Lecture 2: Worksheet

In this lecture, we get acquainted with the most important functions.

Trigonometric functions

The cosine and sine functions can be defined geometrically by the co-ordinates $(\cos(x), \sin(x))$ of a point on the unit circle. The tangent function is defined as $\tan(x) = \sin(x)/\cos(x)$.

$$\begin{aligned} \cos(x) &= \text{ adjacent side/hypothenuse} \\ \sin(x) &= \text{ opposite side/hypothenuse} \\ \tan(x) &= \text{ opposite side/adjadcent side} \end{aligned}$$

Pythagoras theorem gives us the important identity

$$\boxed{\cos^2(x) + \sin^2(x) = 1}$$

Define also $\cot(x) = 1/\tan(x)$. Less important but sometimes used are $\sec(x) = 1/\cos(x)$, $\csc(x) = 1/\sin(x)$.

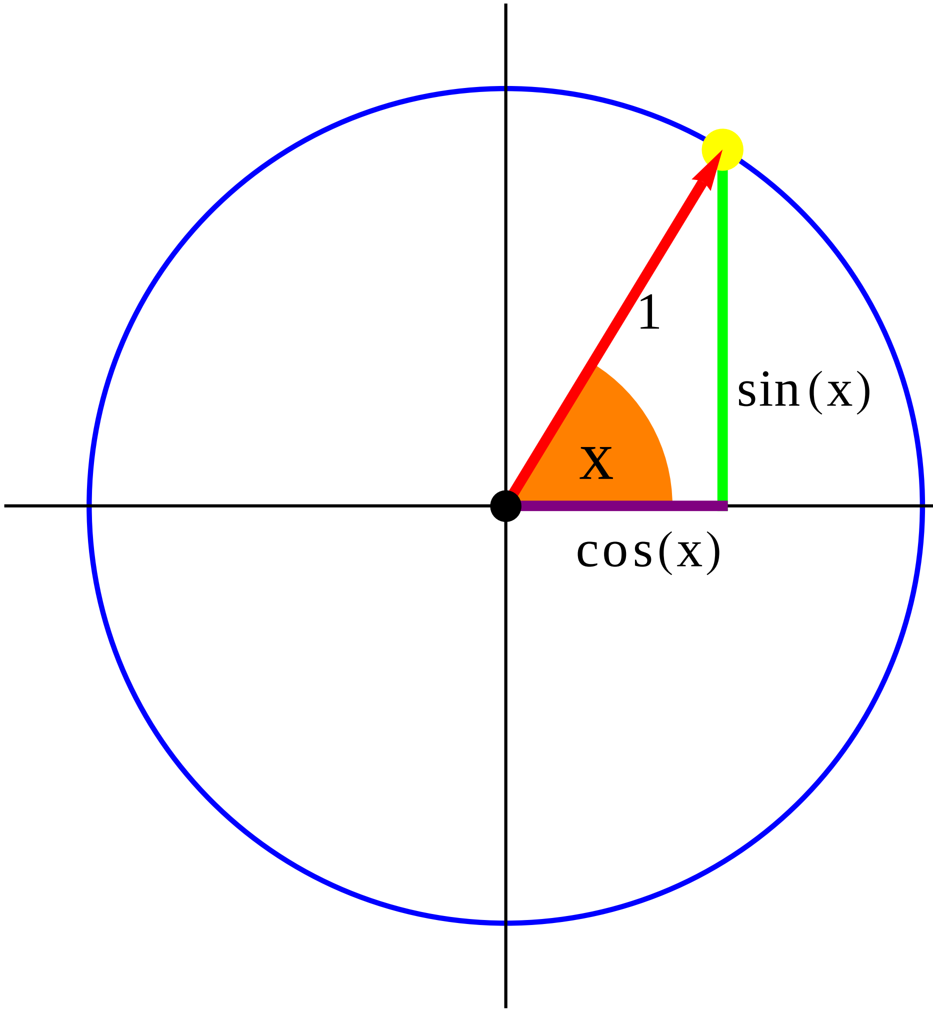
1 Find $\cos(\pi/3)$, $\sin(\pi/3)$.

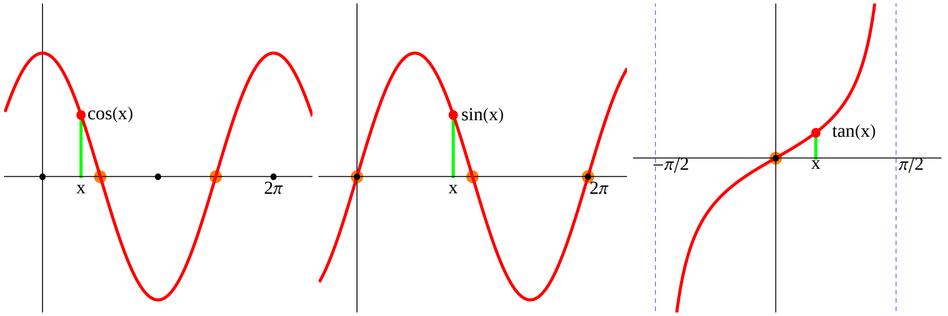
2 Where are the roots os cos and sin?

3 Find $\tan(3\pi/2)$ and $\cot(3\pi/2)$.

4 Find $\cos(3\pi/2)$ and $\sin(3\pi/2)$.

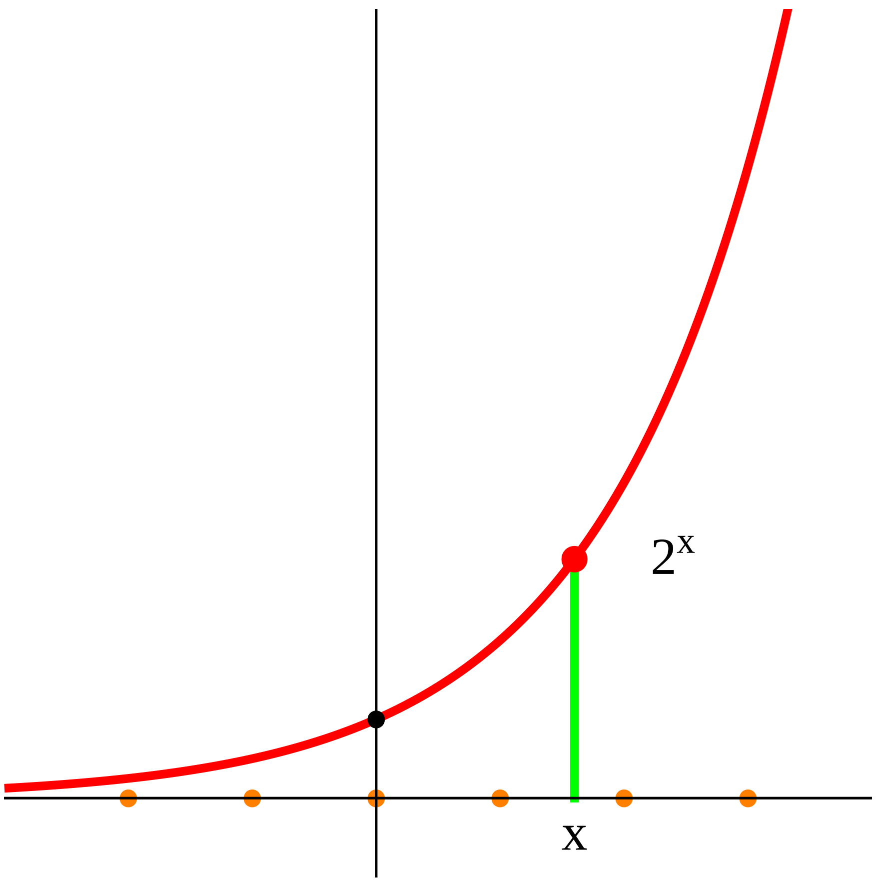
5 Find $\tan(\pi/4)$ and $\cot(\pi/4)$.





The exponential function

The function $f(x) = 2^x$ is first defined for positive integers like $2^{10} = 1024$, then for all integers with $f(0) = 1, f(-n) = 1/f(n)$. Using roots, it can be defined for rational numbers like $2^{3/2} = 8^{1/2} = \sqrt{8} = 2.828....$ Since the function 2^x is monotonone on the set of rationals, we can fill the gaps and define $f(x)$ for any real x. By taking square roots again and again for example, we see $2^{1/2}, 2^{1/4}, 2^{1/8}, ...$ we approach $2^0 = 1$.



There is nothing special about 2 and we can take any positive base a and define the exponential a^x. It satisfies $a^0 = 1$ and the remarkable rule:

$$\boxed{a^{x+y} = a^x \cdot a^y}$$

It is spectacular because it provides a link between addition and multiplication.

We will especially consider the exponential $\boxed{\exp_h(x) = (1+h)^{x/h}}$, where h is a positive parameter. This is a super cool exponential because it satisfies $\exp_h(x+h) = (1+h)\exp_h(x)$ so that

$$[\exp_h(x+h) - \exp_h(x)]/h = \exp_h(x) \ .$$

We will see this relation again. In modern language, we can say that "the quantum derivative of the quantum exponential is the function itself for any Planck constant h".

For $h = 1$, we have the function 2^x we have started with. In the limit $h \to 0$, we get the important exponential function $\exp(x)$ which we also call e^x. For $x = 1$, we get the **Euler number** $e = e^1 = 2.71828....$

1 What is 2^{-5}?

2 Find $2^{1/2}$.

3 Find $27^{1/3}$.

4 Why is $A = 2^{3/4}$ smaller than $B = 2^{4/5}$? Take the 20th power!

5 Assume $h = 2$ find $\exp_h(4)$.