Lecture 6: Fundamental theorem

Calculus is the theory of **differentiation** and **integration**. We explore this concept discrete setup and practice differentiation and integration. We fix a positive constant h and take differences and sums. Without taking limits, we prove already a fundamental theorem of calculus. We can so differentiate and integrate polynomials, exponentials and trigonometric functions. Later, we will do the same with real derivatives and integrals. But now, we can work with arbitrary continuous functions.

Given a function $f(x)$, define the **difference quotient**

$$\boxed{Df(x) = (f(x+h) - f(x))\,\frac{1}{h}}$$

If f is continuous then Df is a continuous. For shorthand, we call it simply "derivative". It will in the limit $h \to 0$ become the derivative we are going to define later in the course but we keep h constant and positive here.

1 Lets take the constant function $f(x) = 5$. We get $Df(x) = (f(x+h) - f(x))/h = (5-5)/h = 0$ everywhere. You can see that in general, if f is a constant function, then $Df(x) = 0$.

2 $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$. You see in general that if f is a linear function $f(x) = ax + b$, then $Df(x) = a$ is constant.

3 If $f(x) = ax + b$, then $Df(x) = \boxed{a}$.

For constant functions, the derivative is zero. For linear functions, the derivative is the slope.

4 For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h$ which is $\boxed{2x + h}$.

Given a function f, define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \leq jh < x$. That is, if k is such that $(k-1)h$ is the largest below x, then

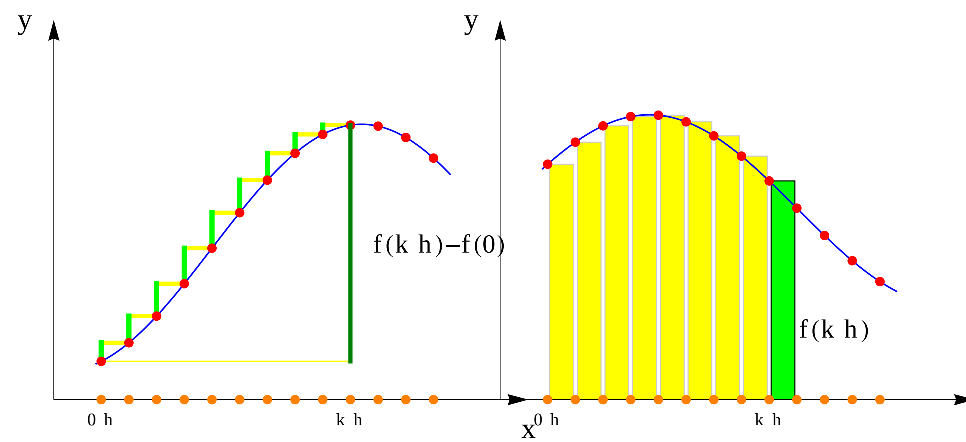
$$\boxed{Sf(x) = h[\, f(0) + f(h) + f(2h) + \cdots + f((k-1)h)\,]}$$

In short hand, we call Sf also the "integral" or "antiderivative" of f. It will become the integral in the limit $h \to 0$ later in the course.

5 Compute $Sf(x)$ for $f(x) = 1$. **Solution**. We have $Sf(x) = 0$ for $x \leq h$, and $Sf(x) = h$ for $h \leq x < 2h$ and $Sf(x) = 2h$ for $2h \leq x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but grows in quantized steps.

The difference	$Df(x)$	will become the	**derivative**	$f'(x)$.
The sum	$Sf(x)$	will become the	**integral**	$\int_0^x f(t)\,dt$.

Df	means	rise over run	and is close to	the	slope of the	graph of f.
Sf	means	areas of rectangles	and is close to	the	area under the	graph of f.



Theorem: Sum the differences and get $SDf(kh) = f(kh) - f(0)$

Theorem: Difference the sum and get $DSf(kh) = f(kh)$

6 For $f(x) = [x]_h^m = x(x-h)(x-2h)...(x-mh+h)$ we have

$f(x+h) - f(x) = (x(x-h)(x-2h)...(x-kh+2h))\,((x+h) - (x-mh+h)) = [x]^{m-1}hm$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. We have obtained the important formula $\boxed{D[x]^m = m[x]^{m-1}}$ We can establish from this differentiation formulas for **polynomials**.

7 If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$.

The fundamental theorem allows us to integrate and get the right values at the points k/n:

8 Find Sf for the same function. The answer is $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero. As a rescaled exponential, it is continuous and monotone. Indeed, using rules of the logarithm we can see $\exp_h(x) = e^{x(\log(1+h)/h)}$. We see that it agrees with the exponential function after a rescaling of x.

9 You have already computed the derivative in a homework. Lets do it again. The function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. **Solution**: $\exp_h(x+h) = (1+h)\exp(x)$ shows that. $\boxed{D\exp_h(x) = \exp_h(x)}$

10 Define $\exp(a \cdot x) = (1+ah)^{x/h}$. It satisfies $\boxed{D\exp_h(a \cdot x) = a\exp_h(a \cdot x)}$ We write a dot because $\exp_h(ax)$ is not equal to $\exp_h(a \cdot x)$. For now, only the differentiation rule for this function is important.

11 You can skip this example. But it is super cool: If we allow a to become complex, we get $\exp(1+ia)(1+aih)^{x/h}$. We still have $D\exp_h^{ai}(x) = ai\exp_h^{ai}(x)$. Taking real and imaginary parts define new functions $\exp_h^{ai}(x) = \cos_h(a\cdot x) + i\sin_h(a\cdot x)$. Despite the fact that we have for a moment escaped to the complex, these functions exist and morph into the familiar cos and sin functions for $h \to 0$. But in general, for any $h > 0$ and any a, we have $\boxed{D\cos_h(a\cdot x) = -a\sin_h(a\cdot x) \text{ and } D\sin_h(a\cdot x) = a\cos_h(a\cdot x).}$ If h is the size of the Planck constant $h = 1.616\cdot 10^{-35}m$, we would notice a difference between the cos and $\cos_h$ only if an x-ray traveling for 13 billion years. It would appear as a gamma ray burst.

Homework

We leave the h away in this homework. To have more fun, also define $\log_h$ as the inverse of $\exp_h$ and define $1/[x]_h = D\log_h(x)$ for $x > 0$. If we start integrating from 1 instead of 0 as usual we write $S_1 f$ and get $S_1 1/[x]_h = \log_h(x)$. We also write simply x^n for $[x]_h^n$ and write $\exp(a\cdot x) = e^{a\cdot x}$ instead of $\exp_h^a(x)$ and $\log(x)$ instead of $\log_h(x)$ because we are among friends. Use the differentiation and integration rules on the right to find derivatives and integrals of the following functions. You are actually doing cutting edge calculus. But you could solve this problem set also if you have already done some calculus and know the derivatives of the basic functions.

1 Find the derivatives $Df(x)$ of the following functions:

a) $f(x) = x^7 - 3x^4 + 5x + 1$
a) $f(x) = x^2 + 8\log(x)$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Df(0)$?

2 Find the integrals $Sf(x)$ of the following functions:

a) $f(x) = x^7$.
b) $f(x) = x^2 + 6x^7 - x$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Sf(1)$ in the case $h = 1$?

3 Find the derivatives $Df(x)$ of the following functions

a) $f(x) = \exp(3\cdot x) + x^6$
b) $f(x) = 4\exp(-3\cdot x) + 9x^6$
c) $f(x) = -\exp(5\cdot x) + x^6$

4 Find the integrals $Sf(x)$ of the following functions

a) $f(x) = \exp(6\cdot x) - 3x^6$
b) $f(x) = \exp(8\cdot x) + x^6$
c) $f(x) = -\exp(5\cdot x) + x^6$

5 Define $f(x) = \sin(4\cdot x) - \exp(2\cdot x) + x^4$. a) Find $Df(x)$
b) Find $Sf(x)$
c) Find $Sf(1)$ in the limit $h \to 0$.

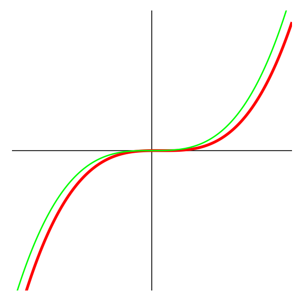
All calculus on 1/3 page

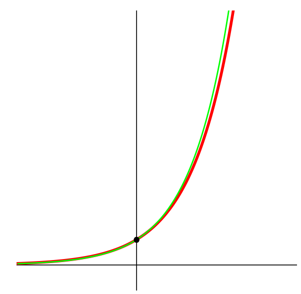
Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

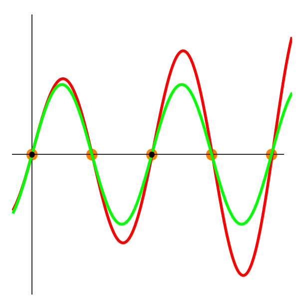
Differentiation rules	Integration rules (for $x = kh$)
$Dx^n = nx^{n-1}$	$Sx^n = x^{n+1}/(n+1)$
$De^{a\cdot x} = ae^{a\cdot x}$	$Se^{a\cdot x} = e^{a\cdot x}/a$
$D\cos(a\cdot x) = -a\sin(a\cdot x)$	$S\cos(a\cdot x) = \sin(a\cdot x)/a$
$D\sin(a\cdot x) = a\cos(a\cdot x)$	$S\sin(a\cdot x) = -\cos(a\cdot x)/a$
$D\log(x) = 1/x$	$S\frac{1}{x} = \log(x)$

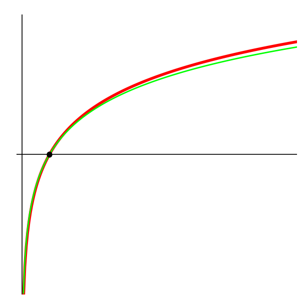
Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x+h)$.

Pictures


$[x]_h^3$ for $h = 0.1$


$\exp_h(x)$ for $h = 0.1$


$\sin_h(x)$ for $h = 0.1$


$\log_h(x)$ for $h = 0.1$