Lecture 23: Improper integrals

In this lecture, we look at integrals on infinite intervals or integrals, where the function can get infinite at some point. These integrals are called **improper integrals**. The area under the curve can remain finite or become infinite.

1 What is the integral

$$\int_1^\infty \frac{1}{x^2}\, dx \; ?$$

Since the anti-derivative is $-1/x$, we have

$$\frac{-1}{x}|_1^\infty = -1/\infty + 1 = 1 \; .$$

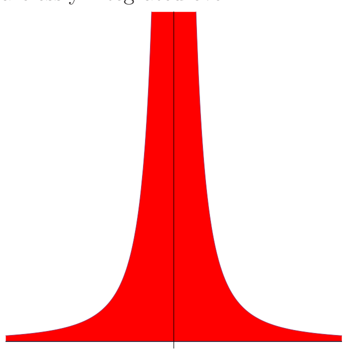
To justify this, compute the integral $\int_1^b 1/x^2 dx = 1 - 1/b$ and see that in the limit $b \to \infty$, the value 1 is achieved.

In a previous lecture, we have seen a chocking example similar to the following one:

2

$$\int_{-1}^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_{-1}^1 = -1 - 1 = -2 \; .$$

This does not make any sense because the function is positive so that the integral should be a positive area. The problem is this time not at the boundary $-1, 1$. The sore point is $x = 0$ over which we have carelessly integrated over.



The next example illustrates the problem with the previous example better:

3 The computation

$$\int_0^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_0^1 = -1 + \infty \; .$$

indicates that the integral does not exist. We can justify by looking at integrals

$$\int_a^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_a^1 = -1 + \frac{1}{a}$$

which are fine for every $a > 0$. But this does not converge for $a \to 0$.

Do we always have a problem if the function goes to infinity at some point?

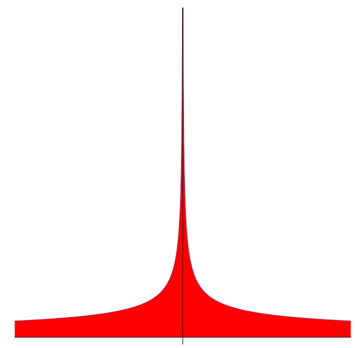
4 Find the following integral

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx \; .$$

Solution: Since the point $x = 0$ is problematic, we integrate from a to 1 with positive a and then take the limit $a \to 0$. Since $x^{-1/2}$ has the antiderivative $x^{1/2}/(1/2) = 2\sqrt{x}$, we have

$$\int_a^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}|_a^1 = 2\sqrt{1} - 2\sqrt{a} = 2(1 - \sqrt{a}) \; .$$

There is no problem with taking the limit $a \to 0$. The answer is 2. Even so the region is infinite its area is finite. This is an interesting example. Imaging this to be a container for paint. We can fill the container with a finite amount of paint but the wall of the region has infinite length.



5 Evaluate the integral $\int_0^1 1/\sqrt{1-x^2}\, dx$. **Solution:** The antiderivative is $\arcsin(x)$. In this case, it is not the point $x = 0$ which produces the difficulty. It is the point $x = 1$. Take $a > 0$ and evaluate

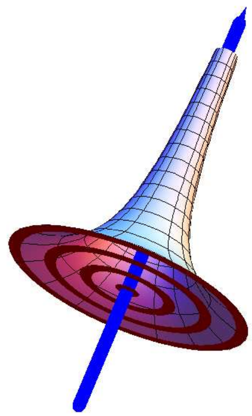
$$\int_0^{1-a} \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x)|_0^{1-a} = \arcsin(1-a) - \arcsin(0) \; .$$

Now $\arcsin(1-a)$ has no problem at limit $a \to 0$. Since $\arcsin(1) = \pi/2$ exists. We get therefore the answer $\arcsin(1) = \pi/2$.

6 Rotate the graph of $f(x) = 1/x$ around the x-axes and compute the volume of the solid between 1 and ∞. The cross section area is π/x^2. If we look at the integral from 1 to a fixed R, we get

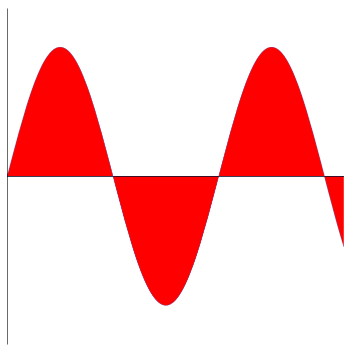
$$\int_1^R \frac{\pi}{x^2}\, dx = -\frac{\pi}{x}|_1^R = -\pi/R + \pi \; .$$

This converges for $R \to \infty$. The volume is π. This famous solid is called **Gabriels trumpet**. This solid is so prominent because if you look at the surface area of the small slice, then it is larger than $dx 2\pi/x$. The total surface area of the trumpet from 1 to R is therefore larger than $\int_1^R 2\pi/x\, dx = 2\pi(\log(R) - \log(1))$. which goes to infinity. We can **fill** the trumpet with a finite amount of paint but we can not **paint** its surface.



Finally, lets look at the following example

7 Evaluate the integral $\int_0^\infty \sin(x)\, dx$. **Solution.** There is no problem at the boundary 0 nor at any other point. We have to investigate however, what happens at ∞. Therefore, we look at the integral $\int_0^b \sin(x)\, dx = -\cos(x)|_0^b = 1 - \cos(b)$. We see that the limit $b \to \infty$ does not exist. The integral fluctuates between 0 and 2.



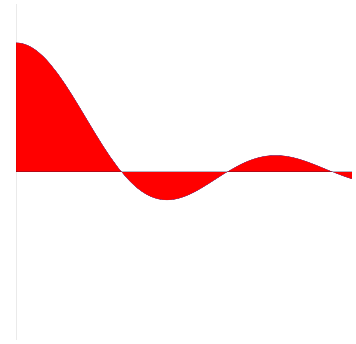
The next example leads to a topic in a follow-up course. It is not covered here, but could make you curious:

8 What about the integral

$$I = \int_0^\infty \frac{\sin(x)}{x}\, dx \ ?$$

Solution. The anti derivative is the Sine integral $Si(x)$ so that we can write $\int_0^b \sin(x)/x\, dx = Si(b)$. It turns out that the limit $b \to \infty$ exists and is equal to $\pi/2$ but this is a topic for a

second semester course like Math 1b. The integral can be written as an alternating series, which converges and there are many ways to compute it: [1]



Lets summarize the two cases of improper integrals: infinitely long intervals and a point where the function becomes infinite.

1) To investigate the improper integral $\int_a^\infty f(x)\, dx$ we look at the limit $\int_a^b f(x)\, dx$ for $b \to \infty$.
1) To investigate improper integral $\int_0^b f(x)\, dx$ where $f(x)$ is not continuous at 0, we take the limit $\int_a^b f(x)\, dx$ for $a \to 0$.

Homework

1 Evaluate the integral $\int_1^2 \frac{5}{\sqrt{x-1}}\, dx$.

2 Evaluate the following integrals
a) $\int_0^1 2x/\sqrt{1-x^2}\, dx$.
b) $\int_0^1 4/\sqrt{1-x^2}\, dx$.
Hint: For a) think about the chain rule $d/dx f(g(x)) = f'(g(x))g'(x)$

3 Evaluate the integral $\int_{-3}^4 (x^2)^{1/3}\, dx$. To make sure that the integral is fine, check whether $\int_{-3}^0$ and $\int_0^4$ work.

4 The integral $\int_{-2}^1 1/x\, dx$ does not exist. We can however take a positive $b > 0$ and look at

$$\int_{-2}^{-b} 1/x\, dx + \int_b^1 1/x\, dx = \log|b| - \log|-2|) + (\log|1| - \log|b|) = \log(2)\ .$$

This value is called the **Cauchy principal value** of the integral. Find the principal value of

$$\int_{-4}^5 3/x^3\, dx$$

using the same process as before, by cutting out $[-a, a]$ and then taking the limit $a \to 0$.

5 a) Evaluate $\int_{-1}^1 \frac{1}{x^4}\, dx$ blindly, without worrying that the function is infinite.
b) Can we give a principal value integral value to $\int_{-1}^1 \frac{1}{x^4}\, dx$? If yes, find the value. If not, tell why not.

[1] Hardy, Mathematical Gazette, 5, 98-103, 1909.