Lecture 21: Area computation

If $f(x) \geq 0$, then $\int_a^b f(x)\ dx$ is the **area under the graph** of $f(x)$ and above the interval $[a, b]$ on the x axes.

If the function is negative, then $\int_a^b f(x)\ dx$ is negative too and the integral is minus the area below the curve:

Therefore, $\int_a^b f(x)\ dx$ is the difference of the area above the graph minus the area below the graph.

More generally we can also look at areas sandwiched between two graphs f and g.

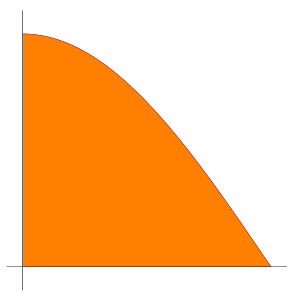
The area of a region G enclosed by two graphs $f \leq g$ and bound by $a \leq x \leq b$ is

$$\int_a^b g(x) - f(x)\ dx$$

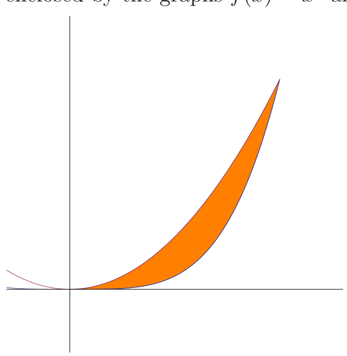
.

Make sure that if you have to compute such an integral that $g \geq f$ before giving it the interpretation of an area.

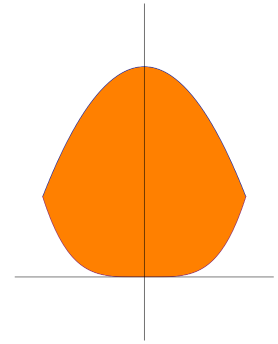
1 Find the area of the region enclosed by the x-axes, the y-axes and the graph of the cos function. **Solution**: $\int_0^{\pi/2} \cos(x)\ dx = 1$.



2 Find the area of the region enclosed by the graphs $f(x) = x^2$ and $f(x) = x^4$.



3 Find the area of the region enclosed by the graphs $f(x) = 1 - x^2$ and $g(x) = x^4$.

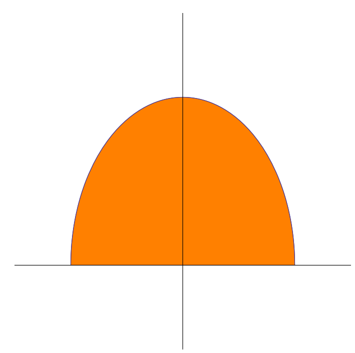


4 Find the area of the region enclosed by a half circle of radius 1. **Solution**: The half circle is the graph of the function $f(x) = \sqrt{1-x^2}$. The area under the graph is

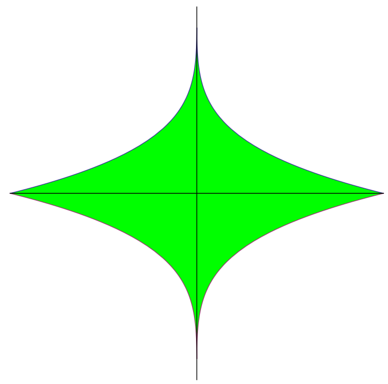
$$\int_{-1}^{1} \sqrt{1-x^2}\ dx\ .$$

Finding the anti-derivative is not so easy. We will find techniques to do so, for now we pop it together: we know that $\arcsin(x)$ has the derivative $1/\sqrt{1-x^2}$ and $x\sqrt{1-x^2}$ has the derivative $\sqrt{1-x^2} - x^2/\sqrt{1-x^2}$. The sum of these two functions has the derivative $\sqrt{1-x^2} - (1-x^2)/\sqrt{1-x^2} = 2\sqrt{1-x^2}$. We find the anti derivative to be $(x\sqrt{1-x^2} + \arcsin(x))/2$. The area is therefore

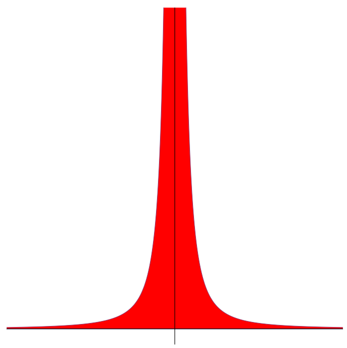
$$\frac{x\sqrt{1-x^2} + \arcsin(x)}{2}\Big|_{-1}^{1} = \frac{\pi}{2}\ .$$



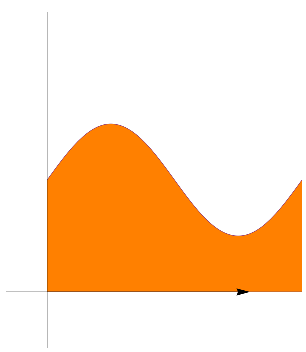
5 Find the area of the region between the graphs of $f(x) = 1 - |x|^{1/4}$ and $g(x) = -1 + |x|^{1/4}$.



6 Find the area under the curve of $f(x) = 1/x^2$ between -6 and 6. Solution. $\int_{-6}^{6} x^{-2}\, dx = -x^{-1}|_{-6}^{6} = -1/6 - 1/6 = -1/3$. There is something fishy with this computation because $f(x)$ is nonnegative so that the area should be positive. But we obtained a negative answer. What is going on?



7 Find the area between the curves $x = 0$ and $x = 2 + \sin(y)$, $y = 2\pi$ and $y = 0$. **Solution:** We turn the picture by 90 degrees so that we compute the area under the curve $y = 0, y = 2 + \sin(x)$ and $x = 2\pi$ and $x = 0$.



8 **The grass problem**. Find the area between the curves $|x|^{1/3}$ and $|x|^{1/2}$. **Solution**. This example illustrates how important it is to have a picture. This is good advise for any "word problem" in mathematics.

Use a picture of the situation while doing the computation.

Homework

1 Find the area of the bounded region enclosed by the graphs $f(x) = 2x^5 - 24x$ and $g(x) = 4x^2$.

2 Find the area of the region enclosed by the four lines $y = x, y = 3 - 2x, y = -2x, y = 3x - 1$.

3 Find the area of the region enclosed by the curves $x = 0, x = \pi/2, y = 4 + \sin(5x), y = 2 + \sin^2(2x)$.

4 Write down an integral which gives the area of the **area 51** region $x^2 + |y|^{51} \leq 1$ by writing the region as a sandwich between two graphs. Evaluate the integral numerically using Wolfram alpha, Mathematica or any other software.

5 The graphs $1 + \sin(x)$ and $2\cos(x) - 1$ intersect at $x = 0, 2\pi$ and a point between. They define a **humming bird** region,consisting of a larger region and a tail region. Find the area of each part and assume the bird has its eye closed.