ and $h(x) = \max(-f(x), 0)$ have the property that $f(x) = g(x) - h(x)$ and that $g(x) \geq 0$ and $h(x) \geq 0$.
b) Draw the graphs of the two functions $g(x), h(x)$ for $f(x) = \cos(3x)$ where $0 \leq x \leq 2\pi$.

5 a) Find $\int_0^3 |x - 1|\ dx$. Distinguish cases.
b) Find $\int_0^3 f(x)\ dx$ for $f(x) = |x - |x - 1||$. Also here, distinguish cases

Lecture 19: Fundamental theorem

In this lecture we prove the **fundamental theorem of calculus** for differentiable functions. This will allow us in general to compute integrals of functions which appear as derivatives.

We have seen earlier that with $Sf(x) = h(f(0) + \cdots + f(kh))$ and $Df(x) = (f(x+h) - f(x))/h$ we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$. This becomes now:

> **Fundamental theorem of calculus:** Assume f is differentiable. Then
>
> $\int_0^x f'(t)\,dt = f(x) - f(0)$ and $\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$

Proof. Using notation of Euler we write $A \sim B$ for "A and B are close" meaning $A - B \to 0$ for $h \to 0$. From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k+1)h$ because f is continuous. We also know $\int_0^x Df(t)\,dt \sim \int_0^x f'(t)\,dt$ because $Df(t) \sim f'(t)$ uniformly for $0 \leq t \leq x$ by the definition of the derivative and the assumption that f' is continuous. We also know $SDf(x) = f(x) - f(0)$ for $x = kh$. By definition of the Riemann integral $Sf(x) \sim \int_0^x f(t)\,dt$ and so $SDf(x) \sim \int_0^x Df(t)\,dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t)\,dt \sim \int_0^x f'(t)\,dt$$

as well as

$$f(x) \sim DSf(x) \sim D\int_0^x f(t)\,dt \sim \frac{d}{dx}\int_0^x f(t)\,dt\,.$$

1 $\int_0^5 3t^7\,dt = \frac{x^8}{8}|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

2 $\int_0^{\pi/2} \cos(t)\,dt = \sin(x)|_0^{\pi/2} = 1$. This is an important example which should become routine in a while.

3 $\int_0^x \sqrt{1+t}\,dt = \int_0^x (1+t)^{1/2}\,dt = (1+t)^{3/2}(2/3)|_0^x = [(1+x)^{3/2} - 1](2/3)$. Here the difficulty was to see that the $1+t$ in the interior of the function does not make a big difference. Keep such examples in mind.

4 Also in this example $\int_0^2 \cos(t+1)\,dt = \sin(x+1)|_0^2 = \sin(3) - \sin(1)$ the additional term $+1$ does not make a big dent.

5 $\int_{\pi/6}^{\pi/4} \cot(x)\,dx$. This is an example where the anti derivative is difficult to spot. It is easy if we know where to look for: the function $\log(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.

6 The example $\int_1^2 1/(t^2-9)\,dt$ is a bit challenging. We need a hint and write $-6/(x^2-9) = 1/(x+3) - 1/(x-3)$. The function $f(x) = \log|x+3| - \log|x-3|$ has therefore $-6/(x^2-9)$ as a derivative. We know therefore $\int_1^2 -6/(t^2-9)\,dt = \log|3+x| - \log|3-x||_1^2 = \log(5) - \log(1) - \log(4) + \log(2) = \log(5/2)$. The original task is now $(-1/6)\log(5/2)$.

7 $\int_0^x \cos(\sin(x))\cos(x)\,dx = \sin(\sin(x))$ because the derivative of $\sin(\sin(x))$ is $\cos(\sin(x))\cos(x)$. The function $\sin(\sin(x))$ is called the **antiderivative** of f. If we differentiate this function, we get $\cos(\sin(x))\cos(x)$.

8 Find $\int_0^\pi \sin(x)\,dx$. **Solution**: This has a very nice answer.

Here is an important notation, which we have used in the example and which might at first look silly. But it is a handy intermediate step when doing the computation.

> $F|_a^b = F(b) - F(a)$.

We give reformulations of the fundamental theorem in ways in which it is mostly used:

> If f is the derivative of a function F then
>
> $$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)\,.$$

In some textbooks, this is called the "second fundamental theorem" or the "evaluation part" of the fundamental theorem of calculus. The statement $\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$ is the "antiderivative part" of the fundamental theorem. They obviously belong together and are two different sides of the same coin.

Here is a version of the fundamental theorem, where the boundaries are functions of x.

> Given functions g, h and if F is a function such that $F' = f$, then
>
> $$\int_{h(x)}^{g(x)} f(t)\,dt = F(g(x)) - F(h(x))\,.$$

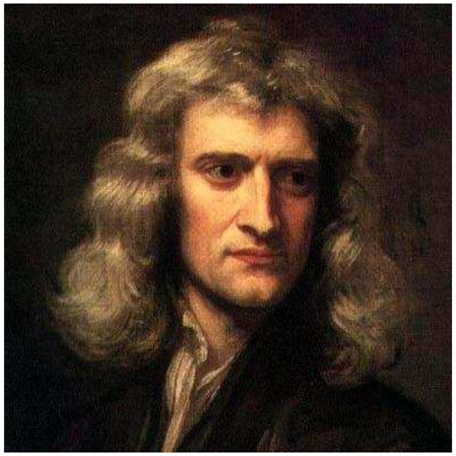
9 $\int_{x^4}^{x^2} \cos(t)\,dt = \sin(x^2) - \sin(x^4)$.

The function F is called an antiderivative. It is not unique but the above formula does always give the right result.

Lets look at a list of important antiderivatives. You should have as many antideriatives "hard wired" in your brain. It really helps. Here are the core functions you should know. They appear a lot.

function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$\frac{-\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x\log(x) - x$

Make your own table!



Meet **Isaac Newton** and **Gottfried Leibniz**. They have discovered the fundamental theorem of calculus. You can see from the expression of their faces, they are honored to find themselves on the same handout with **Austin Powers** and **Doctor Evil**.

Homework

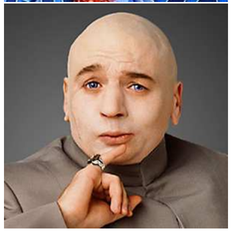
1 For any of the following functions f, find a function F such that $F' = f$.
a) $e^{2x} + \sin(3x) + x^3 + 5x$.
b) $6(x+4)^3$.
c) $3/x + 4/(x-1)$.
d) $\cos(x^2)2x + \sin(x^3)3x^2 + 1/\sqrt{x}$

2 Find the following integrals by finding a function g satisfying $g' = f$. We will learn techniques to find the function. Here, we just use our knowledge about derivatives:
a) $\int_2^3 5x^4 + 4x^3 \, dx$.
b) $\int_{\pi/4}^{\pi/2} \sin(3x) + \cos(x) \, dx$.
c) $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2(x)} \, dx$.
d) $\int_2^3 \frac{1}{x-1} \, dx$.

3 Evaluate the following integrals:
a) $\int_1^2 2^x \, dx$.
b) $\int_{-1}^1 \cosh(x) \, dx$. (Remember $\cosh(x) = (e^x + e^{-x})/2$.)
c) $\int_0^1 \frac{1}{1+x^2} \, dx$.
d) $\int_{1/3}^{2/3} \frac{1}{\sqrt{1-x^2}} \, dx$.

4 a) Compute $F(x) = \int_0^{x^3} \sin(t) \, dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \, dt$ then find $G'(x)$

5 a) **A clever integral:** Evaluate the following integral:
$\int_0^{2\pi} \sin(\sin(x)) \, dx$
Explain the answer you get.



b) **An evil integral**: Can you find the
$\int_1^2 \cos(\log(\tan(x))) \csc(x) \sec(x) \, dx$
You might want to use technology if you do not see the answer.

Lecture 20: Antiderivatives

We have looked at the integral $\int_0^x f(t)\,dt$ and seen that it is the **signed area under the curve**. The area of the region below the curve is counted in a negative way. There is something else to mention:

For $x < 0$, we define $\int_x^0 f(t)\,dt$ as $-\int_0^x f(t)\,dt$. This is compatible with the fundamental theorem $\int_a^b f'(t)\,dt = f(b) - f(a)$.

The function $g(x) = \int_0^x f(t)\,dt + C$ is called the **anti-derivative** of g. The constant C is arbitrary and not fixed. As we will see below, we can often adjust the constant such that some condition is satisfied.

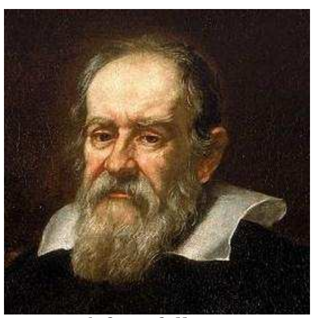
The fundamental theorem of calculus assured us that

The anti derivative is the inverse operation of the derivative. Two different anti derivatives differ by a constant.

Finding the anti-derivative of a function is much harder than finding the derivative. We will learn some techniques but it is in general not possible to give anti derivatives for even very simple functions.

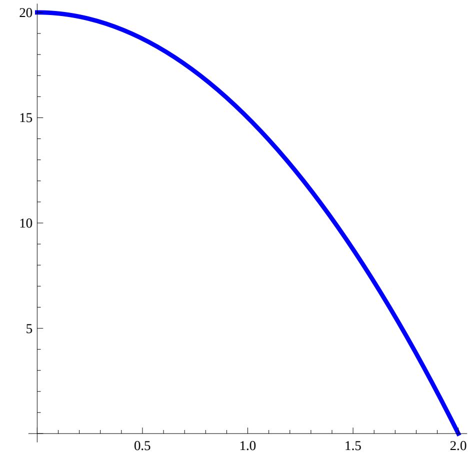
1 Find the anti-derivative of $f(x) = \sin(4x) + 20x^3 + 1/x$. Solution: We can take the anti-derivative of each term separately. The anti derivative is $F(x) = -\cos(4x)/4 + 4x^4 + \log(x) + C$.

2 Find the anti derivative of $f(x) = 1/\cos^2(x) + 1/(1-x)$. **Solution:** we can find the anti derivatives of each term separately and add them up. The result is $F(x) = \cot(x) + \log|1-x| + C$.



Galileo

3 measured **free fall**, a motion with constant acceleration. Assume $s(t)$ is the height of the ball at time t. Assume the ball has zero velocity initially and is located at height $s(0) = 20$. We know that the velocity is $v(t)$ is the derivative of $s(t)$ and the acceleration $a(t)$ is constant equal to -10. So, $v(t) = -10t + C$ is the antiderivative of a. By looking at v at time $t = 0$ we see that $C = v(0)$ is the initial velocity and so zero. We know now $v(t) = -10t$. We need now to compute the anti derivative of $v(t)$. This is $s(t) = -10t^2/2 + C$. Comparing $t = 0$ shows $C = 20$. Now $s(t) = 20 - 5t^2$. The graph of s is a parabola. If we give the ball an additional horizontal velocity, such that time t is equal to x then $s(x) = 20 - 5x^2$ is the visible trajectory. We see that jumping from 20 meters leads to a fall which lasts 2 seconds.



4 The **total cost** is the antiderivative of the **marginal cost** of a good. Both the marginal cost as well as the total cost are a function of the quantity produced. For instance, suppose the total cost of making x shoes is 300 and the total cost of making $x+4$ shoes is 360 for all x. The marginal cost is $60/4 = 15$ dollars. In general the marginal cost changes with the number of goods. There is additional cost needed to produce one more shoe if 300 shoes are produced. **Problem:** Assume the marginal cost of a book is $f(x) = 5 - x/100$ and that producing the first 10 books costs 1000 dollars. What is the total cost of producing 100 books? **Answer:** The anti derivative $5 - x/100$ of f is $F(x) = 5x - x^2/100 + C$ where C is a constant. By comparing $F(10) = 1000$ we get $50 - 100/100 + C = 1000$ and so $C = 951$. the result is $951 + 5 * 100 - 10'000/100 = 1351$. The average book prize has gone down from 100 to 13.51 dollars.

5 The **total revenue** $F(x)$ is the antiderivative of the **marginal revenue** $f(x)$. Also these functions depend on the quantity x produced. We have $F(x) = P(x)x$, where $P(x)$ is the prize. Then $f(x) = F'(x) = P'(x)x + P$. For a **perfect competitive market**, $P'(x) = 0$ so that the prize is equal to the marginal revenue.

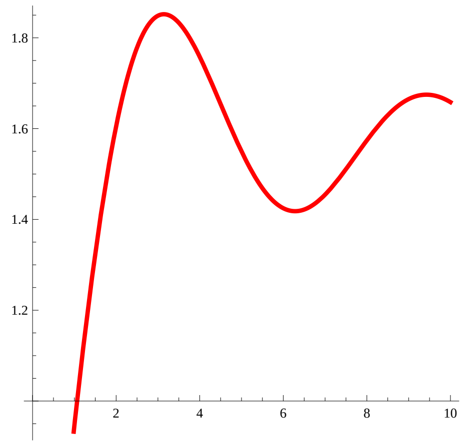
A function f is called **elementary**, if it can be constructed using addition, subtraction, multiplication, division, compositions from polynomials or roots. In other words, an elementary function is built up with functions like $x^3, \sqrt{\cdot}, \exp, \log, \sin, \cos, \tan$ and $\arcsin, \arccos, \arctan$.

6 The function $f(x) = \sin(\sin(\pi + \sqrt{x} + x^2)) + \log(1 + \exp((x^6+1)/(x^2+1)) + (\arctan(e^x))^{1/3}$ is an elementary function.

7 The anti derivative of the sinc function is called the **sine-integral**

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\,dt\,.$$

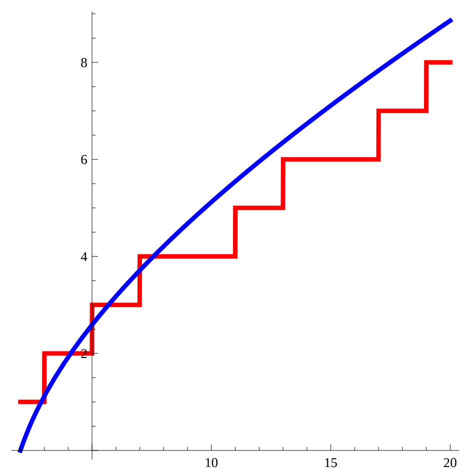
The function $Si(x)$ is not an elementary function.



8 The **offset logarithmic integral** is defined as

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}$$

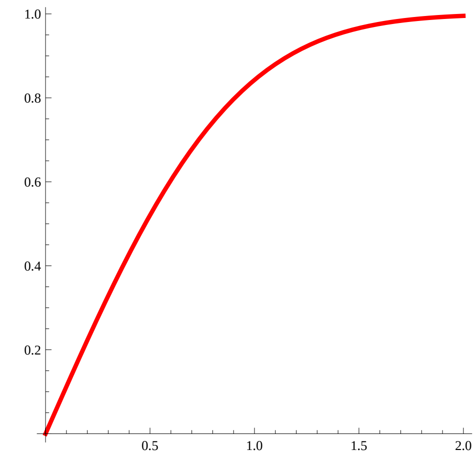
It is a specific anti-derivative. It is a good approximation of the number of prime numbers less than x. The graph below illustrates this. The second stair graph shows the number $\pi(x)$ of primes below x. For example, $\pi(10) = 4$ because $2, 3, 5, 7$ are the only primes below it. The function Li is not an elementary function.



9 The **error function**

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is important in statistics. It is not an elementary function.



The Mathematica command "Integrate" uses about 500 pages of Mathematica code and 600 pages of C code. [1] Before software was doing this, tables of integrals like Gradshteyn and Ryzhik's work were used. This 1200 page book is still useful and contains some integrals, which computer algebra systems have trouble with.

Numerical evaluation

What do we do when we have can not find the integral analytically? We can still compute it numerically. Here is an example: the function $\sin(\sin(x))$ also does not have an elementary anti-derivative. But you could compute the integral numerically with a computer algebra system like Mathematica:

```
NIntegrate[ Sin[Sin[x]],{x,0,10}]
```

Pillow problems

We do not assign homework over spring break but if you have time, here are some integration riddles. We will learn techniques to deal with them. If you can not crack them, no problem. Maybe pick one or two and keep thinking about it over spring break. They make also good pillow problems, problems to think about while falling asleep. Try it. Sometimes, you might know the answer in the morning. Maybe you can guess a function which has $f(x)$ as a derivative.

1 $f(x) = \log(x)/x$.

2 $f(x) = \frac{1}{x^4-1}$.

3 $f(x) = \tan^2(x)$.

4 $f(x) = \cos^4(x)$.

5 $f(x) = \frac{1}{x\log(x)}$.

[1] http://reference.wolfram.com/legacy/v3/MainBook/A.9.5.html

Math 1A: introduction to functions and calculus **Oliver Knill, 2012**

Lecture 21: Area computation

If $f(x) \geq 0$, then $\int_a^b f(x)\ dx$ is the **area under the graph** of $f(x)$ and above the interval $[a,b]$ on the x axes.

If the function is negative, then $\int_a^b f(x)\ dx$ is negative too and this integral is minus the area below the curve:

Therefore, $\int_a^b f(x)\ dx$ as the difference of the area above the graph minus the area below the graph.

More generally we can also look at areas sandwiched between two graphs f and g.

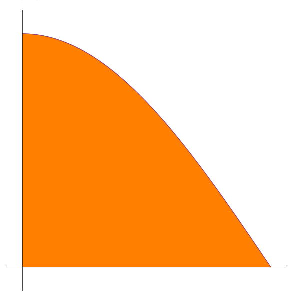
The area of a region G enclosed by two graphs $f \leq g$ and bound by $a \leq x \leq b$ is

$$\int_a^b g(x) - f(x)\ dx$$

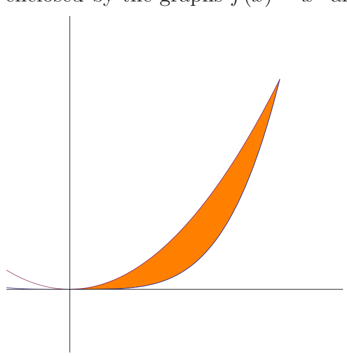
.

Make sure that if you have to compute such an integral that $g \geq f$ before giving it the interpretation of an area.

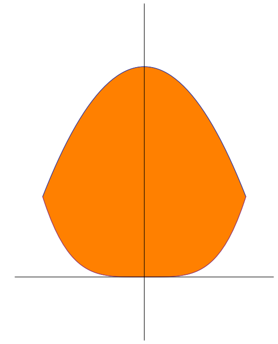
1 Find the area of the region enclosed by the x-axes, the y-axes and the graph of the cos function. **Solution**: $\int_0^{\pi/2} \cos(x)\ dx = 1$.



2 Find the area of the region enclosed by the graphs $f(x) = x^2$ and $f(x) = x^4$.



3 Find the area of the region enclosed by the graphs $f(x) = 1 - x^2$ and $g(x) = x^4$.

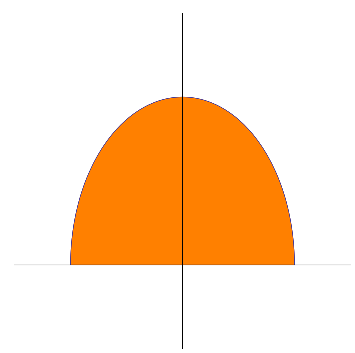


4 Find the area of the region enclosed by a half circle of radius 1. **Solution**: The half circle is the graph of the function $f(x) = \sqrt{1-x^2}$. The area under the graph is

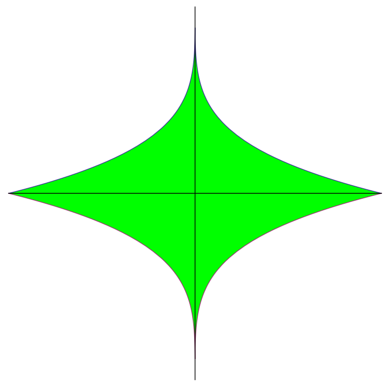
$$\int_{-1}^{1} \sqrt{1-x^2}\ dx\ .$$

Finding the anti-derivative is not so easy. We will find techniques to do so, for now we pop it together: we know that $\arcsin(x)$ has the derivative $1/\sqrt{1-x^2}$ and $x\sqrt{1-x^2}$ has the derivative $\sqrt{1-x^2} - x^2/\sqrt{1-x^2}$. The sum of these two functions has the derivative $\sqrt{1-x^2} - (1-x^2)/\sqrt{1-x^2} = 2\sqrt{1-x^2}$. We find the anti derivative to be $(x\sqrt{1-x^2} + \arcsin(x))/2$. The area is therefore

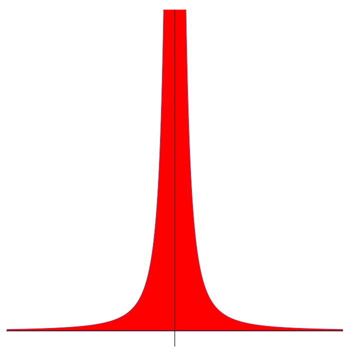
$$\frac{x\sqrt{1-x^2} + \arcsin(x)}{2}\Big|_{-1}^{1} = \frac{\pi}{2}\ .$$



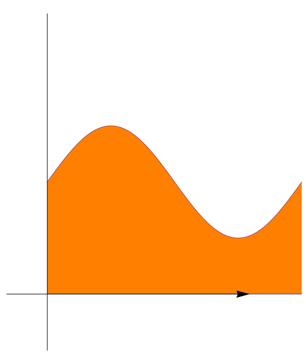
5 Find the area of the region between the graphs of $f(x) = 1 - |x|^{1/4}$ and $g(x) = -1 + |x|^{1/4}$.



6 Find the area under the curve of $f(x) = 1/x^2$ between -6 and 6. Solution. $\int_{-6}^{6} x^{-2}\, dx = -x^{-1}|_{-6}^{6} = -1/6 - 1/6 = -1/3$. There is something fishy with this computation because $f(x)$ is nonnegative so that the area should be positive. But we obtained a negative answer. What is going on?



7 Find the area between the curves $x = 0$ and $x = 2 + \sin(y)$, $y = 2\pi$ and $y = 0$. **Solution:** We turn the picture by 90 degrees so that we compute the area under the curve $y = 0, y = 2 + \sin(x)$ and $x = 2\pi$ and $x = 0$.



8 **The grass problem**. Find the area between the curves $|x|^{1/3}$ and $|x|^{1/2}$. **Solution**. This example illustrates how important it is to have a picture. This is good advise for any "word problem" in mathematics.

Use a picture of the situation while doing the computation.

Homework

1 Find the area of the bounded region enclosed by the graphs $f(x) = x^5 - 12x$ and $g(x) = 2x^2$.

2 Find the area of the region enclosed by the four lines $y = x, y = 3 - 2x, y = -2x, y = 3x - 1$.

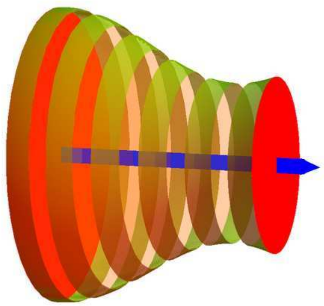
3 Find the area of the region enclosed by the curves $x = 0, x = \pi/2, y = 4 + \sin(3x), y = 2 + \sin^2(2x)$.

4 Write down an integral which gives the area of the region $4x^4 + y^2 \leq 4$ by first writing it as a region between two graphs. Evaluate the integral numerically using Wolfram alpha, Mathematica or any other software.

5 The graphs $1 + \sin(x)$ and $2\cos(x) - 1$ intersect at $x = 0, 2\pi$ and a point between. They define a **humming bird** region,consisting of a larger region and a tail region. Find the area of each part and assume the bird has its eye closed.

Lecture 22: Volume computation

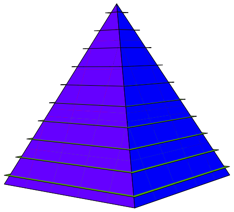
To compute the volume of a solid, we cut it into slices, where each slice is perpendicular to a given line x. If $A(x)$ is the area of the slice and the body is enclosed between a and b then $V = \int_a^b A(x)\, dx$ is the volume. Think of $A(x)dx$ as the volume of a slice. The integral adds them up.



1 Compute the volume of a pyramid with square base length 2 and height 2. **Solution**: we can assume the pyramid is built over the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. The cross section area at height h is $A(h) = (2-h)^2$. Therefore,

$$V = \int_0^2 (2-h)^2 \, dh = \frac{8}{3} .$$

This is base area 4 times height 2 divided by 3.



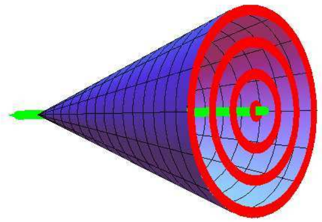
A **solid of revolution** is a surface enclosed by the surface obtained by rotating the graph of a function $f(x)$ around the x-axis.

The area of the cross section at x of a solid of revolution is $A(x) = \pi f(x)^2$. The volume of the solid is $\int_a^b \pi f(x)^2 \, dx$.

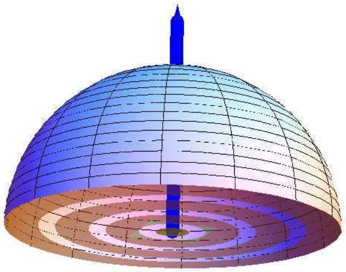
2 Find the volume of a round cone of height 2 and where the circular base has the radius 1. **Solution**. This is a solid of revolution obtained by rotation the graph of $f(x) = x/2$ around the x axes. The area of a cross section is $\pi x^2/4$. Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4 \, dx = \frac{x^3}{4 \cdot 3}|_0^2 = \frac{2\pi}{3} .$$

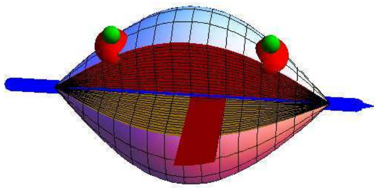
This is the height 2 times the base area π divided by 3.



3 Find the volume of a half sphere of radius 1. **Solution:** The area of the cross section at height h is $\pi(1-h^2)$.



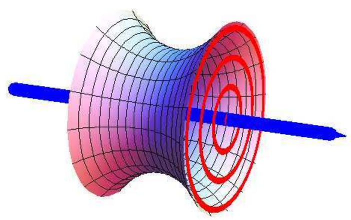
4 We rotate the graph of the function $f(x) = \sin(x)$ around the x axes. But now we cut out a slice of $60 = \pi/3$ degrees out. Find the volume of the solid.
Solution: The area of a slice without the missing piece is $\pi \sin^2(x)$. The integral $\int_0^\pi \sin^2(x)\, dx$ is $\pi/2$ as derived in the lecture. Having cut out 1/6'th the area is $(5/6)\pi \sin^2(x)$. The volume is $\int_0^\pi (5/6)\pi \sin(x)^2 \, dx = (5/6)\pi^2/2$.



Homework

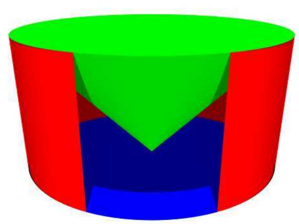
1 Find the volume of the paraboloid for which the radius at position x is $4-x^2$ and x ranges from 0 to 2.

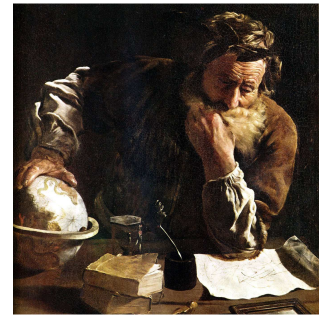
2 A **catenoid** is the surface obtained by rotating the graph of $f(x) = \cosh(x)$ around the x-axes. We have seen that the graph of f is the chain curve, the shape of a hanging chain. Find the volume of of the solid enclosed by the catenoid between $x = -2$ and $x = 2$.
Hint. You might want to check first the identity $2\cosh(x)^2 = 1 + \cosh(2x)$ using the definition $\cosh(x) = (\exp(x) + \exp(-x))/2$.



3 A **tomato** is given by $z^2 + x^2 + 4y^2 = 1$. If we slice perpendicular to the y axes, we get a circular slice $z^2 + x^2 \leq 1 - 4y^2$ of radius $\sqrt{1-4y^2}$.
a) Find the area of this slice.
b) Determine the volume of the tomato.
c) Fix yourself a tomato salad by cutting a fresh tomato into slices and eat it, except for one slice which you staple to your homework paper as proof that you really did it.

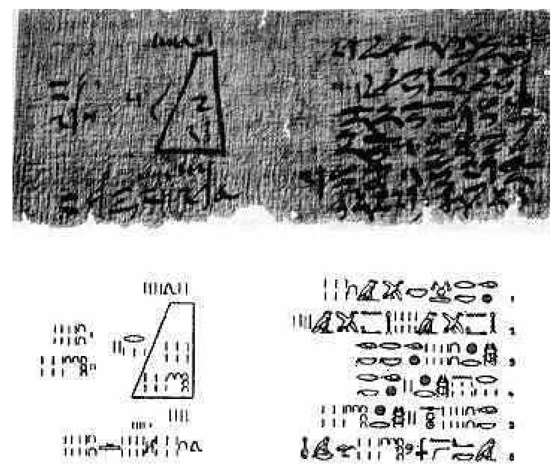
4 As we have seen in the movie of the first class, **Archimedes** was so proud of his formula for the volume of a sphere that he wanted the formula on his tomb stone. He wrote the volume of a half sphere of radius 1 as the difference between the volume of a cylinder of radius 1 and height 1 and the volume of a cone of base radius 1 and height 1. Relate the cross section area of the cylinder-cone complement with the cross section area of the sphere to recover his argument! If stuck, draw in the sand or soak in the bath tub for a while eating your tomato salad. There is no need to streak and scream "Eureka" when the solution is found.





5 Volumes were among the first quantities, Mathematicians wanted to measure and compute. One problem on **Moscow Eqyptian papyrus** dating back to 1850 BC explains the general formula $h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h.
a) Verify that if you slice the frustrum at height z, the area is $(a + (b - a)z/h)^2$.
b) Find the volume using calculus.
Here is the translated formulation from the papyrus: [1] [2]

"You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4 result 16. You are to double 4 result 8. You are to square 2, result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one-third of 6 result 2. You are to take 28 twice, result 56. See it is 56. You will find it right".



[1]Howard Eves, Great moments in mathematics, Volume 1, MAA, Dolciani Mathematical Expositions, 1980, page 10

[2]Image Source: http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian_papyri.html

Lecture 23: Improper integrals

In this lecture, we look at integrals on infinite intervals or integrals, where the function can get infinite at some point. These integrals are called **improper integrals**. The area under the curve can remain finite or become infinite.

1 What is the integral

$$\int_1^\infty \frac{1}{x^2}\, dx \ ?$$

Since the anti-derivative is $-1/x$, we have

$$\frac{-1}{x}|_1^\infty = -1/\infty + 1 = 1 \ .$$

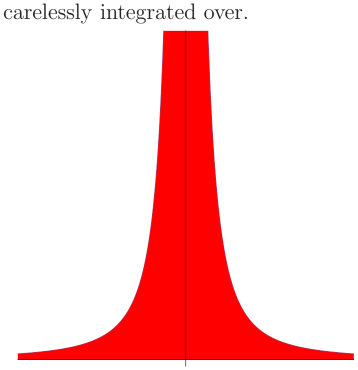
To justify this, compute the integral $\int_1^b 1/x^2 dx = 1 - 1/b$ and see that in the limit $b \to \infty$, the value 1 is achieved.

In a previous lecture, we have seen a chocking example similar to the following one:

2

$$\int_{-1}^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_{-1}^1 = -1 - 1 = -2 \ .$$

This does not make any sense because the function is positive so that the integral should be a positive area. The problem is this time not at the boundary $-1, 1$. The sore point is $x = 0$ over which we have carelessly integrated over.



The next example illustrates the problem with the previous example better:

3 The computation

$$\int_0^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_0^1 = -1 + \infty \ .$$

indicates that the integral does not exist. We can justify by looking at integrals

$$\int_a^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_a^1 = -1 + \frac{1}{a}$$

which are fine for every $a > 0$. But this does not converge for $a \to 0$.

Do we always have a problem if the function goes to infinity at some point?

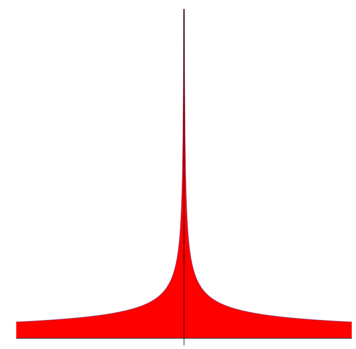
4 Find the following integral

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx \ .$$

Solution: Since the point $x = 0$ is problematic, we integrate from a to 1 with positive a and then take the limit $a \to 0$. Since $x^{-1/2}$ has the antiderivative $x^{1/2}/(1/2) = 2\sqrt{x}$, we have

$$\int_a^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}|_a^1 = 2\sqrt{1} - 2\sqrt{a} = 2(1 - \sqrt{a}) \ .$$

There is no problem with taking the limit $a \to 0$. The answer is 2. Even so the region is infinite its area is finite. This is an interesting example. Imaging this to be a container for paint. We can fill the container with a finite amount of paint but the wall of the region has infinite length.



5 Evaluate the integral $\int_0^1 1/\sqrt{1-x^2}\, dx$. **Solution:** The antiderivative is $\arcsin(x)$. In this case, it is not the point $x = 0$ which produces the difficulty. It is the point $x = 1$. Take $a > 0$ and evaluate

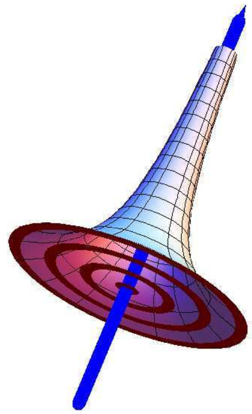
$$\int_0^{1-a} \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x)|_0^{1-a} = \arcsin(1-a) - \arcsin(0) \ .$$

Now $\arcsin(1-a)$ has no problem at limit $a \to 0$. Since $\arcsin(1) = \pi/2$ exists. We get therefore the answer $\arcsin(1) = \pi/2$.

6 Rotate the graph of $f(x) = 1/x$ around the x-axes and compute the volume of the solid between 1 and ∞. The cross section area is π/x^2. If we look at the integral from 1 to a fixed R, we get

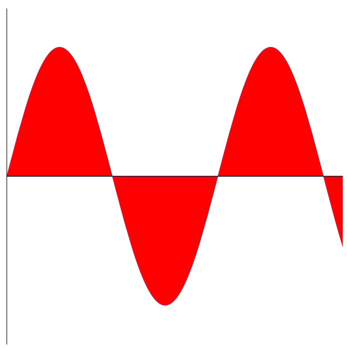
$$\int_1^R \frac{\pi}{x^2}\, dx = -\frac{\pi}{x}|_1^R = -\pi/R + \pi \ .$$

This converges for $R \to \infty$. The volume is π. This famous solid is called **Gabriels trumpet**. This solid is so prominent because if you look at the surface area of the small slice, then it is larger than $dx 2\pi/x$. The total surface area of the trumpet from 1 to R is therefore larger than $\int_1^R 2\pi/x\, dx = 2\pi(\log(R) - \log(1))$. which goes to infinity. We can **fill** the trumpet with a finite amount of paint but we can not **paint** its surface.



Finally, lets look at the following example

7 Evaluate the integral $\int_0^\infty \sin(x)\,dx$. **Solution.** There is no problem at the boundary 0 nor at any other point. We have to investigate however, what happens at ∞. Therefore, we look at the integral $\int_0^b \sin(x)\,dx = -\cos(x)|_0^b = 1 - \cos(b)$. We see that the limit $b \to \infty$ does not exist. The integral fluctuates between 0 and 2.



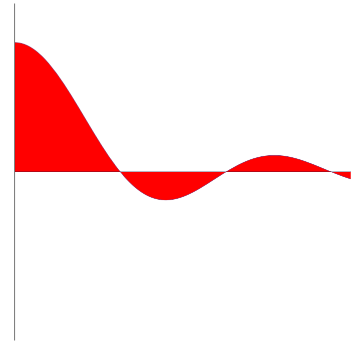
The next example leads to a topic in a follow-up course. It is not covered here, but could make you curious:

8 What about the integral

$$I = \int_0^\infty \frac{\sin(x)}{x}\,dx\ ?$$

Solution. The anti derivative is the Sine integral $Si(x)$ so that we can write $\int_0^b \sin(x)/x\,dx = Si(b)$. It turns out that the limit $b \to \infty$ exists and is equal to $\pi/2$ but this is a topic for a

second semester course like Math 1b. The integral can be written as an alternating series, which converges and there are many ways to compute it: [1]



Lets summarize the two cases of improper integrals: infinitely long intervals and a point where the function becomes infinite.

1) To investigate the improper integral $\int_a^\infty f(x)\,dx$ we look at the limit $\int_a^b f(x)\,dx$ for $b \to \infty$.
1) To investigate improper integral $\int_0^b f(x)\,dx$ where $f(x)$ is not continuous at 0, we take the limit $\int_a^b f(x)\,dx$ for $a \to 0$.

Homework

1 Evaluate the integral $\int_1^2 \frac{5}{\sqrt{x-1}} + \pi\cos(\pi x)\,dx$.

2 Evaluate the following integrals
a) $\int_0^1 2x/\sqrt{1-x^2}\,dx$.
b) $\int_0^1 2/\sqrt{1-x^2}\,dx$.
Hint: For a) think about the chain rule $d/dx f(g(x)) = f'(g(x))g'(x)$

3 Evaluate the integral $\int_{-3}^4 (x^2)^{1/3}\,dx$. To make sure that the integral is fine, check whether $\int_{-3}^0$ and $\int_0^4$ work.

4 The integral $\int_{-2}^1 1/x\,dx$ does not exist. We can however take a positive $b > 0$ and look at

$$\int_{-2}^{-b} 1/x\,dx + \int_b^1 1/x\,dx = \log|b| - \log|-2|) + (\log|1| - \log|b|) = \log(2)\ .$$

This value is called the **Cauchy principal value** of the integral. Find the principal value of

$$\int_{-4}^5 3/x^3\,dx$$

using the same process as before, by cutting out $[-a, a]$ and then taking the limit $a \to 0$.

5 a) Evaluate $\int_{-1}^1 \frac{1}{x^4}\,dx$ blindly, without worrying that the function is infinite.
b) Can we give a principal value integral value to $\int_{-1}^1 \frac{1}{x^4}\,dx$? If yes, find the value. If not, tell why not.

[1] Hardy, Mathematical Gazette, 5, 98-103, 1909.

Lecture 24: Applications of integration

Here is a list of applications for integration. Today, we will do some problems.

- the computation of area
- the computation of volume
- position from acceleration
- cost from marginal cost

Here are some more:

- probabilities and distributions
- averages and expectations
- finding moments of inertia
- work from power

Probability

In **probability theory** functions are used as observables or to define probabilities.

Assuming our probability space to be the real line, an interval $[a,b]$ is called an **event**. Given a nonnegative function $f(x)$ which has the property that $\int_{-\infty}^{\infty} f(x)\, dx = 1$, we call

$$\mathrm{P}[A] = \int_0^b f(x)\, dx$$

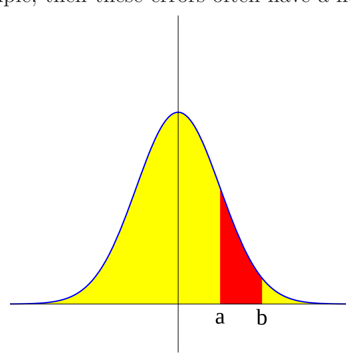
the **probability** of the event. The function $f(x)$ is called the **probability density function**.

The most important probability density is the normal distribution:

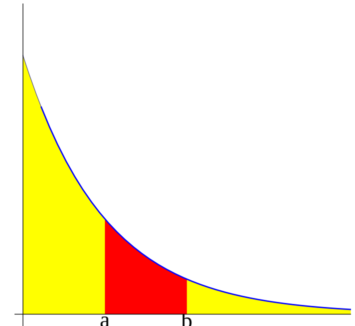
The **normal distribution** has the density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/x} .$$

It is the distribution which appears most often if data can take both positive and negative values. The reason why it appears so often is that if one observes different unrelated quantities with the same statistical properties, then their sum, suitably normalized becomes the normal distribution. If we measure **errors** for example, then these errors often have a normal distribution.



1 The probability density function of the **exponential distribution** is defined as $f(x) = e^{-x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. It is used to used measure lengths of arrival times like the time until you get the next phone call. The density is zero for negative x because there is no way we can travel back in time. What is the probability that you get a phone call between times $x = 1$ and times $x = 2$ from now? The answer is $\int_1^2 f(x)\, dx$.



Assume f is a probability density function (PDF). The antiderivative $F(x) = \int_{-\infty}^x f(t)\, dt$ is called the **cumulative distribution function** (CDF).

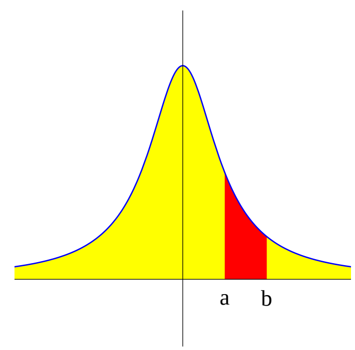
2 For the exponential function the cumulative distribution function is

$$\int_{-\infty}^x f(x)\, dx = \int_0^x f(x)\, dx = -e^{-x}|_0^x = 1 - e^{-x} .$$

The probability density function $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ is called the Cauchy distribution.

3 Find its cumulative distribution function. **Solution:**

$$F(x) = \int_{-\infty}^x f(t)\, dt = \frac{1}{\pi} \arctan(x)|_{-\infty}^x = (\frac{1}{\pi}\arctan(x) + \frac{1}{2}) .$$

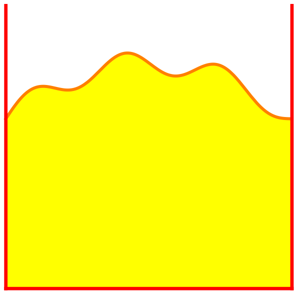


Average

Here is an example for computing the **average**.

4 Assume the level in a **honey jar** over $[0, 2\pi]$ containing crystallized honey is given by a function $f(x) = 3 + \sin(3x)/5 + x(2\pi - x)/10$. In order to restore the honey, it is placed into hot water. The honey melts to its normal state. What height does it have? **Solution:** The average height is $\int_0^{2\pi} f(x)\, dx/(2\pi)$ which is the area divided by the base length.

In probability theory we would call $f(x)$ a **random variable** and the average of f with $\mathrm{E}[f]$ the **expectation**.



Moment of inertia

If we spin a wire of radius L of mass density $f(x)$ around an axes, the **moment of inertia** is defined as $I = \int_0^L x^2 f(x)\, dx$.

The significance is that if we spin it with angular velocity w, then the energy is $Iw/2$.

5 Assume a wire has density $1 + x$ and length 3. Find its moment of inertia. **Solution:**

6 **Flywheels** have a comeback for **powerplants** to absorb energy. If there is not enough power, the flywheels are charged, in peak times, the energy is recovered. They work with 80 percent efficiency. Assume a flywheel is a cylinder of radius 1, density 1 and height 1, then the moment of inertia integral is $\int_0^1 z^2 f(z)\, dx$, where $f(z)$ is the mass in distance z.

Work from power

If $P(t)$ is the amount of power produced at time t, then $\int_0^T P(t)\, dt$ is the **work**=energy produced in the time interval $[0, T]$.

Energy is the anti-derivative of power.

7 Assume a power plant produces power $P(t) = 1000 + \exp(-t) + t^2 - t$. What is the energy produced from $t = 1$ to $t = 10$? **Solution.**

Wouldn't be nice to have one of those bikes with interactive training environments in the gym, allowing to ride in the Peruvian or Swiss Mountains, the California coast or in the Italian Tuscany?

Additionally, there should be some computer game features, racing other riders through beaches, deserts or Texan highways (could be on google earth). Training would be so much more entertaining. Business opportunities everywhere. The first offering such training equipment will make a fortune. Until then we are stuck with TV programs which really suck.

Homework

1 The probability distribution which describes the time you have to wait for your next email is $f(x) = 3e^{-3x}$ where x is time in hours. What is the probability that you get your next email in the next 4 hours?

2 Assume the probability distribution for the waiting time to the next warm day is $f(x) = (1/4)e^{-x/4}$, where x has days as unit. What is the probability to get a warm day between tomorrow and after tomorrow that is between $x = 1$ and $x = 2$?

3 A cone of base radius 1 and height 2 has temperature z^2. What is the average temperature? Remember that if $A(z)$ is the area of a slice at height z then $V = \int_0^2 A(z)\, dz$ is the volume. You have to compute $\int_0^2 z^2 A(z)\, dz/V$.

4 A CD Rom has radius 6. If we would place the material at radius x onto one point, we get a density of $f(x) = 2\pi x$. Find the moment of inertia I of the disc. If we spin it with an angular velocity of $w = 20$ rounds per second. Find the energy $E = Iw^2/2$.

Without credit: Explode a CD: http://www.powerlabs.org/cdexplode.htm. Careful!

5 a) You are on a stationary bike at the Hemenway gym and pedal with power

$$P(t) = 200 + 100\sin(10\pi t) - \frac{t}{300} + \frac{t^2}{19440}$$

(in Watts=W). The periodic fluctuations come from a hilly route. The linear term is the "tiring effect" and the quadratic term is due to **endorphins** kicking in. What energy (Joules J=W s) have you produced in the time $t \in [0, 1800]$ (s=seconds)?

b) Since we do math not physics, we usually ignore all the units but this one is just too much fun. If you divide the result by 4184, you get **kilo calories** = **food calories**. An apple has about 80 food calories. How many apples can you eat after your half hour workout to compensate the spent energy?

Math 1A: introduction to functions and calculus **Oliver Knill, 2012**

Lecture 25: Related rates

Before we continue with integration, we include a short flash-back on differentiation. which allows us to review the **chain rule**

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) .$$

This rule will be useful for us for the integration technique called "substitution". Since the chain rule is often perceived as a difficult concept in calculus, it is good to review it again and have fresh breath before launching into more advanced integration techniques. Related rates problem deal with a relation for variables. Differentiation gives a relation between the derivatives (rate of change). In all these problems, we have an **equation** and a **rate**. You can then solve for the rate which is asked for.

1 Hydrophilic **water gel spheres** have volume $\boxed{V(r(t)) = 4\pi r(t)^3/3}$ and expand at a rate $\boxed{V' = 30}$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.

2 A **wine glass** has a shape $y = x^2$ and volume $\boxed{V(y) = y^2\pi/2}$. Assume we slurp the wine with constant rate $\boxed{V' = -0.1}$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi y y'(t)$ so that $y'(t) = -1/(\pi y)$.

3 A **ladder** has length 1. Assume slips on the ground away with constant speed $\boxed{x' = 2}$. What is the speed of the top part of the ladder sliding down the wall at the time when $x = y$ if $\boxed{x^2(t) + y^2(t) = 1}$. Differentiation gives $2x(t)x'(t) + 2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

4 A **kid slides** down a slide of the shape $\boxed{y = 2/x}$. Assume $\boxed{y' = -7}$. What is $x'(t)$? Evaluate it at $x = 1$. **Solution:** differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.

Image source: http://www.dmfco.com

5 A **canister of oil** releases oil so that the area grows at a constant rate $\boxed{A' = 5}$. With what rate does the radius increase? **Solution.** We have $\boxed{A(r) = r^2\pi}$ and so $5 = A'(r) = 2rr'\pi$. Solving for r' gives $r' = 5/(2r\pi)$.

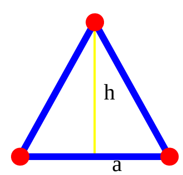
Related rates problems link quantities by a $\boxed{\text{rule}}$. These quantities can depend on time. To solve a related rates problem, differentiate the $\boxed{\text{rule}}$ with respect to time use the given $\boxed{\text{rate of change}}$ and solve for the unknown rate of change. Since related change problems are often difficult to parse. We have the $\boxed{\text{rule}}$ and given $\boxed{\text{rate of change}}$ boxed.

Homework

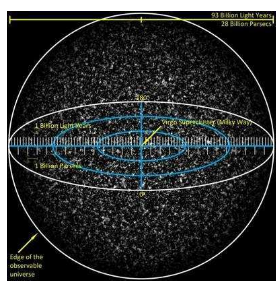
1 The **ideal gas law** $\boxed{pV = T}$ relates pressure p and volume V and temperature T. Assume the temperature $T = 50$ is fixed and $\boxed{V' = -3}$. Find the rate p' with which the pressure increases.

2 Assume the **total production rate** P of a new tablet computer product for kids is constant $P = 100$ and given by the **Cobb-Douglas formula** $\boxed{P = L^{1/3}K^{2/3}}$. Assume labor is increased at a rate $\boxed{L' = 2}$. What is the cost change K'? Evaluate this at $K = 125$ and $L = 64$.

3 You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\boxed{\phi' = 5}$ degree per second (which is $5\pi/180$ radiants per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\boxed{\tan(\phi) = x/h}$.



4 An **isosceles triangle** with base $2a$ and height h has fixed area $\boxed{A = ah = 1}$. Assume the height is decreased by a rate $\boxed{h' = -2}$. With what rate does a increase if $h = 1/2$?



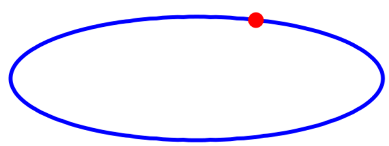
5 There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $\boxed{V = \pi^2 r^4/2}$ increases at a rate $\boxed{V' = 100\pi^2 r^2}$. What is r'? Evaluate it for $r = 47$ (billion light years).

Lecture 26: Implicit differentiation

Implicit differentiation was already crucial to find the derivative of inverse functions. We will review this here because this will give us handy tools for integration.

> The chain rule, related rates and implicit differentiation are all the same concept, but viewed from different angles. You can see implicit differentiation as a special case of related rates where one of the quantities is "time" meaning that this is the variable with respect to which we differentiate.

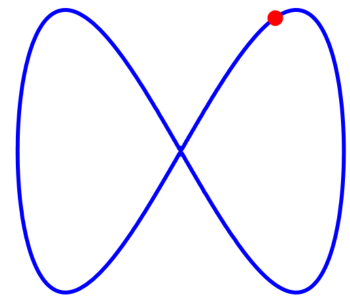
1 Points (x, y) in the plane which satisfy $x^2 + 9y^2 = 10$ form an ellipse. Find the slope y' of the tangent line at the point $(1, 1)$.
Solution: We want to know the derivative dy/dx. We have $2x + 18yy' = 0$. Using $x = 1, y = 1$ we see $y' = -2x/(18y) = -1/9$.
Remark. We could have looked at this as a related rates problem where $x(t), y(t)$ are related and $x' = 1$ Now $2xx' + 9 \cdot 2yy' = 0$ allows to solve for $y' = -2xx'/(9y) = -2/9$.



2 The points (x, y) which satisfy the equation $x^4 - 3x^2 + y^2 = 0$ forms a **figure 8** called **lemniscate of Gerono**. It contains the point $(1, \sqrt{2})$. Find the slope of the curve at that point. **Solution**: We differentiate the law describing the curve with respect to x. This gives

$$4x^3 - 6x + 2yy' = 0$$

We can now solve for $y' = (6x - 4x^3)/(2y) = 1/\sqrt{2}$.



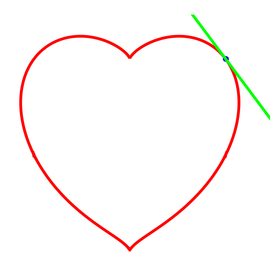
3 The **Valentine equation** $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$ contains the point $(1, 1)$. Near $(1, 1)$, we have $y = y(x)$ so that $(x^2 + y(x)^2 - 1)^3 - x^2y(x)^3 = 0$. Find y' at the point $x = 1$.
Solution: Take the derivative

$$0 = 3(x^2 + y^2 - 1)^2(2x + 2yy') - 2xy^3 - x^2 3y^2 y'(x)$$

and solve for

$$y' = -\frac{3(x^2 + y^2 - 1)2x - 2xy^3}{3(x^2 + y^2 - 1)2y - 3x^2y^2} .$$

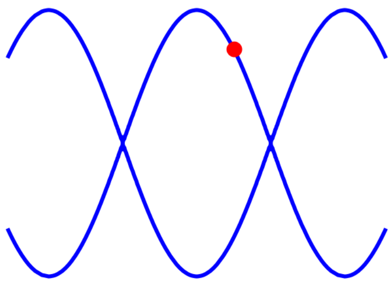
For $x = 1, y = 1$, we get $-4/3$.



4 The energy of a **pendulum** with angle x and angular velocity y is

$$y^2 - \cos(x) = 1$$

is constant. What is y'? We could solve for y and then differentiate. Simpler is to differentiate directly and get $yy' + \sin(x) = 0$ so that $y' = -\sin(x)/y$. At the point $(\pi/2, 1)$ for example we have $y' = -1$.



What is the difference between related rates and implicit differentiation?

Implicit differentiation is the **special case** of related rates where one of the variables is time.

Derivatives of inverse functions

Implicit differentiation has an important application: it allows to compute the derivatives of inverse functions. It is good that we review this, because we can use these derivatives to find anti-derivatives. We have seen this already. Lets do it again.

5 Find the derivative of $\log(x)$ by differentiating $\exp(\log(x)) = x$.
Solution:

$$\begin{aligned} 1 &= \frac{d}{dx} x = \frac{d}{dx} \exp(\log(x)) \\ &= \exp(\log(x)) \frac{d}{dx} \log(x) = x \log'(x) \ . \end{aligned}$$

Solve for $\log'(x) = 1/x$. Since the derivative of $\log(x)$ is $1/x$. The anti-derivative of $1/x$ is $\log(x) + C$.

6 Find the derivative of $\arccos(x)$ by differentiating $\cos(\arccos(x)) = x$.
Solution:

$$\begin{aligned} 1 &= \frac{d}{dx} x = \frac{d}{dx} \cos(\arccos(x)) \\ &= -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1 - \cos^2(\arccos(x))} \arccos'(x) \\ &= -\sqrt{1 - x^2} \arccos'(x) \ . \end{aligned}$$

Solving for $\arccos'(x) = -1/\sqrt{1 - x^2}$. The anti-derivative of $\arccos(x)$ is $-1/\sqrt{1 - x^2}$.

7 Find the derivative of $\arctan(x)$ by differentiating $\tan(\arctan(x)) = x$.
Solution: This is a derivative which we have seen several times by now. We use the identity $1/\cos^2(x) = \tan^2(x) + 1$ to get

$$\begin{aligned} 1 &= \frac{d}{dx} x = \frac{d}{dx} \tan(\arctan(x)) \\ &= \frac{1}{\cos^2(\arctan(x))} \arctan'(x) \\ &= (1 + \tan^2(\arctan(x))) \arctan'(x) \ . \end{aligned}$$

Solve for $\arctan'(x) = 1/(1 + x^2)$. The anti-derivative of $\arctan(x)$ is $1/(1 + x^2)$.

8 Find the derivative of $f(x) = \sqrt{x}$ by differentiating $(\sqrt{x})^2 = x$.
Solution:

$$\begin{aligned} 1 &= \frac{d}{dx} x = \frac{d}{x} (\sqrt{x})^2 \\ &= 2\sqrt{x} f'(x) \end{aligned}$$

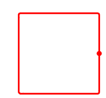
so that $f'(x) = 1/(2\sqrt{x})$.

Homework

1 The equation $y^2 = x^2 - x$ defines the graph of the function $f(x) = \sqrt{x^2 - x}$. Find the slope of the graph at $x = 2$ directly by differentiating f. Then use the implicit differentiation method and differentiate $y^2 = x^2 - x$ assuming $y(x)$ is a function of x and solving for y'.

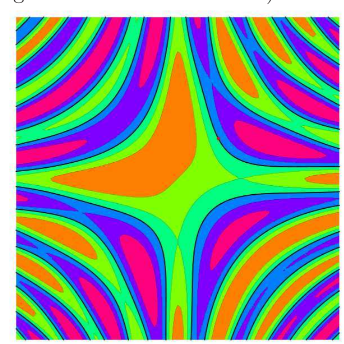
2 The equation $x^2 + y^2 = 5$ defines a circle. Find the slope of the tangent at $(1, 2)$.

3 The equation $x^{100} + y^{100} = 1 + 2^{100}$ defines a curve which looks close to a square. Find the slope of the curve at $(2, 1)$.



4 Derive again the derivative of $\text{arccot}(x)$ again, as we did before and during the first midterm.

5 a) The relation $\sin(x - y) - 2\cos((\pi/2)xy) = 0$ relates $x(t)$ and $y(t)$. Assume $x' = 1$ at $(1, 1)$ what is y'? This is a related rates problem.
b) Now do it directly. Since $x' = 1$ we can use x as the variable. Find $y'(x)$ by implicit differentiation. You should get the same result as in a).



Lecture 28: Review for second midterm

Major points

The **intermediate value theorem** assures that there is $x \in (a,b)$ with $f'(x) = (f(b) - f(a))/(b-a)$. A special case is Rolle's theorem, where $f(b) = f(a)$.

Catastrophes are parameter values where a local minimum disappears. Typically the system jumps then to a lower minimum.

Definite integrals $F(x) = \int_0^x f(x)\ dx$ are defined as a limit of Riemann sums S_n/n.

A function $F(x)$ satisfying $F' = f$ is called the anti-derivative of f. The general anti-derivative is $F + C$ where C is a constant.

The **fundamental theorem of calculus** tells $d/dx \int_0^x f(x)\ dx = f(x)$ and $\int_0^x f'(x)\ dx = f(x) - f(0)$.

The integral $\int_a^b g(x) - f(x)\ dx$ is the **signed area between the graphs** of f and g. Places, where $f < g$ are counted negative.

The integral $\int_a^b A(x)\ dx$ is a **volume** if $A(x)$ is the area of a slice of the solid perpendicular to a point x on an axes.

Write **improper integrals** as limits of definite integrals $\int_1^\infty f(x)\ dx = \lim_{R\to\infty} \int_1^R f(x)\ dx$. We similarly treat points, where f is discontinuous.

Besides **area**, **volume**, **total cost**, or **position**, we can compute **averages**, **inertia** or **work** using integrals.

If x, y are related by $F(x(t), y(t)) = 0$ and $x(t)$ is known we can compute $y'(t)$ using the chain rule. This is **related rates**.

If $f(g(t))$ is known we can compute $g'(x)$ using the chain rule. This works for inverse functions. This is **implicit differentiation**.

To determine the **catastrophes** for a family $f_c(x)$ of functions, determine the critical points in dependence of c and find values c, where a critical point changes from a local minimum to a local maximum.

Important integrals

$\cos(x)$	$\sin(x)$.	$\exp(x)$	$\exp(x)$
$\sin(x)$	$-\cos(x)$.	$\log(x)$	$x\log(x) - x$
$\tan(x)$	$1/\cos^2(x)$.	$1/x$	$\log(x)$
$\arctan(x)$	$1/(1+x^2)$.	$-1/(1+x^2)$	$\text{arccot}(x)$
$1/\sqrt{1-x^2}$	$\arcsin(x)$	$-1/\sqrt{1-x^2}$	$\arccos(x)$

Improper integrals

$\int_1^\infty 1/x^2\ dx$ Prototype of first type improper integral which exists.
$\int_1^\infty 1/x\ dx$ Prototype of first type improper integral which does not exist.
$\int_0^1 1/x\ dx$ Prototype of second type improper integral which does not exist.
$\int_0^1 1/\sqrt{x}\ dx$ Prototype of second type improper integral which does exist.

The fundamental theorem

$\frac{d}{dx} \int_0^x f(x)\ dx = f(x)$

$\int_0^x f'(x)\ dx = f(x) - f(0)$.

This implies

$\int_a^b f'(x)\ dx = f(b) - f(a)$

Without limits of integration, we call $\int f(x)\ dx$ the **anti derivative**. It is defined up to a constant. For example $\int \sin(x)\ dx = -\cos(x) + C$.

Applications

Calculus applies directly if there are situations where one quantity is the derivative of the other.

function	anti derivative
acceleration	velocity
velocity	position
function	area under the graph
length of cross section	area of region
area of cross section	volume of solid
marginal prize	total prize
power	work
probability density function	cumulative distribution function

Tricks

Whenever dealing with an area or volume computation, make a picture.
In related rates problems, make sure you understand what are variables and what are constants.
For volume computations, find the area of the cross section $A(x)$ and integrate.
For area computations find the length of the slice $f(x)$ and integrate.

Most important integrals

The most important integral is the integral

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1}}$$

holds for all n different from 1.

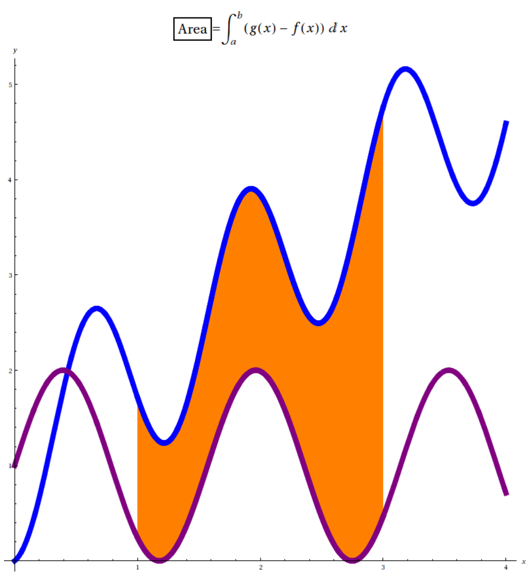
$$\boxed{\int \frac{1}{x} \, dx = \log(x)}$$

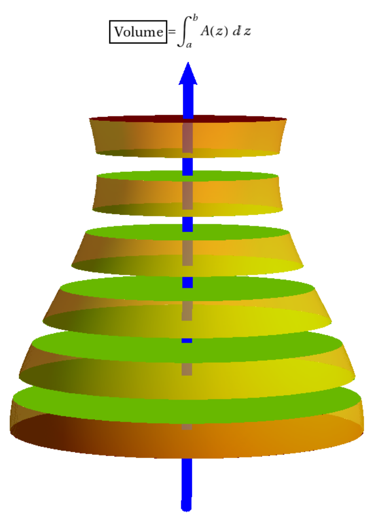
Example: $\int \sqrt{x+7} \, dx = \frac{2}{3}(x+7)^{3/2}$.
Example: $\int \frac{1}{x+5} \, dx = \log(x+5)$
Example: $\int \frac{1}{4x+3} \, dx = \log(4x+3)/4$

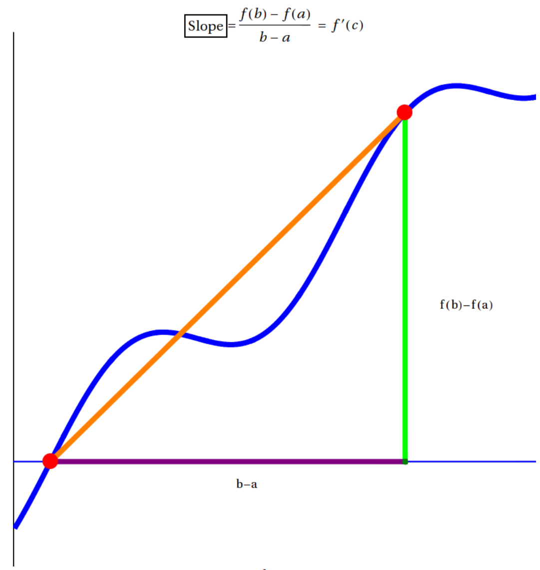
Related rates - Implicit differentiation

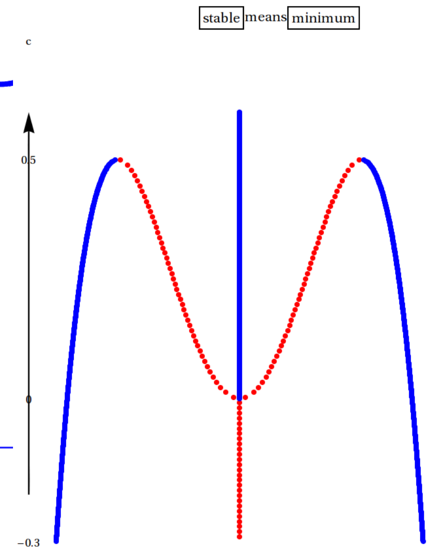
Assume $\cos(xy) + y^4 = 2y$, where x, y both change and $x' = 7$. Find y' at $x = 0, y = 1$. Given$\cos(xy) + y^4 = 2y$. Find $y'(x)$ at $x = 0$.

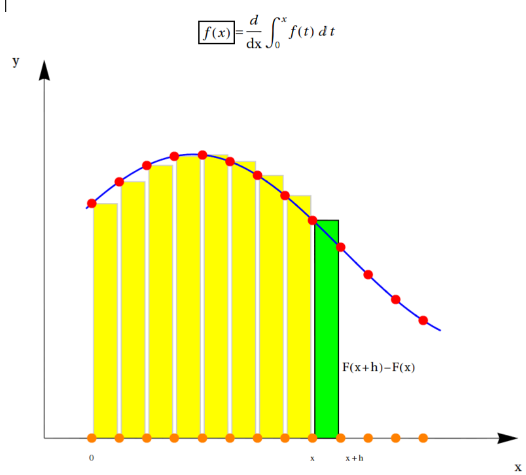
Key pictures

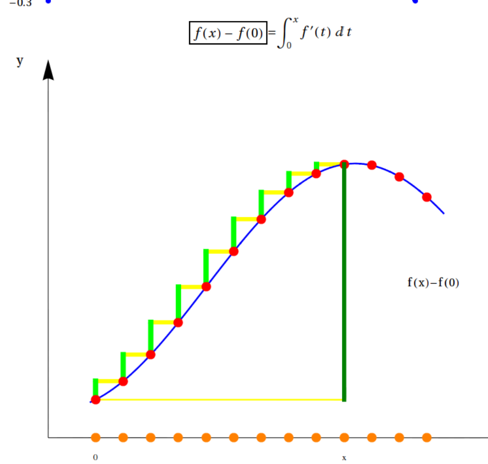












Lecture 27: Substitution

While this lecture is not part of the midterm, it can be useful. You should also without the method described here be able to integrate functions like e^{6x} or $1/(1+x)$. Substitution makes this much easier. If we differentiate the function $\sin(x^2)$ and use the chain rule, we get $\cos(x^2)2x$. By the fundamental theorem of calculus, the anti derivative of $\cos(x^2)2x$ is $\sin(x^2)$. We know therefore

$$\int \cos(x^2)2x \, dx = \sin(x^2) + C \, .$$

Spotting the chain rule

How can we see the integral without knowing the result already? Here is a very important case:

> If we can spot that $f(x) = g(u(x))u'(x)$, then the anti derivative of f is $G(u(x))$ where G is the anti derivative of g.

1 Find the anti derivative of

$$e^{x^4+x^2}(4x^3+2x) \, .$$

Solution: The derivative of the inner function is to the right.

2 Find

$$\int \sqrt{x^5+1}x^4 \, dx \, .$$

Solution. The derivative of x^5+1 is $5x^4$. This is almost what we have there but the constant can be adapted. The answer is $(1/5)(x^5+1)^{3/2}$.

3 Find the anti derivative of

$$\frac{\log(x)}{x} \, .$$

Solution: The derivative of $\log(x)$ is $1/x$. The antiderivative is $\log(x)^2/2$.

4 Find the anti derivative of

$$\cos(\sin(x^2))\cos(x)2x \, .$$

Solution. We see the derivative of $\sin(x^2)$ appear on the right. Therefore, we have $\sin(\sin(x^2))$.

In the next three examples, substitution is actually not necessary. You can just write down the anti derivative, and adjust the constant. It uses the following "speedy rule":

> If $\int f(ax+b) \, dx = F(ax+b)/a$ where F is the anti derivative of f.

5 $\int \sqrt{x+1} \, dx$. **Solution:** $(x+1)^{3/2}(2/3)$.

6 $\int \frac{1}{1+(5x+2)^2} \, dx$. **Solution:** $\arctan(5x+2)(1/5)$.

Doing substitution

Spotting things is sometimes not easy. The method of substitution helps to formalize this. To do so, identify a part of the formula to integrate and call it u then replace an occurrence of $u'dx$ with du.

$$\int f(\text{u(x)})\,\text{u'(x)}\ \ \text{dx} = \int f(\text{u})\ \text{du} \, .$$

Here is a more detailed description: replace a prominent part of the function with a new variable u, then use $du = u'(x)dx$ to replace dx with du/u'. We aim to end up with an integral $\int g(u) \, du$ which does not involve x anymore. Finally, after integration of this integral, replace the variable u again with the function $u(x)$. The last step is called **back-substitution**.

7 Find the anti-derivative

$$\int \log(\log(x))/x \, dx \, .$$

Solution Replace $\log(x)$ with u and replace $u'dx = 1/xdx$ with du. This gives $\int \log(u) \, du = u\log(u) - u = \log(x)\log\log(x)) - \log(x)$.

8 Solve the integral

$$\int x/(1+x^4) \, dx \, .$$

Solution Substitute $u = x^2, du = 2xdx$ to get $(1/2)\int du/(1+u^2) \, du = (1/2)\arctan(u) = (1/2)\arctan(x^2)$.

9 Solve the integral

$$\int \sin(\sqrt{x})/\sqrt{x} \, .$$

Here are some examples which are not so straightforward:

10 Solve the integral

$$\int \sin^3(x) \, dx \, .$$

Solution. We replace $\sin^2(x)$ with $1-\cos^2(x)$ to get

$$\int \sin^3(x)\ dx = \int \sin(x)(1-\cos^2(x))\ dx = -\cos(x) + \cos^3(x)/3\ .$$

11 Solve the integral

$$\int \frac{x^2+1}{\sqrt{x+1}}\ dx\ .$$

Solution: Substitute $u = \sqrt{x+1}$. This gives $x = u^2-1, dx = 2udu$ and we get $\int 2(u^2-1)^2+1\ du$.

12 Solve the integral

$$\int \frac{x^3}{\sqrt{x^2+1}}\ dx\ .$$

Trying $u = \sqrt{x^2+1}$ but this does not work. Try $u = x^2+1$, then $du = 2xdx$ and $dx = du/(2\sqrt{u-1})$. Substitute this in to get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}}\ du = \int \frac{(u-1)}{2\sqrt{u}} = \int u^{1/2}/2 - u^{-1/2}/2\ du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2}\ .$$

Definite integrals

When doing definite integrals, we could find the antiderivative as described and then fill in the boundary points. Substituting the boundaries directly accelerates the process since we do not have to substitute back to the original variables:

$$\int_a^b g(u(x))u'(x)\ dx = \int_{u(a)}^{u(b)} g(u)\ du\ .$$

Proof. This identity follows from the fact that the right hand side is $G(u(b)) - G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has the anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b)) - G(u(a))$.

Top: To keep track which bounds we consider it can help to write $\int_{x=a}^{x=b} f(x)\ dx$.

13 Find the anti derivative of $\int_0^2 \sin(x^3-1)x^2\ dx$. **Solution**.

$$\int_{x=0}^{x=2} \sin(x^3+1)x^2\ dx\ .$$

Solution: Use $u = x^3+1$ and get $du = 3x^2dx$. We get

$$\int_{u=1}^{u=7} \sin(u)du/3 = (1/3)\cos(u)|_1^7 = [-\cos(7)+\cos(1)]/3\ .$$

Also here, we can see the integrals directly

To integrate $f(Ax+B)$ from a to b we get $[F(Ab+B) - F(Aa+B)]/A$, where F is the anti-derivative of f.

14 $\int_0^1 \frac{1}{5x+1}\ dx = [\log(u)]/5|_1^6 = \log(6)/5$.

15 $\int_3^5 \exp(4x-10)\ dx = [\exp(10) - \exp(2)]/4$.

Homework

1 Find the following anti derivatives.
a) $\int 10x\sin(x^2)\ dx$
b) $\int e^{x^5+x}(2x^4+\frac{2}{5})\ dx$
c) $\cos(\cos^2(x))\sin(x)\cos(x)$
d) $e^{5\tan(x)}/\cos^2(x)$.

2 Compute the following definite integrals.
a) $\int_2^5 \sqrt{x^5+x}(x^4+1/5)\ dx$
b) $\int_0^{\sqrt{\pi}} \sin(x^2)x\ dx$.
c) $\int_{1/e}^{e} \frac{\sqrt{\log(x)}}{x}\ dx$.
d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}}\ dx$.

3 a) Find the integral $\int_0^1 x^3\sqrt{1-x^4}\ dx$ using a substitution method.
b) Find the moment of inertia of a rod with density $f(x) = \sqrt{x^3+1}$ between $x=0$ and $x=4$. Remember that the moment of inertia is $\int_0^4 x^2 f(x)\ dx$.

4 a) Integrate

$$\int_0^1 \frac{\arcsin(x)}{\sqrt{1-x^2}}\ dx\ .$$

b) Find the definite integral

$$\int_e^{6e} \frac{dx}{\sqrt{\log(x)}x}\ .$$

5 a) Find the indefinite integral

$$\int \frac{x^5}{\sqrt{x^2+1}}\ dx\ .$$

b) Find the anti-derivative of

$$f(x) = \frac{1}{x(1+\log(x)^2)}\ .$$

Lecture 29: Integration by parts

If we integrate the product rule $(uv)' = u'v + uv'$ we obtain an integration rule called **integration by parts**. It is a powerful tool, which complements substitution. As a rule of thumb, always try first to simplify a function and integrate directly, then give substitution a first shot before trying integration by parts.

$$\int u(x)\, v'(x) dx = u(x)v(x) - \int u'(x)v(x)\, dx .$$

1 Find $\int x \sin(x)\, dx$. **Solution.** Lets identify the part which we want to differentiate and call it u and the part to integrate and call it v'. The integration by parts method now proceeds by writing down uv and subtracting a new integral which integrates $u'v$:

$$\int x \sin(x)\, dx = x\,(-\cos(x)) - \int 1\,(-\cos(x))\, dx = -x\cos(x) + \sin(x) + C\, dx .$$

2 Find $\int xe^x\, dx$. **Solution.**

$$\int x \exp(x)\, dx = x\exp(x) - \int 1 \exp(x)\, dx = x\exp(x) - \exp(x) + C\, dx .$$

3 Find $\int \log(x)\, dx$. **Solution.** There is only one function here, but we can look at it as $\log(x) \cdot 1$

$$\int \log(x)\, 1\, dx = \log(x)x - \int \frac{1}{x} x\, dx = x\log(x) - x + C .$$

4 Find $\int x \log(x)\, dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed like this. We would get through but what if we do not know? Lets differentiate $\log(x)$ and integrate x:

$$\int \log(x)\, x\, dx = \log(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\, dx$$

which is $\log(x)x^2/2 - x^2/4$.

We see that it is better to differentiate log first.

5 **Marry go round:** Find $I = \int \sin(x)\exp(x)\, dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\, dx .$$

Lets do it again:

$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\, dx.$$

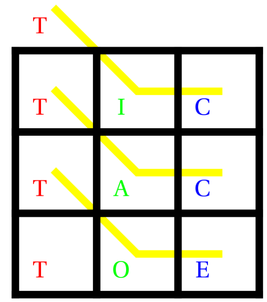
We moved in circles and are stuck! Are we really. We have derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

which we can solve for I and get

$$I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2 .$$

Tic-Tac-Toe



Integration by parts can bog you down if you do it several times. Keeping the order of the signs can be daunting. This is why a **tabular integration by parts method** is so powerful. It has been called "Tic-Tac-Toe" in the movie Stand and deliver. Lets call it Tic-Tac-Toe therefore.

6 Find the anti-derivative of $x^2 \sin(x)$. **Solution:**

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The antiderivative is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C .$$

7 Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

$(x-1)^3$	$\exp(2x)$	
$3(x-1)^2$	$\exp(2x)/2$	$\oplus$
$6(x-1)$	$\exp(2x)/4$	$\ominus$
6	$\exp(2x)/8$	$\oplus$
0	$\exp(2x)/16$	$\ominus$

The anti-derivative is

$$(x-1)^3 e^{2x}/2 - 3(x-1)^2 e^{2x}/4 + 6(x-1)e^{2x}/8 - 6e^{2x}/16 + C .$$

8 Find the anti-derivative of $x^2\cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	$\oplus$
2	$-\cos(x)$	$\ominus$
0	$-\sin(x)$	$\oplus$

The anti-derivative is

$$x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C .$$

Ok, we are now ready for more extreme stuff.

9 Find the anti-derivative of $x^7 \cos(x)$. **Solution:**

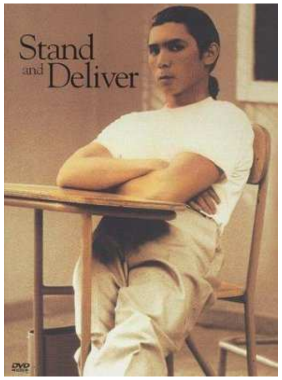
x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	$\oplus$
$42x^5$	$-\cos(x)$	$\ominus$
$120x^4$	$-\sin(x)$	$\oplus$
$840x^3$	$\cos(x)$	$\ominus$
$2520x^2$	$\sin(x)$	$\oplus$
$5040x$	$-\cos(x)$	$\ominus$
5040	$-\sin(x)$	$\oplus$
0	$\cos(x)$	$\ominus$

The anti-derivative is

$$
\begin{aligned}
F(x) &= x^7 \sin(x) \\
&+ 7x^6 \cos(x) \\
&- 42x^5 \sin(x) \\
&- 210x^4 \cos(x) \\
&+ 840x^3 \sin(x) \\
&+ 2520x^2 \cos(x) \\
&- 5040x \sin(x) \\
&- 5040 \cos(x) + C \,.
\end{aligned}
$$

Do this without this method and you see the value of the method.

[1] [2] [3]:



I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to an article of V.N. Murty. The method realizes in a clever way an iterated integration by parts method:

$$
\begin{aligned}
\int fg\,dx &= fg^{(-1)} - f^{(1)}g^{-2} + f^{(2)}g^{(-3)} - \dots \\
&- (-1)^n \int f^{(n+1)} g^{(-n-1)}\, dx
\end{aligned}
$$

which can easily shown to be true by induction and justifies the method: the f function is differentiated again and again and the g function is integrated again and again. You see, where the alternating minus signs come from. You see that we always pair a k'th derivative with a $k+1$'th integral and take the sign $(-1)^k$.

Coffee or Tea?

[1] V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, pages 90-94.
[2] David Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, pages 307-311.
[3] K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, pages 542-543

When doing integration by parts, We want to try first to differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

For coffee lovers, there is an equivalent one: **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials which can be remembered as **LIATE** which is close to "latte" (the coffee).

Whether you prefer to remember it as a "coffee latte" or a "lipton tea" is up to you.

There is even a better method, the "opportunistic method":

> Just integrate what you can integrate and differentiate the rest.

An don't forget to consider integrating 1, if nothing else works.

Homework

1 Integrate $\int x^2 \log(x)\, dx$.
2 Integrate $\int x^5 \sin(x)\, dx$
3 Find the anti derivative of $\int x^6 \exp(x)\, dx$. (*)
4 Find the anti derivative of $\int \sqrt{x} \log(x)\, dx$.
5 Find the anti derivative of $\int \sin(x) \exp(-x)\, dx$.

(*) If you want to go for the record. Lets see who can integrate the largest $x^n \exp(x)$! It has to be done by hand, not with a computer algebra system although.

Lecture 30: Numerical integration

After repeating and before looking at other integration techniques, we briefly look at some numerical techniques. There are variations of Riemann sums which speed up the computation.

Riemann sum with nonuniform spacing

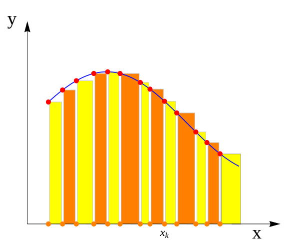
A more general Riemann sum is obtained by choosing n points in $[a, b]$ and defining

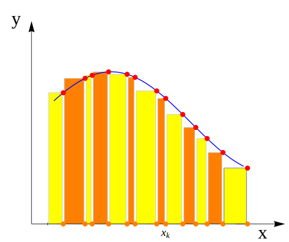
$$S_n = \sum f(y_j)(x_{j+1} - x_j) = \sum_{y_j} f(y_j)\Delta x_j$$

where y_j is in (x_j, x_{j+1}).

This is how Riemann sums are usually introduced in calculus books. The generalization allows to use a small mesh size where the function fluctuates a lot. For theoretical purposes it is mostly equivalent.

The sum $\sum f(x_j)\Delta x_j$ is called the **left Riemann sum**, the sum $\sum f(x_{j+1})\Delta x_j$ the **right Riemann sum**.





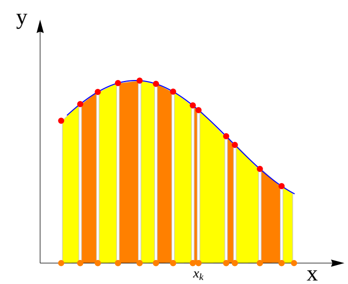
If $x_0 = a, x_n = b$ and $\max_j \Delta x_j \to 0$ for $n \to \infty$ then S_n converges to $\int_a^b f(x)\, dx$.

1 If $x_j - x_k = 1/n$ and $z_j = x_j$, then we have the Riemann sum as we defined it earlier.

2 You numerically integrate $\sin(x)$ on $[0, \pi/2]$ with a Riemann sum. What is better, the left Riemann sum or the right Riemann sum? Look also at the interval $[\pi/2, \pi]$? **Solution:** you see that in the first case, the left Riemann sum is smaller than the actual integral. In the second case, the left Riemann sum is larger than the actual integral.

Trapezoid rule

The average between the left and right hand Riemann sum is called the **Trapezoid rule**. Geometrically, it sums up areas of trapezoids instead of rectangles.



The **Trapezoid rule** computes

$$\frac{1}{2n}\sum_{k=1}^{n}[f(x_k) + f(x_{k+1})] \; .$$

Remark. The Trapezoid rule does not change things much in general. In the case of equal spacing $x_k = a + (b - a)k/n$.

$$\frac{1}{2n}[f(x_0) + f(x_n)] + \frac{1}{n}\sum_{k=1}^{n-1} f(x_k) \; .$$

Simpson rule

Much better is the Simpson method:

The **Simpson rule** computes the sum

$$S_n = \frac{1}{6n}\sum_{k=1}^{n}[f(x_k) + 4f(y_k) + f(x_{k+1})] \; ,$$

where y_k are the midpoints between x_k and x_{k+1}.

The Simpson rule is good because it is exact for quadratic functions: you can check for $f(x) = ax^2 + bx + c$ that the formula

$$\frac{1}{v-u}\int_u^v f(x)\, dx = [f(u) + 4f((u+v)/2) + f(v)]/6$$

holds. To prove it just run the following two lines in Mathematica: (== means "is equal")

```
f[x_] := a x^2 + b x + c;
Simplify[(f[u]+f[v]+4f[(u+v)/2])/6==Integrate[f[x],{x,u,v}]/(v-u)]
```

This actually will imply (as you might see in a course like Math 1b) that the numerical integration for functions which are 4 times differentiable gives numerical results which are n^{-4} close to the actual integral. For 100 division points, this can give accuracy to 10^{-8} already. We see this in a demonstration.

There are other variants which are a bit better but need more function values. If x_k, y_k, z_k, x_{k+1} are equally spaced, then

The **Simpson 3/8 rule** computes

$$\frac{1}{8n}\sum_{k=1}^{n}[f(x_k) + 3f(y_k) + 3f(z_k) + f(x_{k+1})] \; .$$

This formula is again exact for quadratic functions: for $f(x) = ax^2 + bx + c$, the formula

$$\frac{1}{v-u}\int_u^v f(x)\,dx = [f(u) + 3f((2u+v)/3) + 3f((u+2v)/3) + f(v)]/6$$

holds. If you are interested, run the two Mathematica lines:

```
f[x_] := a x^2 + b x + c; L=Integrate[f[x],{x,u,v}]/(v-u);
Simplify[(f[u]+f[v]+3f[(2u+v)/3]+3f[(u+2v)/3])/8==L]
```

This 3/8 method can be slightly better than the first Simpson rule.

Monte Carlo Method

A powerful integration method is to chose n random points x_k in $[a,b]$ and look at the sum divided by n. Because it uses randomness, it is called **Monte Carlo method**.

The **Monte Carlo** integral is the limit S_n to infinity

$$S_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k)\ ,$$

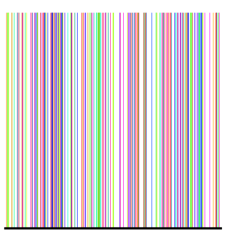
where x_k are n random values in $[a,b]$.

The law of large numbers in probability shows that the **Monte Carlo integral** is equivalent to the **Lebesgue integral** which is more powerful than the Riemann integral. Monte Carlo integration is interesting especially if the function is complicated.

3 The **salt and pepper** function is defined as

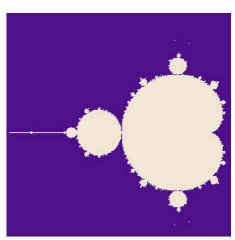
$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

The Riemann integral with equal spacing k/n is equal to 1 for every n. But this is only because we have evaluated the function at rational points, where it is 1.
The Monte Carlo integral gives zero because if we chose a random number in $[0,1]$ we hit an irrational number with probability 1.



The Salt and Pepper function and the Boston Salt and Pepper bridge (Anne Heywood).

4 The following two lines evaluate the **area of the Mandelbrot fractal** using Monte Carlo integration. The function F is equal to 1, if the parameter value c of the quadratic map $z \to z^2 + c$ is in the Mandelbrot set and 0 else. It shoots $100'000$ random points and counts what fraction of the square of area 9 is covered by the set. Numerical experiments give values close to the actual value around 1.51.... One could use more points to get more accurate estimates.




```
F[c_]:=Block[{z=c,u=1},Do[z=N[z^2+c];If[Abs[z]>3,u=0;z=3],{99}];u];
M=10^5; Sum[F[-2.5+3 Random[]+I(-1.5+3 Random[])],{M}]*(9.0/M)
```

Homework

1 Generate $n = 10$ random numbers x_k in $[0,1]$, then sum up the square x_k^2 of these numbers and divide by $n = 10$. Compare your result with $\int_0^1 x^2\,dx$. **Remark.** If using a program, increase the value of n as large as you can. Here is a Mathematica code:

```
n=20; Sum[ Random[]^2,{n}]/n
```

Here is an implementation in Perl. Its still possible to cram the code into one line:

```
#!/usr/bin/perl
$n=20;$s=0;for($i=0;$i<$n;$i++){$f=rand();$s+=$f*$f;} print $s/$n;
```

2 a) Use Simpson's rule to compute $\int_0^\pi \sin(x)\,dx$ using $n = 2$ intervals $[0,\pi/2]$ and $[\pi/2,\pi]$. On each of these intervals $[a,b]$ compute the Simpson sum $[f(a) + 4f((a+b)/2) + f(b)]/6$ with $f(x) = \sin(x)$. Compare with the actual integral.
b) Now use the 3/8 Simpson rule to estimate $\int_0^\pi f(x)\,dx$ using $n = 1$ intervals $[0,\pi]$. Again compare with the actual integral.

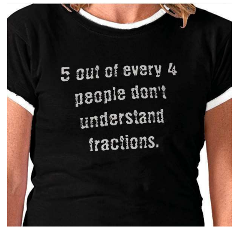
Instead of adding more numerical methods exercices, we want to practice a bit more integration. The challenge in the following problems is to find out which integration method is bets suited. This is good preparation for the final, where we will not reveal which integration method is the best.

3 Integrate $\tan(x)/\cos(x)$ from 0 to $\pi/6$.

4 Find the antiderivative of $x\sin(x)\exp(x)$.

5 Find the antiderivative of $x/\sin(x)^2$.

Lecture 31: Partial fractions



The partial fraction method will be covered in detail follow up calculus courses like Math 1b. Here we just look at some samples to see whats out there. We have learned how to integrate polynomials like $x^4 + 5x + 3$. What about rational functions? We will see here that they are a piece of cake - if you have the right guide of course ...

What we know already

Lets see what we know already:

- We also know that integrating $1/x$ gives $\log(x)$. We can for example integrate
$$\int \frac{1}{x-6}\,dx = \log(x-6) + C \; .$$
- We also have learned how to integrate $1/(1+x^2)$. It was an important integral:
$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C \; .$$
Using substitution, we can do more like
$$\int \frac{dx}{1+4x^2} = \int \frac{du/2}{1+u^2} = \arctan(u)/2 = \arctan(2x)/2 \; .$$
- We also know how to integrate functions of the type $x/(x^2+c)$ using substitution. We can write $u = x^2 + c$ and get $du = 2xdx$ so that
$$\int \frac{x}{x^2+c}\,dx = \int \frac{1}{2u}\,du = \frac{\log(x^2+c)}{2} \; .$$
- Also functions $1/(x+c)^2$ can be integrated using substitution. With $x + c = u$ we get $du = dx$ and
$$\int \frac{1}{(x+c)^2}\,dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{x+c} + C \; .$$

The partial fraction method

We would love to be able to integrate any rational function
$$f(x) = \frac{p(x)}{q(x)} \; ,$$
where p, q are polynomials. This is where **partial fractions come in**. The idea is to write a rational function as a sum of fractions we know how to integrate. The above examples have shown that we can integrate $a/(x+c), (ax+b)/(x^2+c), a/(x+c)^2$ and cases, which after substitution are of this type.

The partial fraction method writes $p(x)/q(x)$ as a sum of functions of the above type which we can integrate.

This is an algebra problem. Here is an important special case:

In order to integrate $\int \frac{1}{(x-a)(x-b)}\,dx$, write
$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \; .$$
and solve for A, B.

In order to solve for A, B, write the right hand side as one fraction again
$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)} \; .$$
We only need to look at the nominator:
$$1 = Ax - Ab + Bx - Ba \; .$$
In order that this is true we must have $A + B = 0, Ab - Ba = 1$. This allows us to solve for A, B.

Examples

1 To integrate $\int \frac{2}{1-x^2}\,dx$ we can write
$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$
and integrate each term
$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x) \; .$$

2 Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write
$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} \; .$$
Now multiply out and solve for A, B:
$$A(x-2) + B(x-3) = 5 - 2x \; .$$
This gives the equations $A + B = -2, -2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. We have not obtained
$$\frac{5-2x}{x^2-5x+6} = -\frac{1}{x-3} - \frac{1}{x-2}$$
and can integrate:
$$\int \frac{5-2x}{x^2-5x+6}\,dx = -\log(x-3) - \log(x-2) \; .$$
Actually, we could have got this one also with substitution. How?

3 Integrate $f(x) = \int \frac{1}{1-4x^2}\,dx$. **Solution.** The denominator is factored as $(1-2x)(1+2x)$. Write
$$\frac{A}{1-2x} + \frac{B}{1+2x} = \frac{1}{1-4x^2} \; .$$
We get $A = 1/4$ and $B = -1/4$ and get the integral
$$\int f(x)\,dx = \frac{1}{4}\log(1-2x) - \frac{1}{4}\log(1+2x) + C \; .$$

Hopital's method

There is a fast method to get the coefficients:

> If a is different from b, then the coefficients A, B in
> $$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} ,$$
> are
> $$A = \lim_{x \to a}(x-a)f(x) = p(a)/(a-b),\ B = \lim_{x \to b}(x-b)f(x) = p(b)/(b-a) .$$

Proof. If we multiply the identity with $x-a$ we get

$$\frac{p(x)}{(x-b)} = A + \frac{B(x-a)}{x-b} .$$

Now we can take the limit $x \to a$ without peril and end up with $A = p(a)/(x-b)$.

Cool, isn't it? This **Hopital method** can save you a lot of time! Especially when you deal with more factors and where sometimes complicated systems of linear equations would have to be solved. Remember

> Math is all about elegance and does not use complicated methods if simple ones are available.

Here is an example:

4 Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = 11/12$, and $B = \frac{2*(-8)+3}{(-8-4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8} .$$

The integral is

$$\frac{11}{12}\log(x-4) + \frac{13}{12}\log(x+8) .$$

Here is an example with three factors:

5 Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2}\log(x-1) - 7\log(x-2) + \frac{13}{2}\log(x-3) .$$

Homework

1 $\int \frac{2dx}{x^2-4}$.

2 $\int \frac{5dx}{4x^2+1}$.

3 $\int \frac{x^3-x+1}{x^2-1}\,dx$.

4 $\int \frac{3x^2}{(x^2+x+1)(x-1)}\,dx$

5 $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)}\,dx$. Use Hopitals method of course!

Hint for 3). Subtract first a polynomial.

Hint for 4). Find the nominator of $\frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$ and set it $3x^2$. To do so, multply out.

Lecture 32: Trig substitutions

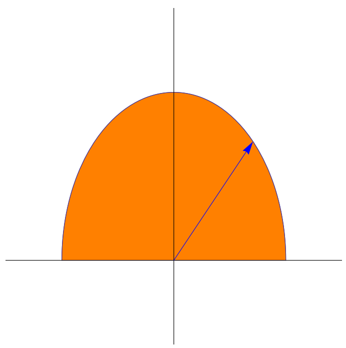
A **Trig substitution** is a special substitution, where x is a trigonometric function of u or u is a trigonometric function of x. Also this topic is covered more in follow up courses like Math 1b. This lecture allows us to practice more the substitution method. Here is an important example:

1 The area of a half circle of radius 1 is given by the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx .$$

Solution. Write $x = \sin(u)$ so that $\cos(u) = \sqrt{1-x^2}$. $dx = \cos(u)du$. We have $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$ the answer is

$$\int_{-\pi/2}^{\pi/2} \cos(u)\cos(u)du = \int_{-\pi/2}^{\pi/2} (1+\cos(2u))/2 = \frac{\pi}{2} .$$



Lets generalize this a bit and do the same computation for a general radius r:

2 Compute the area of a half disc of radius r which is given by the integral

$$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx .$$

Solution. Write $x = r\sin(u)$ so that $r\cos(u) = \sqrt{r^2-x^2}$ and $dx = r\cos(u)\, du$ and $r\sin(-\pi/2) = -r$ and and $r\sin(\pi/2) = r$. The answer is

$$\int_{-\pi/2}^{\pi/2} r^2\cos^2(u)\, du = r^2\pi/2 .$$

Here is an example, we know how to integrate

3 Find the integral

$$\int \frac{dx}{\sqrt{1-x^2}} .$$

We know the answer is $\arcsin(x)$. How can we do that without knowing? **Solution.** We can do it also with a trig substitution. Try $x = \sin(u)$ to get $dx = \cos(u)\, du$ and so

$$\int \frac{\cos(u)\, du}{\cos(u)} = u = \arcsin(x) + C .$$

Here is an example, where $\tan(u)$ is the right substitution. You have to be told that first. It is hard to come up with the idea:

4 Find the following integral:

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

by using the substitution $x = \tan(u)$. **Solution.** Then $1 + x^2 = 1/\cos^2(u)$ and $dx = du/\cos^2(u)$. We get

$$\int \frac{du}{\cos^2(u)\tan^2(u)(1/\cos(u))} = \int \frac{\cos(u)}{\sin^2(u)}\, du = -1/\sin(u) = -1/\sin(\arctan(x)) .$$

Trig substitution is based on the trig identity :

$\cos^2(u) + \sin^2(u) = 1$

Depending on whether you divide this by $\sin^2(u)$ or $\cos^2(u)$ we get

$1 + \tan^2(u) = 1/\cos^2(u)$, $1 + \cot^2(u) = 1/\sin^2(u)$

These identities are worth remembering. Lets look at more examples:

5 Evaluate the following integral

$$\int x^2/\sqrt{1-x^2}\, dx .$$

Solution: Substitute $x = \cos(u), dx = -\sin(u)\, du$ and get

$$\int -\frac{\cos^2(u)}{\sin(u)}\sin(u)du = -\int \cos^2(u)\, du = -\frac{u}{2} - \frac{\sin(2u)}{4} + C = -\frac{\arcsin(x)}{2} + \frac{\sin(2\arcsin(x))}{4} + C .$$

6 Evaluate the integral

$$\int \frac{dx}{(1+x^2)^2} .$$

Solution: we make the substitution $x = \tan(u), dx = du/(\cos^2(u))$. Since $1 + x^2 = \cos^{-2}(u)$ we have

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^2(u)\, du = (u/2) + \frac{\sin(2u)}{4} + C = \frac{\arctan(u)}{2} + \frac{\sin(2\arctan(u)}{4} + C .$$

Here comes an other prototype problem:

7 Find the anti derivative of $1/\sin(x)$. **Solution**: We use the substitution $u = \tan(x/2)$ which gives $x = 2\arctan(u), dx = 2du/(1+u^2)$. Because $1 + u^2 = 1/\cos^2(x/2)$ we have

$$\frac{2u}{1+u^2} = 2\tan(x/2)\cos^2(x/2) = 2\sin(x/2)\cos(x/2) = \sin(x) .$$

Plug this into the integral

$$\int \frac{1}{\sin(x)}\, dx = \int \frac{1+u^2}{2u}\frac{2du}{1+u^2} = \int \frac{1}{u}\, du = \log(u) + C = \log(\tan(\frac{x}{2})) + C .$$

Unlike before, where x is a trig function of u, now u is a trig function of x. This example shows that the substitution $u = \tan(x/2)$ is magic. Because of the following identities

0. $u = \tan(x/2)$

[1] $dx = \frac{2du}{(1+u^2)}$

[2] $\sin(x) = \frac{2u}{1+u^2}$

[3] $\cos(x) = \frac{1-u^2}{1+u^2}$

It allows us to reduce any rational function involving trig functions to rational functions.[1]

Any function $p(x)/q(x)$ where p, q are trigonometric polynomials can be integrated using elementary functions.

It is usually a lot of work but here is an example:

8 To find the integral

$$\int \frac{\cos(x) + \tan(x)}{\sin(x) + \cot(x)} \, dx$$

for example, we replace $dx, \sin(x), \cos(x), \tan(x) = \sin(x)/\cos(x), \cot(x) = \cos(x)/\sin(x)$ with the above formulas we get a rational expression which involves u only This gives us an integral $\int p(u)/q(u) \, du$ with polynomials p, q. In our case, this would simplify to

$$\int \frac{2u\,(u^4 + 2u^3 - 2u^2 + 2u + 1)}{(u-1)(u+1)\,(u^2+1)\,(u^4 - 4u^2 - 1)} du$$

The method of partial fractions provides us then with the solution.

[1]Proofs: [1] differentiate to get $du = dx/(2\cos^2(x/2)) = dx(1+u^2)/2$. [2] use double angle $\sin(x) = 2\tan(x/2)\cos^2(x/2)$ and then $1/\cos^2(x/2) = 1 + \tan^2(x/2)$. [3] use double angle $\cos(x) = \cos^2(x/2) - \sin^2(x/2) = (1 - \sin^2(x/2)/\cos^2(x/2))\cos^2(x/2)$ and again $1/\cos^2(x/2) = 1 + \tan^2(x/2)$.

Homework

1 Find the antiderivative:

$$\int \sqrt{1 - 4x^2} \, dx \, .$$

2 Find the antiderivative:

$$\int (1 - x^2)^{3/2} \, dx \, .$$

3 Find the antiderivative:

$$\int \frac{\sqrt{1 - x^2}}{x^2} \, dx \, .$$

4 Integrate

$$\int \frac{dx}{1 + \sin(x)} \, .$$

Use the substitution $u = \tan(x/2)$.

5 Compute

$$\int \frac{dx}{\cos(x)}$$

using the substitution $u = \tan(x/2)$.

Math 1A: introduction to functions and calculus **Oliver Knill, 2012**

Lecture 33: Calculus and Music

A music piece is a function

Calculus plays a role in music because every music piece just is a **function**. If you have a loudspeaker with a membrane at position $f(t)$ at time t, then you can listen to the music. The pressure variations in the air are sound waves which reach your ear, where your eardrum oscillates with the function $f(t-T)+g(t)$ where $g(t)$ is background noise and T is a time delay for the sound reach your ear. Plotting and playing works the same way. In Mathematica, we can play a function with

```
Play[ Sin[2Pi 1000 x^2],{x,0,10}]
```

This function contains all the information about the music piece. A music ".WAV" file contains sampled values of the function. A sample rate of 44'100 per second is usual. Since our ear does not hear frequencies larger than 20'000 KHz, a samplying rate of 44 K is good enough by a **theorem of Nyquist-Shannon**. In .MP3 files essential values are encoded in a compressed way. To get from the sample values $f(n)$ the function back, the sinc function is used. The **Whittaker-Shannon interpolation formula**

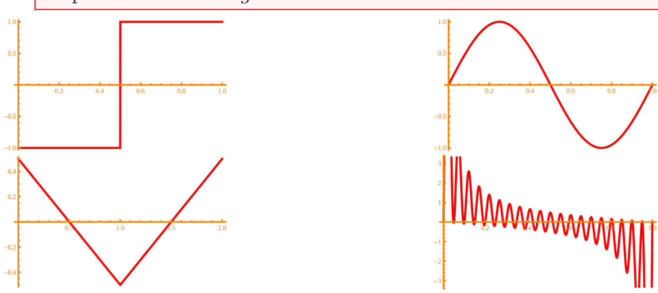
$$f(t) = \sum_n f(n)\text{sinc}(t+n)$$

is especially good. We take this lecture as an opportunity to review some facts about functions. We especially see that log, exp and trigonometric functions play an important role in music.

The wave form and hull

A periodic signal is the building block of sound. Assume $g(x)$ is a 2π periodic function, we can generate a sound of 440 Hertz when playing the function $f(x)=g(440\cdot 2\pi x)$. If the function does not have a smaller period, then we hear the A tone with 440 Hertz.

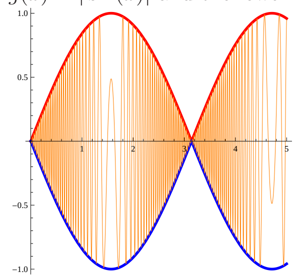
A periodic function g is called a **wave form**.



The wave form makes up the **timbre** of a sound which allows to model music instruments with macroscopic terms like "attack, vibrato, coloration, noise, echo, reverbation" and other characteristics.

The upper **hull function** is defined as the interpolation of successive local maxima of f. The lower **hull function** is the interpolation of the local minima.

For the function $f(x)=\sin(100x)$ for example, the upper hull function is $g(x)=1$ and the lower hull function is $g(x)=-1$. For $f(x)=\sin(x)\sin(100x)$ the upper hull function is approximately $g(x)=|\sin(x)|$ and the lower hull function is approximately $g(x)=-|\sin(x)|$.



We can not hear the actual function because the function changes too fast that we can notice individual vibrations. But we can hear the hull function. Simplest examples are change of dynamics in music like **creshendi** or **diminuendi** or a vibrato. We can generate a beautiful hull by playing two frequencies which are close. You hear **interference**.

The scale

Western music uses a discrete set of frequencies. This scale is based on the exponential function. The frequency f is an exponential function of the scale s. On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

The **Midi numbering** of musical notes is

$$s = 69 + 12\cdot\log_2(f/440)$$

1 What is the frequency of the Midi tone 100? **Solution**. We have to solve the above equation for f and get the **piano scale function**

$$f(s) = 440\cdot 2^{(s-69)/12}\ .$$

Evaluated at 100 we get 2637.02 Hz.

The piano scale function

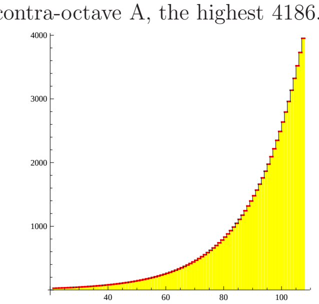
$$f(s) = 440\cdot 2^{(s-69)/12}\ .$$

is an exponential function $f(s)=be^{as}$ which satisfies $f(s+12)=2f(s)$.

2 Find the discrete derivative $Df(x)=f(x+1)-f(x)$ of the Piano scale function. **Solution:** The function is of the form $f(x)=A2^{ax}$. We have $f(x+1)=2^a f$ and so $Df(x)=(2^a-1)f$ with $a=1/12$. Lets get reminded that such discrete relations lead to the important property $\boxed{\frac{d}{dx}\exp(ax)=a\exp(x)}$ for the exponential function.

```
midifrequency[m_] := N[440 2^((m - 69)/12)]
```

The classical piano covers the 88 Midi tone scale from 21 to 108. The lowest frequency is $27.5Hz$, the sub-contra-octave A, the highest $4186.01Hz$, the 5-line octave C.



Decomposition in overtones: low and high pass filter Every wave form can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can here melodies. Here is an example of a decomposition: $f(x) = \sin(x)+\sin(2x)/2+sin(3x)/3+\sin(4x)/4+\sin(5x)/5$. With infinitely many terms, one can also discribe discontinuous functions.
Filtering and tuning: pitch and autotune An other advantage of a decomposition of a function into basic building blocks is that one can leave out frequencies which are not good. Examples are **low pass** or **high pass** filters. A popular filter is **autotune** which does not filter but moves the frequencies around so that you can no more sing wrong. If 440 Herz (A) and 523.2 Herz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4\cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4\cos(2\pi 523.2x)$. This filtering is done on the wave form scale.
Mixing different functions: rip and remix If f and g are two functions which represent songs, we can look at $(f+g)/2$ which is the **average** of the two songs. In real life this is done using **tracks**. Different instruments can be recorded independently for example and then mixed together. One can for example get guitare $g(t)$, voice $v(t)$ and piano $p(t)$ and form $f(t) = ag(t) + bv(t) + c(p(t)$, where the constants a, b, c are chosen.
Differentiate functions: reverb and echo If f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x + h) - f(x)]/h$. Such a differentiation is easy to achieve with a real song. It turns out that for small h, like of order of $h = 1/1000$, the song does not change much. The reason is that a frequency $\sin(kx)$ or hearing the derivative $\cos(kx)$ produces the same song. However, if we allow h to be larger, then a **reverb** or **echo** effect is produced.

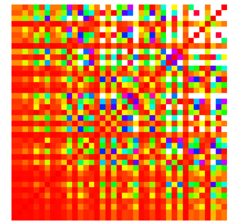
Other relations with math

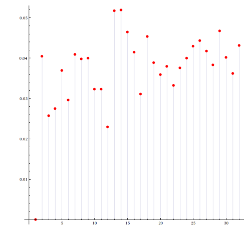
We might not have time for this during the lecture.
Symmetries. Symmetries play an important role in art and science. In geometry we know rotational, translational symmetries or reflection symmetries. Like in geometry, symmetries play a role both in Calculus as well as in Music. We see some examples in the presentation.
Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**, **Diophatine problems** like how good frequency rations are approximated by rationals: Why is the **chromatic scale** based on the twelfth root of 2 so good? Indian music for example uses **microtones** and a scale of 22. The 12 tone scale is good because many powers $2^{k/12}$ are close to rational numbers. I once defined the "scale fitness" function

$$M(n) = \sum_{k=1}^{n} \min_{p,q} |2^{k/n} - \frac{p}{q}| G(p,q)$$

which is a measure on how good a music scale is. It uses Euler's **gradus suavis** (="degree of pleasure") function $G(n,m)$ of a fraction n/m which is $G(n,m) = 1+E(nm/\gcd(n,m))$, where the **Euler gradus** function $E(n) = \sum_{p|n} e(p)(p-1)$ and p runs over all prime factors p of n and $e(p)$ is the multiplicity. The picture to the left shows Euler's function $G(n,m)$, the right hand side the scale fitness function in dependence on n. You see that $n = 12$ is clearly the winner. This analysis could be refined to include scales like Stockhausens $5^{k/25}$ scale. You can listen to the Stockhausen's scale with $f(t) = \sin(2\pi t 100 \cdot 5^{[t]/25})$, where $[t]$ is

the largest integer smaller than t. Our familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 100 \cdot 2^{[t]/12})$.





3 The perfect fifth 3/2 has the gradus suavis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth 4/3 for which $1 + E(12) = 1 + (2 - 1)(3 - 1)$. You can listen to the perfect fifth $f(x) = \sin(1000x) + \sin(1500x)$ or the perfect fourth $\sin(1000x) + \sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x) + \sin(1333x) + \sin(1500x) + \sin(2000x)$.

Homework

1 **Modulation.** Draw the hull function of the following functions.
a) $f(x) = \sin(1000x) - \sin(1001x)$
b) $f(x) = \sin(x) + \cos(\tan(1000\sqrt{x}))$
c) $f(x) = \sqrt{x}\cos(10000x)$
d) $f(x) = \cos(x)\sin(e^{2x})/2$

Here is how to play a function with Mathematica. It will play for 9 seconds:

```
Play[Cos[x] Sin[Exp[2 x]]/x, {x, 0, 10}]
```

Hint. You can play functions online with Wolfram Alpha. Here is an example:

```
play  sin(1000 x)
```

2 **Amplitude modulation (AM)**: If you listen to $f(x) = \sin(x)\sin(1000x)$ you hear an amplitude change. Draw the hull function. How many increase in amplitudes to you hear in 10 seconds?

3 **Frequency Modulation (FM)**: If we play $f(x) = x\sin(1000\sin(x))$, there are points, where the frequency is low. This is a frequency change. Draw the hull function.

4 **Smoothness**: If we play the function $f(x) = \tan(\sin(3000\sin(x)))$, the sound sounds pretty nice. If we change that to $f(x) = \tan(2\sin(3000\sin(x)))$, the sound is awful. Can you see why? To answer this, you might want to plot a similar function where 3000 is replaced by 3.

5 **A mystery sound**: How would you describe the sound $f(x) = \sin(1/\sin(2\pi 3x))$? Our ear can not hear frequencies below 20 Hertz. Why can one still hear something? To answer this, you might want to plot the function from $x = 0$ to $x = 10$.

Lecture 35: Feedback

Simon would like to get some feedback from you about the lecture:

1 Was the lecture clear?

2 How was my blackboard work?

3 Were you engaged?

4 Was the pace ok?

5 Any further comments?

Thanks!

Math 1A: introduction to functions and calculus **Oliver Knill, 2012**

Lecture 34: Calculus and Statistics

In this lecture, we look at an application of calculus to statistics. We have already defined the probability density function f and its anti-derivative, the cumulative distribution function.

Probability density

Recall that a probability density function is a function f satisfying $\int f(x)\,dx = 1$ and which has the property that it is ≥ 0 everywhere. We say f is a probability density function on an interval $[a, b]$ if $\int_a^b f(x)\,dx = 1$ and $f(x) \geq 0$ there. In such a case, we assume that f is zero outside the interval.

Recall also that we called the antiderivative of f the cumulative distribution function $F(x)$ (CDF).

Expectation

The **expectation** of probability density function f is

$$m = \int_{-\infty}^{\infty} x f(x)\,dx .$$

In the case, when the probability density function is zero outside some interval, we have

The **expectation** of probability density function f defined on some interval $[a, b]$ is

$$m = \int_a^b x f(x)\,dx .$$

Variance and Standard deviation

The **variance** of probability density function f is

$$\int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx ,$$

where m is the expectation.

Again, if the probability density function is defined on some interval $[a, b]$ then

The **variance** of probability density function f is

$$\int_a^b (x-m)^2 f(x)\,dx ,$$

where m is the expectation of f.

The square root of the variance is the **standard deviation**.

Examples

In the lecture, we will compute this in some examples. Here is some sample.

1 The expectation of the geometric distribution $f(x) = e^{-x}$

$$\int x e^{-x}\,dx = 1 .$$

The variance of the geometric distribution $f(x) = e^{-x}$ is 1 and the standard deviation 1 too.

Remember that we can compute also with Tic-Tac-Toe:

$$\int x^2 e^{-x}\,dx$$

x^2	e^{-x}	
$2x$	$-e^{-x}$	$\oplus$
2	e^{-x}	$\ominus$
0	e^{-x}	$\oplus$

2 The expectation of the standard Normal distribution $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$

$$\int_0^{\infty} x(2\pi)^{-1/2} e^{-x^2/2}\,dx = 0 .$$

Homework

1 The function $f(x) = \cos(x)/2$ on $[-\pi/2, \pi/2]$ is a probability density function. Its mean is 0. Find its variance

$$\int_{-\pi/2}^{\pi/2} x^2 \cos(x)\,dx .$$

2 The **uniform distribution on** $[a, b]$ is a distribution, where any real number between a and b is equally likely to occur. The probability density function is $f(x) = 1/(b-a)$ for $a \leq x \leq b$ and 0 elsewhere. Verify that $f(x)$ is a valid probability density function.

3 Verify that the function which is 0 for $x < 0$ and equal to

$$f(x) = \frac{1}{\log(2)} \frac{e^{-x}}{1+e^{-x}}$$

for $x \geq 0$ is a probability density function.

4 A particular **Cauchy distribution** has the probability density

$$f(x) = \frac{1}{\pi} \frac{1}{(x-1)^2+1} .$$

Verify that $f(x)$ is a valid probability density function.

5 Find the cumulative distribution function (CDF) $F(x)$ of f in the previous problem.

Lecture 34: Calculus and Economics

In this lecture we look more at applications of calculus to **economics**. This is an opportunity to review extrema problems.

Marginal and total cost

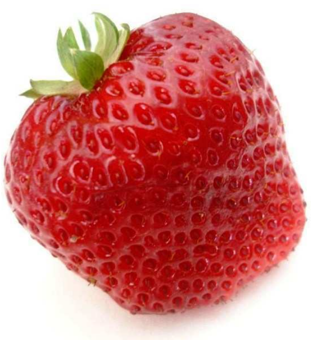
Recall that the **marginal cost** was defined as the derivative of the **total cost**. Both, the marginal cost and total cost are functions of the quantity of goods produced.

1 Assume the total cost function is $C(x) = 10x + 0.01x^2$. Find the marginal cost and the place where the total cost is maximal. **Solution**. Differentiate.

2 You sell spring water. The marginal cost to produce depends on the season and given by $f(x) = 10 - 10\sin(2x)$. For which x is the total cost maximal?

3 The following example is adapted from the book "Dominik Heckner and Tobias Kretschmer: Don't worry about Micro, 2008", where the following strawberry story appears: (verbatim citation in italics):

Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

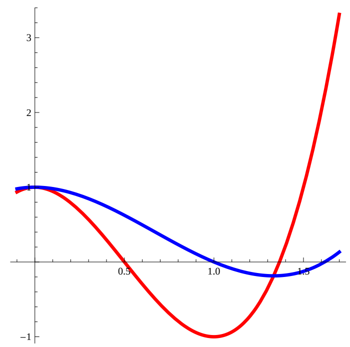


Again, if $F(x)$ is the **total cost function** in dependence of the quantity x, then $F' = f$ is called the **marginal cost**.

The function $g(x) = F(x)/x$ is called the **average cost**.

A point where $f = g$ is called a **break even point**.

4 If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, lets call it the "strawberry theorem":

Strawberry theorem: We have $g'(x) = 0$ if an only if $f = g$.

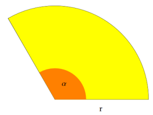
Proof.

$$g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g) \ .$$

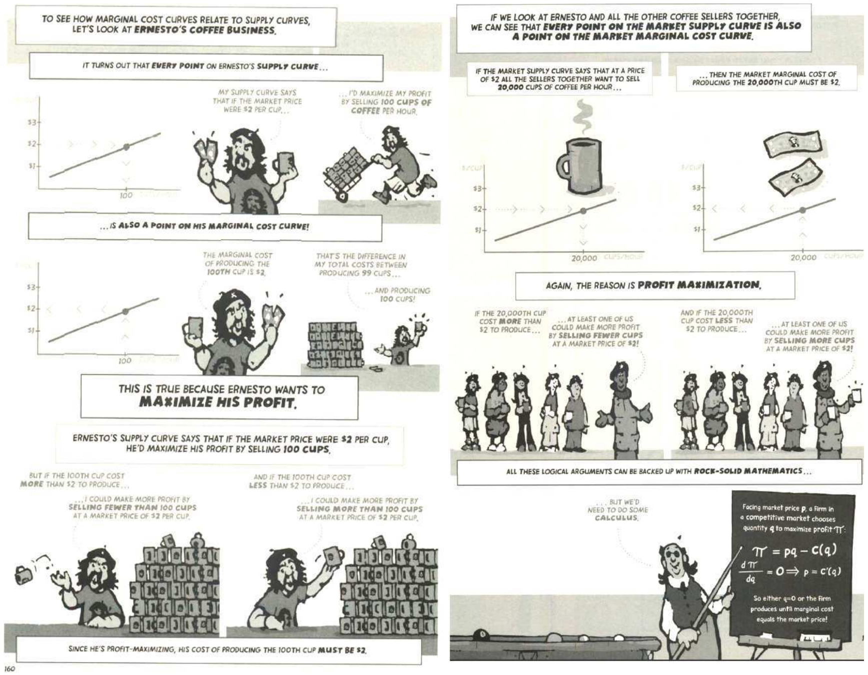
More extremization problems

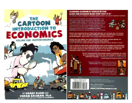
1 Find the rhomboid with side length 1 which has maximal area. Use an angle α to extremize.

2 Find the sector of radius$r = 1$ and angle α which has minimal circumference $r2r + r\alpha$ if the area $r^2\alpha/2 = 1$ is fixed.



3 Find the ellipse of length $2a$ and width $2b$ which has fixed area $\pi ab = \pi$ and for which the sum of diameters $2a + 2b$ is maximal.





Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang. You can detect the strawberry theorem ($g' = 0$ is equivalent to $f = g$) can be seen on the blackboard.

Homework

1 Verify the Strawberry theorem in the case when the marginal cost is $f(x) = \cos(x)$. This means that you have to compute the total cost $F(x)$ and the average cost $g(x)$ and see whether things pan out in that case.

2 The **production function** in an office gives the production $Q(L)$ in dependence of labor L. Assume $Q(L) = 5000L^3 - 3L^5$. Find L which gives the maximal production.

This can be typical: For smaller groups, production usually increases when adding more workforce. After some point, bottle necks occur, not all resources can be used at the same time, management and bureaucracy is added, each person has less impact and feels less responsible, meetings slow down production etc. In this range, adding more people will decrease the productivity.

3 **Marginal revenue** f is the rate of change in **total revenue** F. As total and marginal cost, these are functions of the **cost** x. Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the point, where the total revenue has a local maximum.

4 Find a line $y = mx$ through the points $(3,4), (6,3), (2,5)$. which minimize the function

$$f(m) = (3m-4)^2 + (6m-3)^2 + (2m-5)^2 \ .$$

5 A function f on $[0,1]$ which has the property that its range is contained in $[0,1]$ and which has the property that $|f'(x)| < 1$ for all $x \in (0,1)$ is called a "contraction". Brower's fixed point theorem tells that a contraction has a fixed point.
a) Verify that $f(x) = 1 - x^2/2$ is a contraction.
b) Find the fixed point.

Here is a citation from the book "Mathematical Economics" by Michael Carter : *Fixed point theorems are powerful tools for the economic theorist. They are used to demonstrate the existence of a solution to an economic model, which establishes the consistency of the model and highlights the requirements minimal requirements to ensure a solution. The classic applications of fixed point theorems in economics involve the existence of market equilibria in an economy and the existence of Nash equilibria in strategic games. They are also applied in dynamic models, a fundamental tool in macroeconomic analysis.*

Lecture 36: Integrals and AI

Can we build an artificial calculus teacher?

Machines assist us already in many domains: heavy work is done by **machines and robots**, accounting by **computers** and fighting by **drones**. Lawyers and doctors are assisted by artificial intelligence. There is no reason why teaching is different. The **web** has become a "gigantic brain" to which virtually any question can be asked or googled: "Dr Know" in Spielberg's movie "AI" is humbled: enter symptoms for an illness and get a diagnosis, enter a legal question and find previous cases. Enter a calculus problem and get an answer. Building an **artificial calculus teacher** involves calculus itself: such a bot must connect dots on various levels: understand questions, read and grade papers and exams, write good and original exam questions, know about learning and pedagogy. Ideally, it should also have "ideas" like to "make a lecture on artificial intelligence". But first of all, our AI friend needs to know calculus and be able to generate and solve calculus problems. [1]

Generating calculus problems

Having been involved in a linear algebra book project once, helping to generating solutions to problems, I know that some calculus books are written with help of computer algebra systems. They generate problems and solutions. This applies mostly to drill problems. In order to generate problems, we first must build **random functions**. Our AI engine "sofia" knew how to generate random problems with solutions. Random functions are involved when asked "give me an example of a function". This is easy: the system would generate functions of reasonable complexity:

Call the 10 functions $\{\sin, \cos, \log, \exp, \tan, \mathrm{sqt}, \mathrm{pow}, \mathrm{inv}, \mathrm{sca}, \mathrm{tra}\ \}$ **basic functions**.

Here $\mathrm{sqt}(x) = \sqrt{x}$ and $\mathrm{inv}(x) = 1/x^k$ for a random integer k between -1 and -3, $\mathrm{pow}(x) = x^k$ for a random integer k between 2 and 5. $\mathrm{sca}(x) = kx$ is a scalar multiplication for a random nonzero integer k between -3 and 3 and $\mathrm{tra}(x) = x+k$ translates for a random integer k between -4 and 4.

Second, we use addition, subtraction multiplication, division and composition to build more complicated functions:

A **basic operation** is an operation from the list $\{f \circ g, f+g, f*g, f/g, f-g\ \}$.

The operation x^y is not included because it is equivalent to $\exp(x\log(y)) = \exp\circ(x \cdot \log)$. We can now build functions of various complexities:

[1] In the academic year of 2003/2004, thanks to a grant from the Harvard Provost, I could work with undergraduates **Johnny Carlsson**, **Andrew Chi** and **Mark Lezama** on a "calculus chat bot". We spent a couple of hours per week to enter mathematics and general knowledge, build interfaces to various computer algebra systems like Pari, Mathematica, Macsyma and build a web interface. We fed our knowledge to already known chat bots and newly built ones and even had various bots chat with each other. We conceptionally explored the question of automated learning of the bots from the conversations as well as to add context to the conversation, since bots needs to remember previous topics mentioned to understand some questions. We learned how immense the task is. In the mean time it has become business. Companies like **Wolfram research** have teams of mathematicians and computer scientists working on content for the "Wolfram alpha" engine. Having recently seen a group at work here in Cambridge on Mass Av, I guess they generate probably in one day as much content as our Sofia group could do in a week for our "pet project".

A **random function** of complexity n is obtained by taking n random basic functions $f_1, \dots, f_n$, and n random basic operators $\oplus_1, \dots, \oplus_n$ and forming $f_n \oplus_n f_{n-1} \oplus_{n-1} \cdots \oplus_2 f_1 \oplus_1 f_0$ where $f_0(x) = x$ and where we start forming the function from the right.

1 **Visitor:** "Give me an easy function": Sofia looks for a function of complexity one: like $x\tan(x)$, or $x + \log(x)$, or $-3x^2$, or $x/(x-3)$.

2 **Visitor:** "Give me a function": Sofia returns a random function of complexity two: $x\sin(x) - \tan(x)$, or $-e^{\sqrt{x}} + \sqrt{x}$ or $x\sin(x)/\log(x)$ or $\tan(x)/x^4$.

3 **Visitor:** "Give me a difficult function": Sofia builds a random function of complexity four like $x^4 e^{-\cos(x)}\cos(x)+\tan(x)$, or $x-\sqrt{x}-e^x+\log(x)+\cos(x)$, or $(1+x)(x\cot(x)-\log(x))/x^2$, or $(-x+\sin(x+3)-3)\csc(x)$

Now, we can build a random calculus problem. To give you an idea, here are some templates for integration problems:

A **random integration problem** of complexity n is a sentence from the sentence list { "Integrate $f(x) = F(x)$", "Find the anti derivative of $F(x)$", "What is the integral of $f(x) = F(x)$?", "You know the derivative of a function is $f'(x) = F(x)$. Find $f(x)$." }, where F is a random function of complexity n.

4 **Visitor** "Give me a differentiation problem". **Sofia:** Differentiate $f(x) = x\sin(x) - \frac{1}{x^2}$. The answer is $\frac{2}{x^3} + \sin(x) + x\cos(x)$.

5 **Visitor**: "Give me a difficult integration problem". **Sofia**: Find f if $f'(x) = \frac{1}{x} + \left(3\sin^2(x) + \sin(\sin(x))\right)\cos(x)$. The answer is $\log(x) + \sin^3(x) - \cos(\sin(x))$.

6 **Visitor**: "Give me an easy extremization problem". **Sofia**: Find the extrema of $f(x) = x/\log(x)$. The answer is $x = e$.

7 **Visitor**: Give me an extremization problem". **Sofia**: Find the maxima and minima of $f(x) = x - x^4 + \log(x)$. The extrema are

$$-\frac{\sqrt{\left(9+\sqrt{3153}\right)^{2/3} - 8\sqrt[3]{6}} + \sqrt{8\sqrt[3]{6} - \left(9+\sqrt{3153}\right)^{2/3}\left(1 + 6\sqrt{\frac{2}{9+\sqrt{3153}-8\sqrt[3]{6\left(9+\sqrt{3153}\right)}}}\right)}}{2 2^{5/6}\sqrt[3]{3}\sqrt[6]{9+\sqrt{3153}}}.$$

The last example shows the perils of random generation. Even so the function had decent complexity, the solution was difficult. Solutions can even be transcendental. This is not a big deal: just generate a new problem. By the way, all the above problems and solutions have been generated by Sofia. The dirty secret of calculus books is that there are maybe a thousand different type of questions which are usually asked. This is a reason why textbooks have become boring clones of each other and companies like "Aleks", "demidec" etc exist which constantly mine the web and course sites like this and homework databases like "webwork" which contain thousands of pre-compiled problems in which randomness is already built in.

Automated problem generation is the "fast food" of teaching and usually not healthy. But like "fast food" has evolved, we can expect more and more computer assisting in calculus teaching.

Be assured that for this course, the problems have been written by hand (I sometimes use Mathematica to see whether answers are reasonable). Handmade problems can sometimes a bit "rough" but hopefully some were more interesting. I feel that it is not fair to feed computer generated problems to humans. It is possible to write a program giving an answer to "Write me a final exam", but the exam would be uninspiring.

Corner detection

How do we detect corners in pictures? This is necessary to understand pictures, drawings. It might also be needed to see whether a given function is reasonably shaped. There should not be too many "wiggles" for example. There are various techniques to measure that. One of the best methods in computer vision uses the notion of **curvature**:

Given a function $f(x)$, define the **curvature** as

$$k(x) = \frac{f''(x)}{(1+f'(x)^2)^{3/2}} .$$

Is is a measure on how much the curve is bent at the point $(x, f(x))$. Positive curvature means the curve is concave up, otherwise concave down.

8 For a quadratic function $f(x) = x^2$, we have $\kappa(x) = 1/(1+x^2)$. We see that the curvature is maximal at the lowest part of the parabola.

9 For the function $f(x) = \sqrt{1-x^2}$, we have $f'(x) = -2x/\sqrt{1-x^2}$ and $f''(x) = -(1-x^2)^{-3/2}$. We have $(1+f'(x)^2) = 1/(1-x^2)$ and $k(x) = -1$.

10 **Problem:** Find the curvature for the graph of $f(x) = x^5/5 - x$. Where is the curvature maximal?

Here is a cool theorem:

If we integrate up the curvature along a graph of f on $[a, b]$ so that we always travel with constant speed on the graph, we get the angle difference $\beta - \alpha$.

Proof. $f'(x)$ is the slope of the curve and $g(x) = \arctan(f'(x))$ the angle. We have $\alpha = g(a) = \arctan(f'(a))$ and $\beta = g(b) = \arctan(f'(b))$. The fundamental theorem of calculus tells $\int_a^b g'(x)\, dx = g(b) - g(a)$. But $g'(x) = f''(x)/(1+f'(x)^2)$. If we travel so that $\sqrt{1+f'(x)^2} = 1$ then this is curvature.

It follows that if we integrate up curvature along the boundary of a region in the plane, we get 2π. This is a simple version of the Gauss-Bonnet theorem called Hopf Umlaufsatz.

Connecting the dots

We want to connect points $P_1, \dots, P_n$ by a smooth graph. This "connecting the dots" problem is quite frequent. Our brain does this automatically. We need to see a few glances to "see" the motion of an object and predict where it will end. We need to connect dots if we drive a car, if we interpret a picture etc. On a more abstract level, we need to connect dots in the landscape of ideas whenever we solve a problem. We want to go from A to B and need to construct intermediate steps.
Here is a simple method found by G. Chaikin in 1974 [2] which generates a smooth curve through a few points.

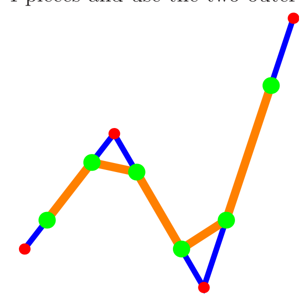
[2] G. Chaikin, An algorithm for high speed curve generation. Computer Graphics and Image Processing 3 (1974), 346-349.

Given a sequence of n points $P_1, ..., P_n$ define a new sequence of $2n-2$ points $R_2, ..., R_{2n-1}$ by

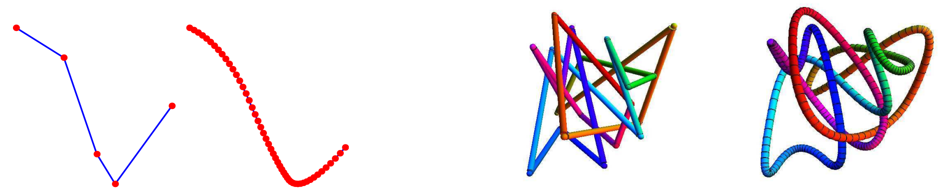
$$R_{2i} = \frac{3}{4}P_i + \frac{1}{4}P_{i+1}, \qquad R_{2i+1} = \frac{1}{4}P_i + \frac{3}{4}P_{i+1}$$

for $i = 1, ..., n-1$.

One such a step defines a **Chaikin step**. The limiting curve is called the **Chaikin curve** defined by the original points. The picture should explain how we get the new points from the old ones: divide each segment into 4 pieces and use the two outer points to get new points.



The Chaikin steps produce a smooth curve approximating a given set of points.



The method can be used for example to study the complexity of **random knots**. To answer the question stated initially:

AI will assist teachers in the future and help them to be more efficient.

Homework

1 A calculus bot wants to build a differentiation problem by combining log and sin and exp. Differentiate all of the 6 combinations $\log(\sin(\exp(x)))$, $\log(\exp(\sin(x)))$,$\exp(\log(\sin(x)))$, $\exp(\sin(\log(x)))$, $\sin(\log(\exp(x)))$ and $\sin(\exp(\log(x)))$.

2 Four of the 6 combinations of log and sin and exp can be integrated as elementary functions. Do these integrals.

3 Find the curvature of the sin curve at $x = 0$, $x = \pi/2$ and $x = 3\pi/2$.

4 Draw the points $(0, \sin(0)), (\pi/2, \sin(\pi/2)), (\pi, \sin(\pi), (3\pi/2, \sin(3\pi/2)), (2\pi, \sin(2\pi))$ and connect them with lines. Now do Chaikin iteration for 2 generations on paper.

5 Make a list of the 10 most important integrals you currently not know, look the integrals up and list the table here.

Lecture 37: Final remarks

The holographic picture

How can one one summarize calculus in one sentence?

"Calculus establishes that two operations on functions f are related: first the derivative of f which is the rate of change of f second the integral of f which is the area under the graph of f."

In one paragraph:

The development of calculus by Newton and Leibniz is a major achievement of the past millennium. The core of this course introduces differential and integral calculus. Differential calculus studies the "rate of change" f' of a function, integral calculus treats "accumulation" $\int_0^x f(x)\,dx$ which can be interpreted as "area under the curve". The fundamental theorem of calculus links the two: it tells that

$$\int_0^x f'(t)\,dt = f(x) - f(0), \frac{d}{dx}\int_0^x f(t)\,dt = f(x)\ .$$

The subject can be applied to problems from other scientific disciplines like economics (the strawberry theorem for total, average and marginal costs), reasoning for relating quantities (like estimating the speed of an airplane from angle change, psychology (catastrophes explaining revolutionary changes or flips in human perception), geometry (volume and area computation), statistics (distribution and cumulative distribution functions) or everyday life (the wobbly table theorem).

In one page?

1. Calculus relates two fundamental operations, the derivative measuring the rate of change of a function and the derivative which measures the area under the graph.
2. Taking derivatives is done with the chain, product and quotient rules, taking intervals with substitution including trig substitution, integration by parts and partial fraction rules.
3. Basic functions are polynomials and $\exp, \log, \sin, \cos, \tan$. We can add, subtract, multiply, divide and compose functions, we can differentiate and integrate them.
4. A function is continuous at p if $f(x) \to f(b)$ for $x \to p$. It is differentiable at p if $(f(x+h) - f(x))/h$ has a limit for $h \to 0$. Limits from the right and left should agree.
5. To extremize a function, we look at points where $f'(x) = 0$. If $f''(x) > 0$ we have a local minimum, if $f''(x) < 0$ we have a local maximum. There can be critical points $f'(x) = 0$ without being extremum like x^3.
6) To relate how different quantities change in time, we differentiate the formula relating the quantities using the chain rule. If there is a third time variable, then this is the story of related rates, if one of the variables is the parameter, then this is implicit differentiation.

One could look at math from a historical perspectives or read original things. One could do projects, use more computer algebra systems, practice visualization and visualize things. You have studied maybe 300 hours for this course including homework, reading, and discussing the

material. Years would be needed to study it more on a research level. New calculus is constantly developed. I myself have been working mostly on more probabilistic versions of calculus which allows to bypass some of the difficulties when discretizing calculus. The loss of symmetries obtained by discretization can be compensated differently.

The future of calculus

Calculus will without doubt look different in 50 years. Many changes have already started, not only on the context level, also from outside: Calculus books will be gone, electronic paper which will be almost indistinguishable from real paper has replaced it. Text, computations, graphics are all fluid in that we can at any point adjust the amount of details. Similarly than we can zoom into a map or picture by pinching the screen, we can triple pinch a text or proof or picture. As we do so, more details are added, more steps of a calculation added, more information included into a graph etc. Every picture is interactive can turn in a movie, an animation, parameters can be changed, functions deformed with the finger. Every picture is a little laboratory. Questions can be asked directly to the text and answers provided. The text can at any time be set back to an official textbook version of the course. The teacher has the possibility to set global preferences and toss around topics. Examples, homework problems and exam problems will be adjusted automatically disallowing for example to treat integration by parts before the product rule. Much of this is not science fiction, there are electronic interactive books already now available for tablet computers which have impressive experimental and animation features. Impossible because it is too difficult to achieve? Remember the last lecture 36. We will have AI on our side and much of this grunt work to compress and expand knowledge can be done computer assisted.

Calculus courses after 1a

To prepare for this course, I set myself the task to formulate the main topic in one short sentence and then single out 4 major goals for the course, then build titles for each lecture etc Here are 4 calculus courses at Harvard drawn out at the level of a "4 point summary". At other schools of higher education, there are similar courses.

The course 1A	from extremization to the fundamental theorem
functions	polynomials, exp, log, trig functions
limits	velocity, tangents, infinite limits
derivatives	product, chain rule with related rates, extremization
integrals	techniques, area, volume, fundamental theorem
The course 1B	from series and integration to differential equations
integration	integration: parts, trig substitution, partial fractions, indefinite
series	convergence, Power, Taylor and Dirichlet series
diff equations	separation of variables, systems like exponential and logistic equations
systems diff eq	equilibria, nullclines, analysis
The course 21A	geometry, extremization and integral theorems in space
geometry	analytic geometry of space, geometric objects, distances
differentiation	curves and surfaces, gradient, curl, divergence
integration	double and triple integrals, other coordinate systems
integral theorems	line and flux integrals, Green, Stokes and Gauss
The course 21B	matrix algebra, eigensystems, dynamical systems and Fourier
equations and maps	Gauss-Jordan elimination, kernel, image, linear maps
matrix algebra	determinants, eigenvalues, eigenspaces, diagonalization
dynamical systems	difference and differential equations with various techniques
fourier theory	Fourier series and dynamical systems on function spaces

There is also a 19a/19b track. The 19a course focuses on models and applications in biology, the 19b course replaces differential equations from 21b with probability theory. The Math 20 course covers linear algebra and multivariable calculus for economists in one semester but covers less material than the 21a/21b track.

The lighter side of calculus

Sofia, our bot had also to know a lot of jokes, especially about math. Here are some relevant to calculus in some way. I left out the inappropriate ones.

1 Why do you rarely find mathematicians at the beach? Because they use sine and cosine to get a tan.

2 **Theorem:** The less you know, the more you make. **Proof:** We know Power = Work/Time. Since Knowledge = Power and Time = Money we know Knowledge = Work/Money. Solve for Money to get Money = Work/Knowledge. If Knowledge goes to zero, money approaches infinity.

3 Why do they never serve beer in a calculus class? Because you can't drink and derive.

4 Descartes comes to a bar. Barmen: An other beer? Descartes: I think not. And disappears.

5 If it's zero degrees outside today. Tomorrow it will be twice as cold. How cold will it be?

6 There are three types of calculus teachers: those who can count and those who can not.

7 Calculus is like love; a simple idea, but it can be complicated.

8 A mathematician and an engineer are on a desert island with two palm trees and coconuts. The engineer climbs up, gets its coconut gets down and eats. The mathematician climbs up the other, gets the coconut, climbs the first tree and deposits it. "I've reduced the problem to a solved one".

9 Pickup line: You are so x^2. Can I be $x^3/3$, the area under your curves?

10 The Evolution of calculus teaching:
1960ies: A peasant sells a bag of potatoes for 10 dollars. His costs are 4/5 of his selling price. What is his profit?
1970ies: A farmer sells a bag of potatoes for 10 dollars. His costs are 4/5 of his selling price, that is, 8 dollars. What is his profit?
1980ies: A farmer exchanges a set P of potatoes with a set M of money. The cardinality of the set M is equal to 10, and each element of M is worth one dollars Draw ten big dots representing the elements of M. The set C of production costs is composed of two big dots less than the set M. Represent C as a subset of M and give the answer to the question: What is the cardinality of the set of profits?
1990ies: A farmer sells a bag of potatoes for 10 dollars. His production costs are 8 dollars, and his profit is 2 dollars. Underline the word "potatoes" and discuss it with your classmates.
2000ies: A farmer sells a bag of potatoes for 10 dollars. His or her production costs are 0.80 of his or her revenue. On your calculator, graph revenue vs. costs and run the program POTATO to determine the profit. Discuss the result with other students and start blog about other examples in economics.
2010ies: A farmer sells a bag of potatoes for 10 dollars. His costs are 8 dollars. Use the Potato theorem to find the profit. Then watch the wobbling potato movie.

11 **Q:** What is the first derivative of a cow? **A:** Prime Rib!

12 **Q:** What does the zero say to the eight? **A:** Nice belt!

13 **Theorem.** A cat has nine tails. **Proof.** No cat has eight tails. Since one cat has one more tail than no cat, it must have nine tails.

14 **Q:** How can you tell that a mathematician is extroverted? **A:** When talking to you, he looks at your shoes instead of at his.

15 **Q:** What does the little mermaid wear? **A:** An algae-bra.

16 In a dark, narrow alley, a function and a differential operator meet: "Get out of my way - or I'll differentiate you till you're zero!" "Try it - I'm e^x ..." Same alley, same function, but a different operator: "Get out of my way - or I'll differentiate you till you're zero!" "Try it - I'm e^x..." "Too bad... I'm d/dy."

17 **Q:** How do you make 1 burn? **A:** Fire differentiation at a log.

18 An investment firm hires. In the last round, a mathematician, an engineer, and a business guy are asked what starting salary expectations they had: mathematician: "Would 30,000 be too much?" engineer: "I think 60,000 would be OK." Finance person: "What about 300,000?" Officer: "A mathematician will do the same work for a tenth!" Business guy: "I thought of 135,000 for me, 135,000 for you and 30,000 for the mathematician to do the work.

19 Theorem. Every natural number is interesting. Proof. Assume there is an uninteresting one. Then there is smallest one. But as the smallest, it is interesting. Contradiction!