

## Lecture 9: The product rule

In this lecture, we look at the derivative of a product of functions. The product rule is also called **Leibniz rule** named after **Gottfried Leibniz**, who found it in 1684. It is an important rule because it allows us to differentiate many more functions. We will be able to compute so the derivative of  $f(x) = x \sin(x)$  for example without taking the limit the limit  $\lim(f(x+h) - f(x))/h$ . We are too lazy for that. Lets start with the identity



$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)] .$$

It can be written as  $D(fg) = Dfg^+ + fDg$  with  $g^+(x) = g(x+h)$ . This **quantum Leibniz rule** can also be seen geometrically: the rectangle of area  $(f + df)(g + dg)$  is the union of rectangles with area  $f \cdot g$ ,  $f \cdot dg$  and  $df \cdot g^+$ . Divide this relation by  $h$  to see

$$\begin{aligned} \frac{[f(x+h) - f(x)]}{h} \cdot g(x+h) &\rightarrow f'(x) \cdot g(x) \\ f(x) \cdot \frac{[g(x+h) - g(x)]}{h} &\rightarrow f(x) \cdot g'(x) . \end{aligned}$$

We get the extraordinarily important **product rule**:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) .$$

**Remark:** the discrete Leibniz rule is especially true in the **Babylonian calculus** developed in the first hour.

1 Find the derivative function  $f'(x)$  for  $f(x) = x^3 \sin(x)$ . **Solution:** We know how to differentiate  $x^3$  and  $\sin(x)$  so that  $f'(x) = 3x^2 \sin(x) + x^3 \cos(x)$ .

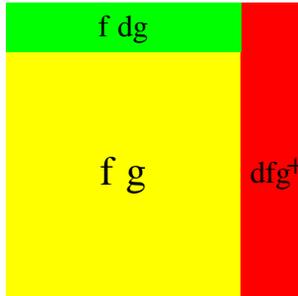
2 While we know

$$\frac{d}{dx}x^5 = 5x^4 ,$$

lets compute this with the Leibniz rule and write  $x^5 = x^3 \cdot x^2$ . We have

$$\frac{d}{dx}x^3 = 3x^2, \frac{d}{dx}x^2 = 2x .$$

The Leibniz rule gives us  $d/dx^5 = 3x^4 + 2x^4 = 5x^4$ .



3 Lets look at a few derivatives related to functions where we know the answer already but where we can check things using the product formula:

- $\frac{d}{dx}(x^3 \cdot x^5)$
- $\frac{d}{dx}e^{3x}e^{5x}$
- $\frac{d}{dx}\sqrt{x}/\sqrt{x}$
- $\frac{d}{dx}\sin(x)\cos(x)$

Before we look at the quotient rule which allows to differentiate  $f(x)/g(x)$  we can also write the later as  $f(x) \cdot 1/g(x)$  and use a rule telling us how to differentiate  $1/g(x)$ . This is the **reciprocal rule**:

If  $g(x) \neq 0$ , then

$$\frac{d}{dx} \frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2} .$$

In order to see this  $h = 1/g$  and differentiate the equation  $1 = g(x)h(x)$  on both sides. The product rule gives  $0 = g'(x)h(x) + g(x)h'(x)$  so that  $h'(x) = -h(x)g'(x)/g(x) = -g'(x)/g^2(x)$ .

4 Find the derivative of  $f(x) = 1/x^4$ . **Solution:**  $f'(x) = -4x^3/x^8 = -4/x^5$ . The same computation shows that  $\frac{d}{dx}x^n = nx^{n-1}$  holds for all integers  $n$ .

The formula  $\frac{d}{dx}x^n = nx^{n-1}$  holds for all integers  $n$ .

The **quotient rule** is obtained by applying the product rule to  $f(x) \cdot (1/g(x))$  and using the reciprocal rule:

If  $g(x) \neq 0$ , then

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{[f'(x)g(x) - f(x)g'(x)]}{g^2(x)} .$$

5 Find the derivative of  $f(x) = \tan(x)$ . **Solution:** because  $\tan(x) = \sin(x)/\cos(x)$  we have

$$\tan'(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} .$$

6 Find the derivative of  $f(x) = \frac{2-x}{x^2+x^4+1}$ . **Solution.** We apply the quotient rule and get  $[(-1)x^2 + x^4 + 1 + (2-x)(2x+4x^3)]/(x^2+x^4+1)$ .

Here are some more problems with solutions:

- 7 Find the second derivative of  $\tan(x)$ . **Solution.** We have already computed  $\tan'(x) = 1/\cos^2(x)$ . Differentiate this again with the quotient rule gives

$$\frac{-\frac{d}{dx} \cos^2(x)}{\cos^4(x)}.$$

We still have to find the derivative of  $\cos^2(x)$ . The product rule gives  $\cos(x)\sin(x) + \sin(x)\cos(x) = 2\cos(x)\sin(x)$ . Our final result is

$$2\sin(x)/\cos^3(x).$$

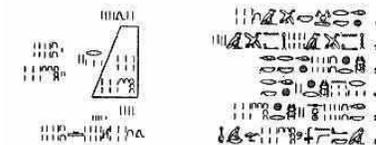
- 8 A cylinder has volume  $V = \pi r^2 h$ , where  $r$  is the radius and  $h$  is the height. Assume the radius grows like  $r(t) = 1 + t$  and the height shrinks like  $1 - \sin(t)$ . Does the volume grow or decrease at  $t = 0$ ?

**Solution:** The volume  $V(t) = \pi(1 + t)^2(1 - \sin(t))$  is a product of two functions  $f(t) = \pi(1 + t)^2$  and  $g(t) = (1 - \sin(t))$ . We have  $f(0) = 1, g'(0) = 2, f'(0) = 2, g(0) = 1$ . The product rule gives  $V'(0) = \pi \cdot 1 \cdot (-1) + \pi \cdot 2 \cdot 1 = \pi$ . The volume increases in volume at first.

On the **Moscow papyrus** dating back to 1850 BC, the general formula  $V = h(a^2 + ab + b^2)/3$  for a truncated pyramid with base length  $a$ , roof length  $b$  and height  $h$  appeared. Assume  $h(t) = 1 + \sin(t), a(t) = 1 + t, b(t) = 1 - 2t$ . Does the volume of the truncated pyramid grow or decrease at first? **Solution.** We could fill in



- 9  $a(t), b(t), h(t)$  into the formula for  $V$  and compute the derivative using the product rule. A bit faster is to write  $f(t) = a^2 + ab + b^2 = (1 + t)^2 + (1 - 3t)^2 + (1 + t)(1 - 3t)$  and note  $f(0) = 3, f'(0) = -6$  then get from  $h(t) = (1 + \sin(t))$  the data  $h(0) = 1, h'(0) = 1$ . So that  $V'(0) = (h'(0)f(0) - h(0)f'(0))/3 = (1 \cdot 3 - 1(-6))/3 = -1$ . The pyramid shrinks in volume at first.



- 10 We pump up a balloon and let it fly. Assume that the thrust increases like  $t$  and the resistance decreases like  $1/\sqrt{1-t}$  since the balloon gets smaller. The distance traveled is  $f(t) = t/\sqrt{1-t}$ . Find the velocity  $f'(t)$  at time  $t = 0$ .

## Homework

- 1 Find the derivatives of the following functions:

- $f(x) = \sin(5x)\cos(2x)$ .
- $f(x) = \cos^2(x)/x^2$ .
- $f(x) = x^2\sin(x)\cos(x)$ .
- $f(x) = 1/\sqrt{x}$ .
- $f(x) = \cot(x) + \tan(x)$ .

- 2 a) Verify that for  $f(x) = g(x)h(x)k(x)l(x)$  the formula  $f' = g'hkl + gh'kl + ghk'l + ghkl'$  holds.

b) Verify the following formula for derivative of  $f(x) = g(x)^4$ :  $f'(x) = 4g^3(x)g'(x)$ . We will derive this later using the chain rule. Don't use that rule yet.

- 3 If  $f(x) = \text{sinc}(x) = \sin(x)/x$ , find its derivative  $g(x) = f'(x)$  and then the derivative of  $g(x)$ . Then evaluate it numerically at  $x = 0$ .

- 4 Find the derivative of

$$\frac{\sin(x)}{1 + \cos(x) + \frac{x^4}{\sin(x)}}$$

at  $x = 0$ .

- 5 a) We have already computed the derivative of  $g(x) = \sqrt{x}$  in the last homework. Introduce  $f(x) = x^{1/4}$  and apply the product rule to  $g(x) = f(x)f(x)$  to get the derivative of  $f$ .

b) Use problem 2b) applied to the identity  $x = f(x)^4$  to get the derivative of  $f$ .

Remark: Also this last problem 5) is a preparation for the chain rule, we see next Monday. Avoid using the chain rule already here.