# 5/11/2013: Practice final C

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| 11 | | 10 |
| 12 | | 10 |
| 13 | | 10 |
| Total: | | 140 |

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F** If a function $f(x)$ has a critical point 0 and $f''(0) = 0$ then 0 is neither a maximum nor minimum.

**Solution:**
We can have $f(x) = x^4$ for example.

2) T **F** If $f' = g$ then $\int_0^x g(x) = f(x)$.

**Solution:**
Not necessarily. The right answer is $\int_0^x g(x)\, dx = f(x) - f(0)$.

3) T **F** The function $f(x) = 1/x$ has the derivative $\log(x)$.

**Solution:**
No! We went in the wrong direction. The anti derivative is $\log(x)$.

4) T **F** The function $f(x) = \arctan(x)$ has the derivative $1/\cos^2(x)$.

**Solution:**
It is $\tan(x)$ which has the derivative $1/\cos^2(x)$, not arctan.

5) **T** F The fundamental theorem of calculus implies that $\int_a^b f'(x)\, dx = f(b) - f(a)$.

**Solution:**
Right on! The most important result in calculus is the fundamental theorem of calculus. Hit me again.

6) T **F** $\lim_{x\to 8} 1/(x-8) = \infty$ implies $\lim_{x\to 3} 1/(x-3) = \omega$.

**Solution:**
This is a classical calculus joke. The fact that it is funny does not make it true.

7) T **F** A continuous function which satisfies $\lim_{x \to -\infty} f(x) = 3$ and $\lim_{x \to \infty} f(x) = 5$ has a root.

**Solution:**
It would be a consequence of the intermediate value theorem if the sign would be different but the signs are not different .

8) **T** F The function $f(x) = (x^7 - 1)/(x - 1)$ has a limit at $x = 1$.

**Solution:**
This is a classical case for healing: we can factor out $(x - 1)$ and see $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ for $x$ different than 1. The limit now exists and is 7. Alternatively, we could have brought the function to Hopital and cured it there. Differentiating the top and bottom gives 7 too.

9) **T** F If $f_c(x)$ is an even function with parameter $c$ and $f'(0) = 0$ and for $c < 3$ the function is concave up at $x = 0$ and for $c > 3$ the function is concave down at $x = 0$, then $c = 3$ is a catastrophe.

**Solution:**
Yes, an even function has a minimum at 0 if it is concave up and a maximum at 0 if it is concave down. At the parameter value $c = 3$ the nature of the critical point changes. This implies that it is a catastrophe.

10) T **F** The function $f(x) = +\sqrt{x^2}$ has a continuous derivative 1 everywhere.

**Solution:**
Agreed, thats a nasty question, but the plus sign should have taken the sting out of it. The function $f(x)$ satisfies $f(x) = |x|$ and has no derivative at $x = 0$.

11) **T** F A rower rows on the Charles river leaving at 5 PM at the Harvard boat house and returning at 6 PM. If $f(t)$ is the distance of the rower at time $t$ to the boat house, then there is a point where $f'(t) = 0$.

**Solution:**
This is a consequence of Rolles theorem.

12) T **F** A global maximum of a function $f(x)$ on the interval $[0, 1]$ is a critical point.

**Solution:**
The extremum could be at the boundary.

13) T F

A continuous function on the interval $[2, 3]$ has a global maximum and global minimum.

**Solution:**
Every continuous function on a closed interval has a maximum as well as a minimum somewhere.

14) T F

The intermediate value theorem assures that if $f$ is continuous on $[a, b]$ then there is a root of $f$ in $(a, b)$.

**Solution:**
One has to assume that $f(a), f(b)$ have different signs.

15) T F

On an arbitrary floor, a square table can be turned so that it does not wobble any more.

**Solution:**
Yes, this is the greatest theorem ever.

16) T F

The derivative of $\log(x)$ is $1/x$.

**Solution:**
Now we are talking. A previous question above had it wrong.

17) T F

If $f$ is the marginal cost and $F = \int_0^x f(x)\ dx$ the total cost and $g(x) = F(x)/x$ the average cost, then points where $f = g$ are called "break even points".

**Solution:**
Yes, this is precisely the definition of a break-even point. The Strawberry theorem assured that $g' = 0$ at those points.

18) T F

At a function party, Log talks to Tan and the couple Sin and Cos, when she sees her friend Exp alone in a corner. Log: "What's wrong?" Exp: "I feel so lonely!" Log: "Go integrate yourself!" Exp sobbs: "Won't change anything." Log: "You are so right".

**Solution:**
Yes, $\exp(x)' = \exp(x)$. There was a happy end nevertheless: Exp later met Cot and had a good time too.

19) | T | F | If a car's position at time $t$ is $f(t) = t^3 - t$, then its acceleration at $t = 1$ is 6.

**Solution:**
Differentiate twice.

20) | T | F | For trig substitution, the identities $u = \tan(x/2)$, $dx = \frac{2du}{(1+u^2)}$, $\sin(x) = \frac{2u}{1+u^2}$, $\cos(x) = \frac{1-u^2}{1+u^2}$ are useful.

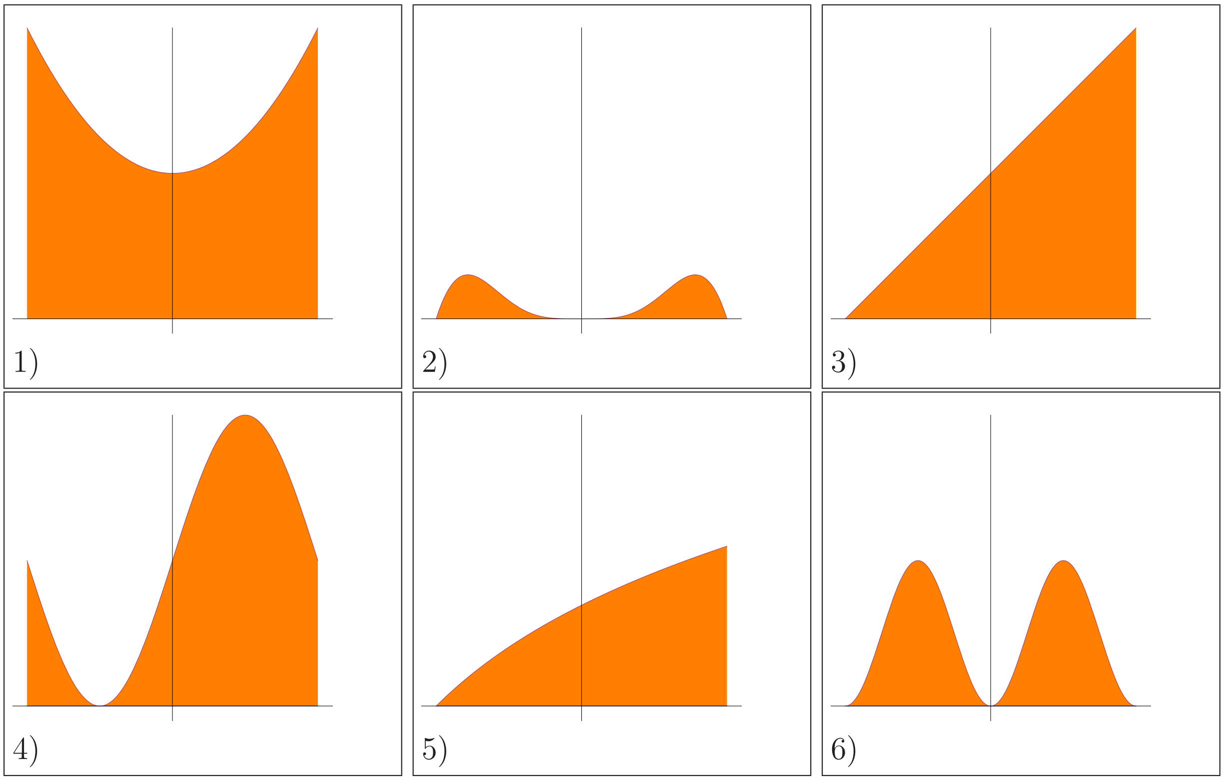
**Solution:**
They are and they are magic too.

Problem 2) Matching problem (10 points) No justifications are needed.

a) Match the following integrals with the graphs and (possibly signed) areas.

| Integral | Enter 1-6 |
|---|---|
| $\int_{-1}^{1} \sin(\pi x) x^3 \, dx$. | |
| $\int_{-1}^{1} \log(x+2) \, dx$. | |
| $\int_{-1}^{1} x + 1 \, dx$. | |

| Integral | Enter 1-6 |
|---|---|
| $\int_{-1}^{1} (1 + \sin(\pi x)) \, dx$. | |
| $\int_{-1}^{1} \sin^2(x) \, dx$. | |
| $\int_{-1}^{1} x^2 + 1 \, dx$. | |




**Solution:**
2) 5) 3) and 4) 6) 1)

Problem 3) Matching problem (10 points) No justifications are needed.

Determine from each of the following functions, whether discontinuities appears at $x = 0$ and if, which of the three type of discontinuities it is at 0.

| Function | Jump discontinuity | Infinity | Oscillation | No discontinuity |
|---|---|---|---|---|
| $f(x) = \log(\|x\|^5)$ | | | | |
| $f(x) = \cos(5/x))$ | | | | |
| $f(x) = \cot(1/x)$ | | | | |
| $f(x) = \sin(x^2)/x^3$ | | | | |
| $f(x) = \arctan(\tan(x - \pi/2))$ | | | | |
| $f(x) = 1/\tan(x)$ | | | | |
| $f(x) = 1/\sin(x)$ | | | | |
| $f(x) = 1/\sin(1/x)$ | | | | |
| $f(x) = \sin(\exp(x))/\cos(x)$ | | | | |
| $f(x) = 1/\log \|x\|$ | | | | |

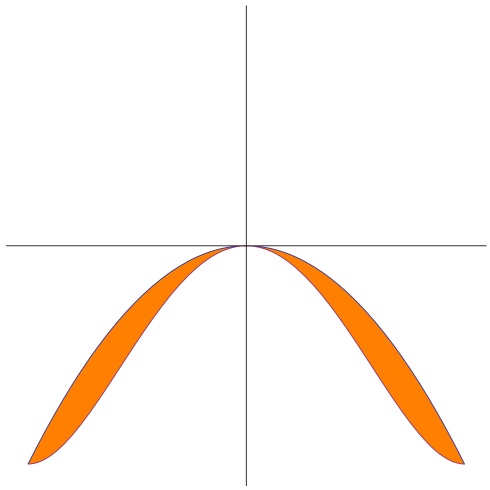
**Solution:**

| Function | Jump discontinuity | Infinity | Oscillation | No discontinuity |
|---|---|---|---|---|
| $f(x) = \log(\|x\|^5)$ | | X | | |
| $f(x) = \cos(5/x))$ | | | X | |
| $f(x) = \cot(1/x)$ | | | X | |
| $f(x) = \sin(x^2)/x^3$ | | X | | |
| $f(x) = \arctan(\tan(x - \pi/2))$ | X (*) | | | |
| $f(x) = 1/\tan(x)$ | | X | | |
| $f(x) = 1/\sin(x)$ | | X | | |
| $f(x) = 1/\sin(1/x)$ | | | X | |
| $f(x) = \sin(\exp(x))/\cos(x)$ | | | | X |
| $f(x) = 1/\log \|x\|$ | | | | X |

(*) this one is a bit tricky and be correct when marking it "no discontinuity" . The reason is that $f(x) = x - \pi/2$ But if arctan is confined to the branch $(-\pi/2, \pi/2)$, then the function will have jumps.

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the graphs of the function $f(x) = x^4 - 2x^2$ and the function $g(x) = -x^2$.



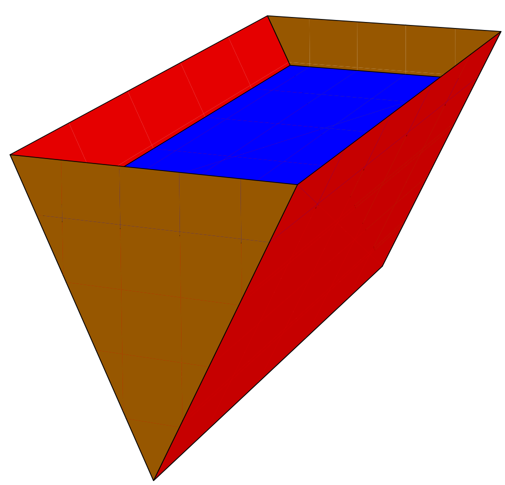
**Solution:**
The difficulty is to see what is above and what is below. The function $-x^2$ is above. The picture has helped us here but we could also plug in some values like $x = 1/2$ to see. Furthermore, we also have to find the intersection points which are $-1$ and $1$. Now we can set up the integral

$$\int_{-1}^{1} -x^2 - (x^4 - 2x^2)\, dx = 8/5 \; .$$

Problem 5) Volume computation (10 points)

A farmer builds a bath tub for his warthog "Tuk". The bath has triangular shape of length 10 for which the width is $2z$ at height $z$. so that when filled with height $z$ the surface area of the water is $20z$. If the bath has height 1, what is its volume?

P.S. Don't ask how comfortable it is to soak in a bath tub with that geometry. The answer most likely would be "Noink Muink".



**Solution:**

$$\int_0^1 20z\ dz = 20z^2/2|0^1 = 10\ .$$

Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (3 points) $\int_1^2 \sqrt{x} + x^2 - 1/\sqrt{x} + 1/x\ dx$.

b) (3 points) $\int_1^2 2x\sqrt{x^2-1}\ dx$

c) (4 points) $\int_1^2 2/(5x-1)\ dx$

**Solution:**
a) $x^{3/2}(2/3) + x^3/3 - x^{1/2}2 + \log(x)|_1^3 = -1/3 + 10\sqrt{2}/3 + \log(2)$.
b) $(x^2-1)^{3/2}(2/3)|_1^2 = 2\sqrt{3}$.
c) $2\log(5x-1)/5|_1^2 = 2(4/5)\log(3/2)$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (3 points) $\int \frac{3}{1+x^2} + x^2\ dx$

b) (3 points) $\int \frac{\tan^2(x)}{\cos^2(x)}\ dx$

c) (4 points) $\int \log(5x)\ dx$.

**Solution:**
a) $3\arctan(x) + x^3/3 + C$.
b) $\tan^3(x)/3 + C$.
c) Use $u = 5x$, $du = 5dx$ and get $\int \log(u)\ du = (u\log(u) - u)/5 = x\log(5x) - x$.

Problem 8) Implicit differentiation/Related rates (10 points)

A juice container of volume $V = \pi r^2 h$ changes radius $r$ but keeps the height $h = 2$ fixed. Liquid leaves at a constant rate $V'(t) = -1$. At which rate does the radius of the bag shrink when $r = 1/2$?

**Solution:**
Differentiate the equation $V(r) = 2\pi r^2(t)$ and use the chain rule: $-1 = V'(r) = 4\pi r r'$. We get $r' = -1/(4\pi r) = -1/(2\pi)$.

Problem 9) Global extrema (10 points)

We build a chocolate box which has 4 cubical containers of dimension $x \times x \times h$. The total material is $f(x, h) = 4x^2 + 12xh$ and the volume is $4x^2h$. Assume the volume is 4, what geometry produces the minimal cost?


**Solution:**
$4x^2g = 4$ implies $h = 1/x^2$. We have to minimize the function $f(x) = 4x^2 + 12/x$. Its derivative is $8x - 12/x^2$ which is zero for $x = (3/2)^{1/3}$.

Problem 10) Integration techniques (10 points)

Which integration technique works? It is enough to get the right technique and give the first step, not do the actual integration:

a) (2 points) $\int (x^2 + x + 1)\sin(x)\, dx$.

b) (2 points) $\int x/(1 + x^2)\, dx$.

c) (2 points) $\int \sqrt{4 - x^2}\, dx$.

d) (2 points) $\int \sin(\log(x))/x$.

e) (2 points) $\int \frac{1}{(x-6)(x-7)}\, dx$.

**Solution:**
a) integration by parts. Take $u = x^2 + x + 1$ and $v = \sin(x)$.
b) substitution $1 + x^2 = u$.
c) trig substitution $x^2 = 4\sin^2(u)$.
d) substitution $\log(x) = u$.
e) partial fractions.

Problem 11) Hopital's rule (10 points)

Find the following limits as $x \to 0$ or state that the limit does not exist.

a) (2 points) $\frac{\tan(x)}{x}$

b) (2 points) $\frac{x}{\cos(x)-x}$.

c) (2 points) $x\log(1+x)/\sin(x)$.

d) (2 points) $x\log(x)$.

e) (2 points) $x/(1-\exp(x))$.

**Solution:**
a) Use Hopital and differentiate both sides. Leading to $1/1 = 1$. Alternatively, write it as $(1/\cos(x))\sin(x)/x$. Because $1/\cos(x) \to 1$ and $\sin(x)/x \to 1$, the limit is 1.
b) No Hopital is needed (because we do not divide by 0). The limit is 0.
c) Use Hopital, differentiate and see that the limit is 0.
d) Use Hopital, differentiate and see the limit is 0.
e) Use Hopital, the limit is $-1$.

Problem 12) Applications (10 points)

The cumulative distribution function on $[0, 1]$

$$F(x) = \frac{2}{\pi}\arcsin(\sqrt{x})$$

defines the **arc-sin** distribution.

a) Find the probability density function $f(x)$ on $[0, 1]$.

b) Verify that $\int_0^1 f(x)\, dx = 1$.
Remark. The arc sin distribution is important chaos theory and probability theory.

**Solution:**
a) the probability density function is the derivative which is

$$\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}\ .$$

b) Since we know the antiderivative already there is no need to integrate this again. We know it is $F(1) - F(0) = 1 - 0 = 1$.