Math 1A: introduction to functions and calculus **Oliver Knill, 2013**

Lecture 12: Global extrema

In this lecture we are interested in the points where a function is maximal overall. These **global extrema** can occur at critical points of f or at the boundary of the domain, where f is defined.

> A point p is called a **global maximum** of f if $f(p) \geq f(x)$ for all x. A point p is called a **global minimum** of f if $f(p) \leq f(x)$ for all x.

How do we find global maxima? The answer is simple: make a list of all local extrema and boundary points, then pick the largest. Global maxima or minima do not need to exist however. The function $f(x) = x^2$ has a global minimum at $x = 0$ but no global maximum. The function $f(x) = x^3$ has no global extremum at all. We can however look at global maxima on finite intervals.

1 Find the global maximum of $f(x) = x^2$ on the interval $[-1, 2]$. **Solution**. We look for local extrema at critical points and at the boundary. Then we compare all these extrema to find the maximum or minimum. The critical points are $x = 0$. The boundary points are $-1, 2$. Comparing the values $f(-1) = 1, f(0) = 0$ and $f(4) = 4$ shows that f has a global maximum at 2 and a global minimum at 0.

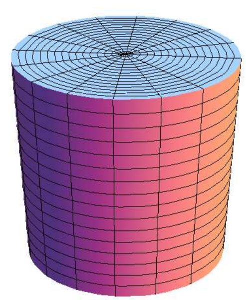
> **Extreme value theorem** A continuous function f on a finite interval $[a, b]$ attains a global maximum and a global minimum.

Here is the argument: Because the function is continuous, the image of the interval $[a, b]$ is a closed interval $[c, d]$. [1] There is a point such that $f(x) = c$, which is a global minimum and a point where $f(x) = d$ which is a global maximum.

Note that the global maximum or minimum can also also on the boundary or points where the derivative dos not exist.

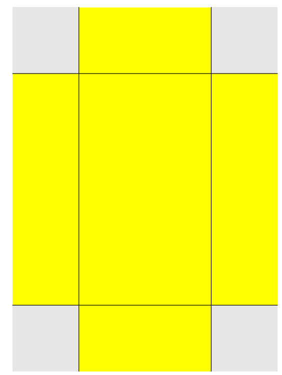
2 Find the global maximum and minimum of the function $f(x) = |x|$. The function has no absolute maximum as it goes to infinity for $x \to \infty$. The function has a global minimum at $x = 0$ but the function is not differentiable there. The point $x = 0$ is a point which does not belong to the domain of f'.

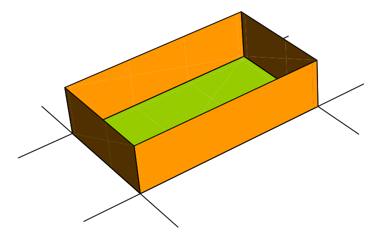
3 A **soda can** is a cylinder of volume $\pi r^2 h$. The surface area $2\pi r h + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is 2π, we can solve the equation for $h = (1 - r^2)/r = 1/r - r$ **Solution:** The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.



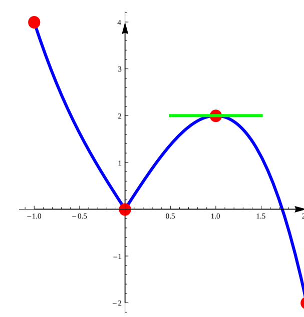
[1]This statement needs more justification but is intuitive enough that we can accept it.

4 Take a card of 2×2 inches. If we cut out 4 squares of equal side length x at the corners, we can fold up the paper to a tray with width $(2 - 2x)$ length $(2 - 2x)$ and height x. For which $x \in [0, 1]$ is the tray volume maximal?
Solution The volume is $f(x) = (2 - 2x)(2 - 2x)x$. To find the maximum, we need to compare the critical points which is at $x = 1/3$ and the boundary points $x = 0$ and $x = 1$.





5 Find the global maxima and minima of the function $f(x) = 3|x| - x^3$ on the interval $[-1, 2]$.
Solution. For $x > 0$ the function is $3x - x^3$ which can be differentiated. The derivative $3 - 3x^2$ is zero at $x = 1$. For $x < 0$ the function is $-3x - x^3$. The derivative is $-3 - x^2$ and has no root. The only critical points are 1. There is also the point $x = 0$ which is not in the domain where we can differentiate the function. We have to deal with this point separately. We also have to look at the boundary points $x = -1$ and $x = 2$. Making a list of function values at $x = -1, x = 0, x = 1, x = 2$ gives the maximum.



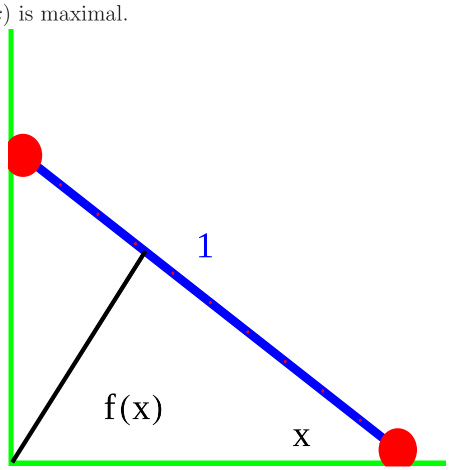
Homework

1 Find the global maxima and minima of the function $f(x) = 2x + (x-2)^3$ on the interval $[-1, 2]$.

2 Find the global maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 36x$ on the interval $[-4, 4]$

3 A candy manufacturer builds spherical candies. Its effectiveness is $A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r. Find the radius, where $f(r) = A(r) - V(r)$ has a global maximum for $r \geq 0$.

4 A ladder of length 1 is one side at a wall and on one side at the floor. First verify that the distance from the ladder to the corner is $f(x) = \sin(x)\cos(x)$. Find the angle x for which $f(x)$ is maximal.



5 a) The function $S(x) = -x \log(x)$ is called the **entropy function**. Find the probability $0 < x \leq 1$ which maximizes entropy. important principle in all science is

that nature tries to maximize entropy. In some sense we compute here the number of maximal entropy.
b) We can write $1/x^x = e^{-x \log(x)}$. Find the positive value x, where x^{-x} has a local maximum. [2]

Entropy has been introduced by Bolzman. It is important in physics and chemistry.

[2]We have used the identity $a^b = e^{b \log(a)}$