Lecture 6: Worksheet

The exponential function

We illuminate the fundamental theorem for the exponential function $\exp(x) = (1+h)^{x/h}$. We look again at the case $h = 1$ in which case $\exp(x) = 2^x$ maps positive integers to positive integers.

$$D\exp(x) = \exp(x) \ .$$

From the fundamental theorem, we get $SD\exp(x) = S\exp(x) = \exp(x) - \exp(0)$ we see

$$S\exp(x) = \exp(x) - 1 \ .$$

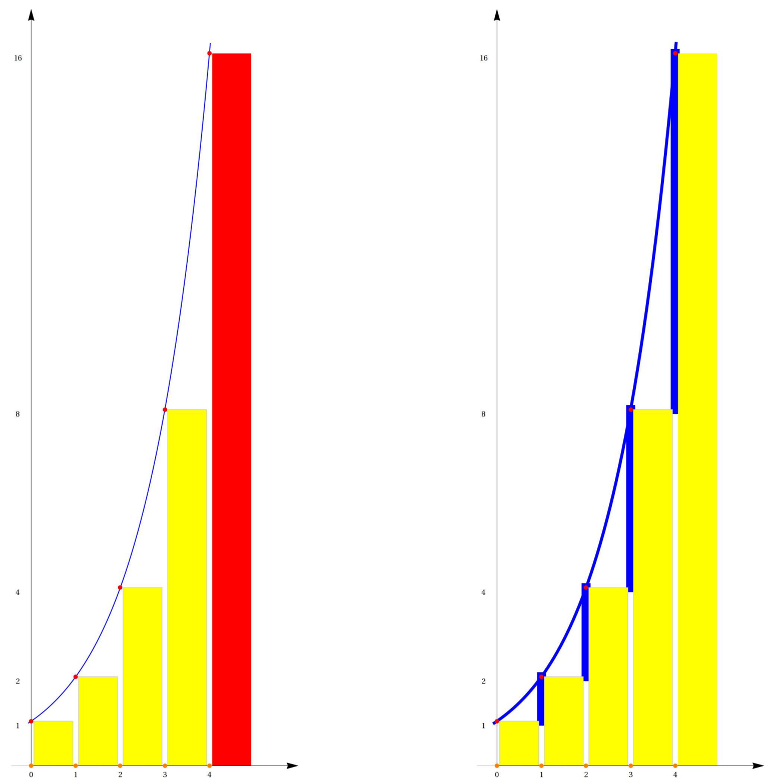
In other words, for the exponential function, we know both the derivative and the integral.

1 The formula $D\exp(x) = \exp(x)$ tells for $x = 3$ that $16-8 = 8$.

2 The formula $S\exp(x) = \exp(x) - 1$ tells for $x = 4$ that $1+2+4+8 = 16-1$.

3 The identity $DS\exp(x) = \exp(x)$ is illustrated in the left picture for $x = 4$.

4 The identity $SD\exp(x) = \exp(x) - 1$ is illustrated to the right for $x = 4$.



Here are some worked out examples, similar to what we expect you to do for the homework of lecture 6: The homework should be straightforward, except when finding $Sf(x)$, we want to add a constant such that $Sf(0) = 0$. In general, you will not need to evaluate functions and can leave terms like $\sin(5 \cdot x)$ as they are. If you have seen calculus already, then you could do this exercice by writing

$$\frac{d}{dx} f(x)$$

instead of $Df(x)$ and by writing

$$\int_0^x f(x)\, dx$$

instead of $Sf(x)$. Since we did not introduce the derivative df/dx nor the integral $\int_0^x$ yet, for now, just use the differentiation and integrations rules in the box to the right to solve the problems.

1 **Problem:** Find the derivative $Df(x)$ of the function $f(x) = \sin(5 \cdot x) + x^7 + 3$.
Answer: From the differentiation rules, we know $\boxed{Df(x) = 5\cos(5 \cdot x) + 7x^6}$.

2 **Problem:** Find the derivative $Df(0)$ of the same function $f(x) = \sin(5 \cdot x) + 5x^7 + 3$.
Answer: We know $Df(x) = 5\cos(5 \cdot x) + 35x^6$. Plugging in $x = 0$ gives $\boxed{5}$.

3 **Problem:** Find the integral $Sf(x)$ of the function $f(x) = \sin(5 \cdot x) + 5x^7 + 3$.
Answer: From the integration rules, we know $\boxed{Sf(x) = -\cos(5 \cdot x)/5 + 5x^8/8 + 3x}$.

4 **Problem:** Find the integral $Sf(1)$ of the function $f(x) = x^2 + 1$.
Answer: From the integration rules, we know $Sf(x) = x^3/3 + x$. Plugging in $x = 1$ gives $\boxed{1/3 + 1}$ if we use the functions in the limit $h \to 0$. For positive h, we have to evaluate $x(x - h)(x - 2h)/3 + x$ for $x = 1$ which is $(1 - h)(1 - 2h)/3 + 1$

5 **Problem:** Find the integral $Sf(1)$ of the function $f(x) = \exp(4 \cdot x)$.
Answer: From the integration rules, we know $Sf(x) = \exp(4 \cdot x)/4 - 1/4$. We have added a constant such that $Sf(0) = 0$. Plugging in $x = 1$ gives $\boxed{\exp(4)/4 - 1/4}$.

6 **Problem:** Assume $h = 1/1000$. Determine the value of

$$\frac{1}{1000}[f(\frac{0}{1000}) + f(\frac{1}{1000}) + ... + f(\frac{999}{1000})]$$

for the function $f(x) = -\sin(7x) + \exp(3x)$.
Answer: The problem asks for $Sf(1)$. We first compute $Sf(x)$ taking care that $Sf(0) = 0$.

$$Sf(x) = \cos(7x)/7 + \exp(3x)/3 - (1/7 + 1/3)\ .$$

Now plug in $x = 1$ to get $\boxed{\cos(7)/7 + \exp(3)/3 - (1/7 + 1/3)}$.