Lecture 14: Newton's method

Recall that a point a is called a **root** of a function f if $f(a) = 0$. We were able to find the roots of functions using a "divide and conquer" technique: start with an interval $[a, b]$ for which $f(a) < 0$ and $f(b) > 0$. If $f((a+b)/2)$ is positive, then use the interval $[a, (a+b)/2]$ otherwise $[(a+b)/2, b]$. After n steps, we are $(b-a)/2^n$ close to the root.
If the function f is differentiable, we can do much better. We can use the value of the derivative to get closer to a point $y = T(x)$. Lets find y. If we draw a tangent at $(x, f(x))$, then

$$f'(x) = \frac{f(x)}{x-y} .$$

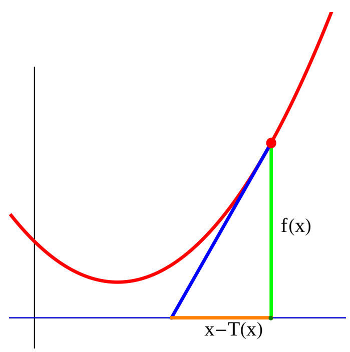
because $f'(x)$ is the slope of the tangent and the right hand side is "rise" over "run". If we solve for y we get

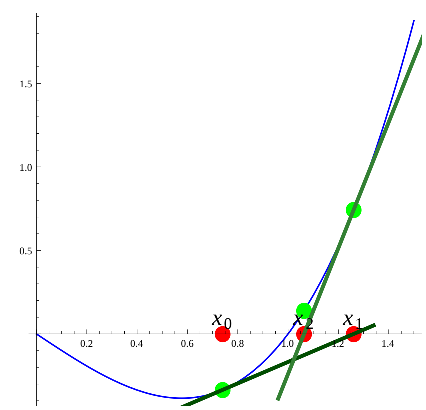
The **Newton map** is defined as

$$y = T(x) = x - \frac{f(x)}{f'(x)} .$$

Newton's method is the process of applying this map again and again until we are sufficiently close to the root. It is an extremely fast method to find the root of a function. Start with a point x, then compute a new point $x_1 = T(x)$, then $x_2 = T(x_1)$ etc.

If p is a root such that $f'(p) \neq 0$, and x_0 is close enough to p, then $x_1 = T(x), x_2 = T^2(x)$ converges to the root p.





1 If $f(x) = ax + b$, we reach the root in one step.

2 If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0 but not as fast as the method promises. Indeed, the root 0 is also a critical point of f. This slows us down.

3 The Newton map brings us to infinity if we start at a critical point.

Newton used this method to find the roots of polynomials. It is amazingly fast: Starting 0.1 close to the point, we have after one step 0.01 after 2 steps 0.0001 after 3 steps 0.00000001 and after 4 steps 0.0000000000000001.

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$ if we start sufficiently close to the root.

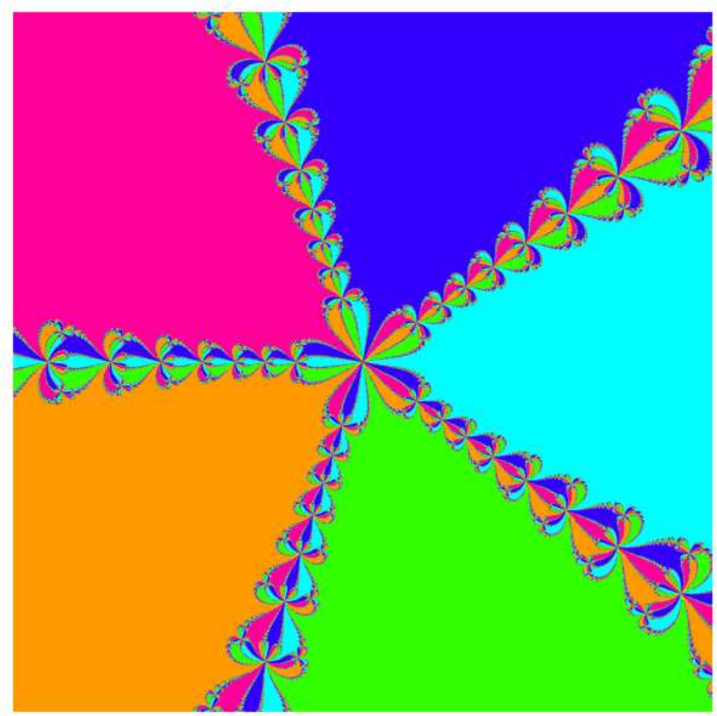
In 10 steps we can get a $2^{10} = 1024$ digits accuracy. Having a fast method to compute roots is useful. For example in computer graphics, where things can not be fast enough. Also in number theory, when working with integers having thousands of digits the Newton method can help. There is much theoretical use of the method. It goes so far as to explain stability of planetary motion or stability of plasma in fusion reactors.

4 Verify that the Newton map $T(x)$ in the case $f(x) = (x-1)^3$ has the property that we approach the root $x = 1$. **Solution**. You see that the approach is not that fast: we get $T(x) = x + (1-x)/3 = (1+2x)/3$. It converges exponentially fast, but not super exponential. The reason is that the derivative at $x - 1$ is not zero. That slows us down.

If we have several roots, and we start at some point, to which root will the Newton method converge? Does it at all converge? This is an interesting question. It is also historically intriguing because it is one of the first cases, where "chaos" can be observed at the end of the 19'th century.

5 Find the Newton map in the case $f(x) = x^5 - 1$. **Solution** $T(x) = x - (x^5 - 1)/(5x^4)$.

If we look for roots in the complex like for $f(x) = x^5 - 1$ which has 5 roots in the complex plane, the basin of attraction of each of the points is a complicated set, a so called **Newton fractal**. Here is the picture:



6 Lets compute $\sqrt{2}$ to 12 digits accuracy - by hand! We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned} T(1) &= 1 - (1-2)/2 = 3/2 \\ T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\ T(17/12) &= 577/408 \\ T(577/408) &= 665857/470832 \end{aligned}$$

This is already $1.6 \cdot 10^{-12}$ close to the real root!

7 To find the cube root of 10 we have to find a root of $f(x) = x^3 - 10$. The Newton map is $T(x) = x - (x^3 - 10)/(3x^2)$. If we start with $x = 2$, we get the following steps $2, 13/6, 3277/1521, 105569067476/49000820427$. After three steps we have a result which is already $2.2 \cdot 10^{-9}$ close to the root.

The Newton method is scrumtrulescent!

Will Ferrell's sketch: "Inside the Actors Studio" at Saturday Night live created the term "scrumtrulescent". See http://www.hulu.com/watch/3524

Homework

1 Find a formula for the Newton map $T(x) = x - f(x)/f'(x)$ in the following cases
a) $f(x) = (x-1)^2$
b) $f(x) = e^{3x}$
c) $f(x) = e^{-x^2}$
d) $f(x) = \tan(x)$.

2 a) The sinc function $f(x) = \sin(x)/x$ has a root between 1 and 4. We get closer to the root by doing a Newton step starting with $x = \pi/2$. Illustrate this step graphically and then add an other step.

3 The Newton map is handy to compute square roots. Assume we cant to find the square root of 99. We have to solve $\sqrt{99} = x$ or $f(x) = x^2 - 99 = 0$. Perform two Newton steps $T(x) = x - (x^2 - 99)/(2x)$ starting at $x = 10$.

4 a) Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$. What happens if you apply the Newton step again and again?
b) Find the Newton step $T(x)$ in general if $f(x) = x^\alpha$, where α is a real number.

5 a) Verify that the Newton map in the case $f(x) = (4 - 3/x)^{1/3}$ is the quadratic map $T(x) = 4x(1-x)$.
b) This is an example of a chaotic map. The Newton step does not converge. Apply a few steps with a calculator.