

## Lecture 10: The chain rule

How do we take the derivative of a composition of functions like  $f(x) = \sin(x^7)$ ? The product rule does not work here. The functions are "chained", we evaluate first  $x^7$  then apply  $\sin$  to it. In order to differentiate, we take the derivative of the first function we evaluate  $x^7$  then multiply this with the derivative of the function  $\sin$  at  $x^7$ . The answer is  $7x^6 \cos(x^7)$ .

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

The chain rule follows from the identity

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h}.$$

Write  $H(x) = g(x+h) - g(x)$  in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h}.$$

As  $h \rightarrow 0$ , we also have  $H \rightarrow 0$  and the first part goes to  $f'(g(x))$  and the second factor has  $g'(x)$  as a limit.

**1** Find the derivative of  $f(x) = (4x - 1)^{17}$ . **Solution** The inner function is  $g(x) = 4x - 1$ . It has the derivative 4. We get therefore  $f'(x) = 17(4x - 1)^6 \cdot 4 = 28(4x - 1)^6$ . Remark. We could have expanded out the power  $(4x - 1)^{17}$  first and avoided the chain rule. Avoiding the **chain rule** is called the **pain rule**.

**2** Find the derivative of  $f(x) = \sin(\pi \cos(x))$  at  $x = 0$ . **Solution:** applying the chain rule gives  $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$ .

**3** For linear functions  $f(x) = ax + b$ ,  $g(x) = cx + d$ , the chain rule can readily be checked. We have  $f(g(x)) = a(cx + d) + b = acx + ad + b$  which has the derivative  $ac$ . Indeed this is the definition of  $f$  times the derivative of  $g$ . You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

**4** Find the derivative of the natural logarithm function  $\log(x)$ . **Solution** Differentiate the identity  $\exp(\log(x)) = x$ . On the right hand side we have 1. On the left hand side the chain rule gives  $\exp(\log(x)) \log'(x) = x \log'(x) = 1$ . Therefore  $\log'(x) = 1/x$ .

<sup>1</sup>We always write  $\log(x)$  for the natural log. The  $\ln$  notation is old fashioned and only used in obscure places like calculus books and calculators from the last millenium.

$$\frac{d}{dx} \log(x) = 1/x.$$

Denote by  $\arccos(x)$  the inverse of  $\cos(x)$  on  $[0, \pi]$  and with  $\arcsin(x)$  the inverse of  $\sin(x)$  on  $[-\pi/2, \pi/2]$ .

**5** Find the derivative of  $\arcsin(x)$ . **Solution.** We write  $x = \sin(\arcsin(x))$  and differentiate.

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

**6** Find the derivative of  $\arccos(x)$ . **Solution.** We write  $x = \cos(\arccos(x))$  and differentiate.

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

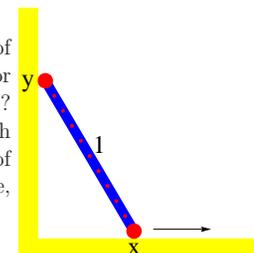
**7**  $f(x) = \sin(x^2 + 3)$ . Then  $f'(x) = \cos(x^2 + 3)2x$ .

**8**  $f(x) = \sin(\sin(\sin(x)))$ . Then  $f'(x) = \cos(\sin(\sin(x))) \cos(\sin(x)) \cos(x)$ .

Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

**9** Lets compute the derivative of  $\sin(\sqrt{x^5 - 1})$  for example. **Solution:** This is a composition of three functions  $f(g(h(x)))$ , where  $h(x) = x^5 - 1$ ,  $g(x) = \sqrt{x}$  and  $f(x) = \sin(x)$ . The chain rule applied to the function  $\sin(x)$  and  $\sqrt{x^5 - 1}$  gives  $\cos(\sqrt{x^5 - 1}) \frac{d}{dx} \sqrt{x^5 - 1}$ . Apply now the chain rule again for the derivative on the right hand side.

Here is a famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point  $(0, y)$  on the wall with  $(x, 0)$  on the floor. We want to express  $y$  as a function of  $x$ . We have  $y = f(x) = \sqrt{1 - x^2}$ . Taking the derivative, assuming  $x' = 1$  gives  $f'(x) = -2x/\sqrt{1 - x^2}$ .



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle**  $\theta = \arcsin((2v^2/(3g))^{2/3})$ , where  $g$  is the gravitational acceleration and  $v = x'$  is the velocity.

**11** For the brave: find the derivative of  $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))$ )).

## Practice

- 12 Take the derivative of  $f_3(x) = e^{e^{e^x}}$ .

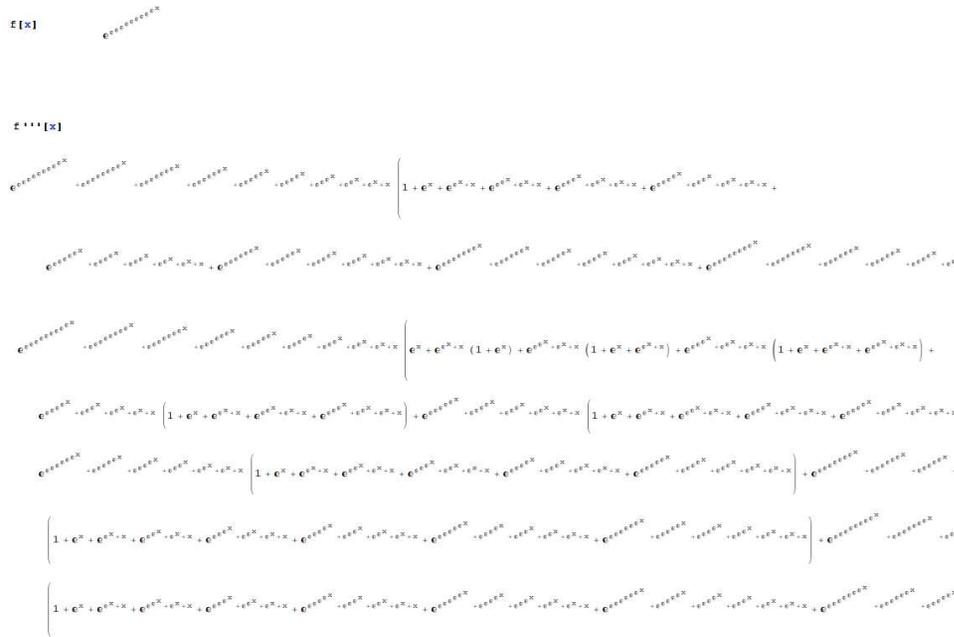
**Solution** We can also write this as  $\exp(\exp(\exp(x)))$ . The derivative is

$$\exp(\exp(\exp(x))) \exp(\exp(x)) \exp(x) .$$

- 13 Lets push that to the extreme and differentiate

$$f(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x))))))))))$$

three times. No kidding, here is the picture of the formula obtained when running this in Mathematica:



- 14 Find the derivative of  $1/\sin(x)$  using the quotient rule.

**Solution**  $-\cos(x) \cdot 1/\sin^2(x)$ .

- 15 Find the derivative of  $f(x) = 1/\sin(x)$  using the chain rule.

**Solution.** The outer function is  $f(x) = 1/x$ . Therefore  $f'(x) = -\cos(x)/\sin^2(x)$ .

## Homework

- 1 Find the derivatives of the following functions.

a)  $f(x) = \sin(\log(x))$

c)  $f(x) = \exp(1/(1+x))$

b)  $f(x) = \tan(x^7)$

d)  $(2 + \sin(x))^{-5}$

- 2 Find the derivatives of the following functions at  $x = 1$ .

a)  $f(x) = x^8 \log(x)$ . (where log is natural log)

c)  $(1 + x^2 + x^4)^{100}$

b)  $\sqrt{x^5 + 1}$

d)  $\frac{5x^4}{2\sqrt{x^5+1}}$

- 3 a) Find the derivative of  $f(x) = 1/x$  by differentiating the identity  $xf(x) = 1$ .

b) Find the derivative of  $f(x) = \operatorname{arccot}(x)$  by differentiating  $\cot(\operatorname{arccot}(x)) = x$ .

- 4 a) Find the derivative of  $f(x) = \sqrt{x}$  by differentiating the identity  $f(x)^2 = x$ .

b) Find the derivative of  $f(x) = x^{m/n}$  by differentiating the identity  $f(x)^n = x^m$ .

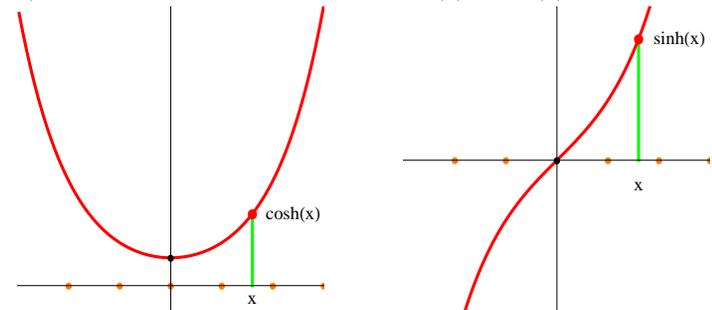
The function  $f(x) = [\exp(x) + \exp(-x)]/2$  is called  $\cosh(x)$ .

The function  $f(x) = [\exp(x) - \exp(-x)]/2$  is called  $\sinh(x)$ .

They are called **hyperbolic cosine** and **hyperbolic sine**. The first is even, the second is odd. You can see directly using  $\exp'(x) = \exp(x)$  and  $\exp'(-x) = -\exp(-x)$  that  $\sinh'(x) = \cosh(x)$  and  $\cosh'(x) = \sinh(x)$ . Furthermore  $\exp = \cosh + \sinh$  writes  $\exp$  as a sum of an even and odd function.

- 5 a) Find the derivative of the inverse  $\operatorname{arccosh}(x)$  of  $\cosh(x)$ .

b) Find the derivative of the inverse  $\operatorname{arsinh}(x)$  of  $\sinh(x)$ .



The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine.

