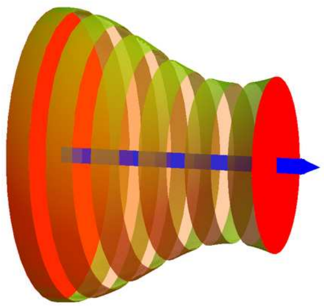
Math 1A: introduction to functions and calculus **Oliver Knill, 2013**

Lecture 22: Volume computation

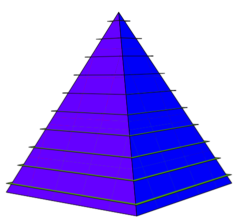
To compute the volume of a solid, we cut it into slices, where each slice is perpendicular to a given line x. If $A(x)$ is the area of the slice and the body is enclosed between a and b then $V = \int_a^b A(x)\, dx$ is the volume. Think of $A(x)dx$ as the volume of a slice. The integral adds them up.



1 Compute the volume of a pyramid with square base length 2 and height 2. **Solution**: we can assume the pyramid is built over the square $-1 \le x \le 1$ and $-1 \le y \le 1$. The cross section area at height h is $A(h) = (2-h)^2$. Therefore,

$$V = \int_0^2 (2-h)^2\, dh = \frac{8}{3} .$$

This is base area 4 times height 2 divided by 3.



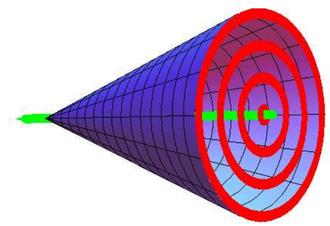
A **solid of revolution** is a surface enclosed by the surface obtained by rotating the graph of a function $f(x)$ around the x-axis.

The area of the cross section at x of a solid of revolution is $A(x) = \pi f(x)^2$. The volume of the solid is $\int_a^b \pi f(x)^2\, dx$.

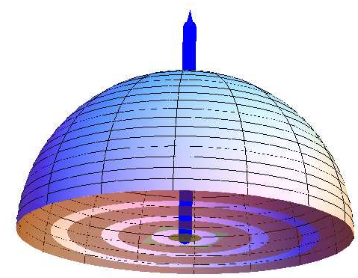
2 Find the volume of a round cone of height 2 and where the circular base has the radius 1. **Solution**. This is a solid of revolution obtained by rotation the graph of $f(x) = x/2$ around the x axes. The area of a cross section is $\pi x^2/4$. Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4\, dx = \frac{x^3}{4 \cdot 3}\Big|_0^2 = \frac{2\pi}{3} .$$

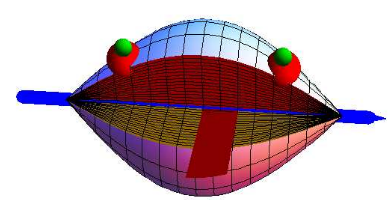
This is the height 2 times the base area π divided by 3.



3 Find the volume of a half sphere of radius 1. **Solution:** The area of the cross section at height h is $\pi(1-h^2)$.



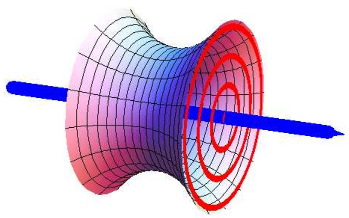
4 We rotate the graph of the function $f(x) = \sin(x)$ around the x axes. But now we cut out a slice of $60 = \pi/3$ degrees out. Find the volume of the solid.
Solution: The area of a slice without the missing piece is $\pi \sin^2(x)$. The integral $\int_0^\pi \sin^2(x)\, dx$ is $\pi/2$ as derived in the lecture. Having cut out 1/6'th the area is $(5/6)\pi \sin^2(x)$. The volume is $\int_0^\pi (5/6)\pi \sin(x)^2\, dx = (5/6)\pi^2/2$.



Homework

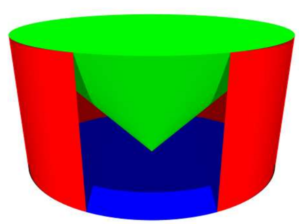
1 Find the volume of the paraboloid for which the radius at position x is $9-x^2$ and x ranges from 0 to 3.

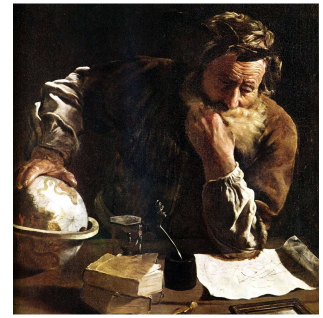
2 A **catenoid** is the surface obtained by rotating the graph of $f(x) = \cosh(x)$ around the x-axes. We have seen that the graph of f is the chain curve, the shape of a hanging chain. Find the volume of of the solid enclosed by the catenoid between $x = -2$ and $x = 2$.
Hint. You might want to check first the identity $2\cosh(x)^2 = 1 + \cosh(2x)$ using the definition $\cosh(x) = (\exp(x) + \exp(-x))/2$.



3 A **tomato** is given by $z^2 + x^2 + 4y^2 = 1$. If we slice perpendicular to the y axes, we get a circular slice $z^2 + x^2 \leq 1 - 4y^2$ of radius $\sqrt{1-4y^2}$.
a) Find the area of this slice.
b) Determine the volume of the tomato.
c) Fix yourself a tomato salad with sliced tomatos. Staple one to your homework paper as proof that you really did it.

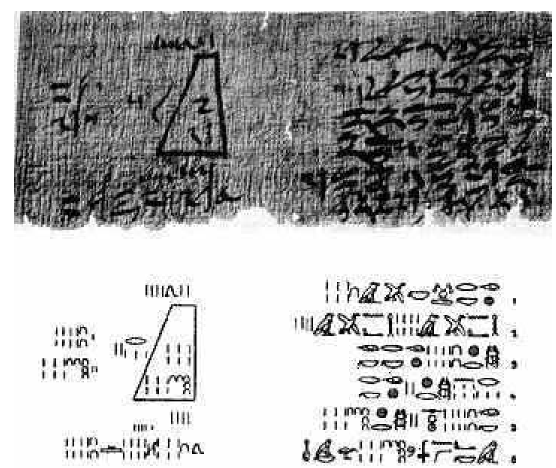
4 As we have seen in the movie of the first class, **Archimedes** was so proud of his formula for the volume of a sphere that he wanted the formula on his tomb stone. He wrote the volume of a half sphere of radius 1 as the difference between the volume of a cylinder of radius 1 and height 1 and the volume of a cone of base radius 1 and height 1. Relate the cross section area of the cylinder-cone complement with the cross section area of the sphere to recover his argument! If stuck, draw in the sand, soak in the bath tub or eat your tomato salad. No credit is given for streaking and screaming "Eureka".





5 Volumes were among the first quantities, Mathematicians wanted to measure and compute. One problem on **Moscow Egypt papyrus** dating back to 1850 BC explains the general formula $h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h.
a) Verify that if you slice the frustrum at height z, the area is $(a + (b-a)z/h)^2$.
b) Find the volume using calculus.
Here is the translated formulation from the papyrus: [1] [2]

"You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4 result 16. You are to double 4 result 8. You are to square 2, result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one-third of 6 result 2. You are to take 28 twice, result 56. See it is 56. You will find it right".



[1] Howard Eves, Great moments in mathematics, Volume 1, MAA, Dolciani Mathematical Expositions, 1980, page 10
[2] Image Source: http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian_papyri.html