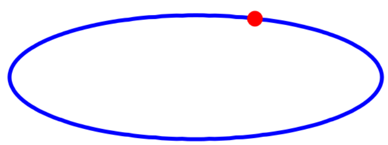
Lecture 26: Implicit differentiation

Implicit differentiation had bin crucial for finding the derivative of inverse functions. We will review this here because this will give us handy tools for integration.

> The chain rule, related rates and implicit differentiation belong all to the same concept. But each sees it from a different angle. You can see implicit differentiation as a special case of related rates, where one of the quantities is "time" meaning that this is the variable with respect to which we differentiate. There is considerable overlap with what we do here with what we have done last time.

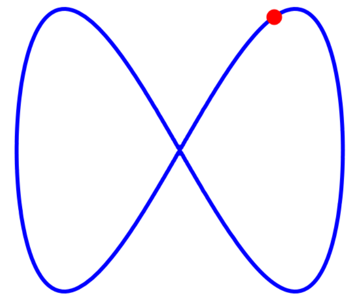
1 Points (x, y) in the plane which satisfy $x^2 + 9y^2 = 10$ form an ellipse. Find the slope y' of the tangent line at the point $(1, 1)$.
Solution: We want to know the derivative dy/dx. We have $2x + 18yy' = 0$. Using $x = 1, y = 1$ we see $y' = -2x/(18y) = -1/9$.
Remark. We could have looked at this as a related rates problem where $x(t), y(t)$ are related and $x' = 1$ Now $2xx' + 9 \cdot 2yy' = 0$ allows to solve for $y' = -2xx'/(9y) = -2/9$.



2 The points (x, y) which satisfy the equation $x^4 - 3x^2 + y^2 = 0$ forms a **figure 8** called **lemniscate of Gerono**. It contains the point $(1, \sqrt{2})$. Find the slope of the curve at that point. **Solution**: We differentiate the law describing the curve with respect to x. This gives

$$4x^3 - 6x + 2yy' = 0$$

We can now solve for $y' = (6x - 4x^3)/(2y) = 1/\sqrt{2}$.



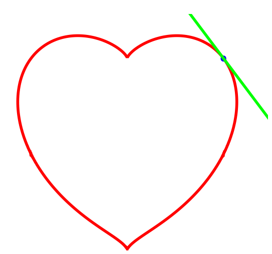
3 The **Valentine equation** $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$ contains the point $(1, 1)$. Near $(1, 1)$, we have $y = y(x)$ so that $(x^2 + y(x)^2 - 1)^3 - x^2y(x)^3 = 0$. Find y' at the point $x = 1$.
Solution: Take the derivative

$$0 = 3(x^2 + y^2 - 1)^2(2x + 2yy') - 2xy^3 - x^2 3y^2 y'(x)$$

and solve for

$$y' = -\frac{3(x^2 + y^2 - 1)2x - 2xy^3}{3(x^2 + y^2 - 1)2y - 3x^2y^2} .$$

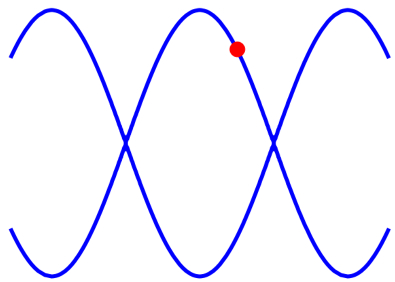
For $x = 1, y = 1$, we get $-4/3$.



4 The energy of a **pendulum** with angle x and angular velocity y is

$$y^2 - \cos(x) = 1$$

is constant. What is y'? We could solve for y and then differentiate. Simpler is to differentiate directly and get $yy' + \sin(x) = 0$ so that $y' = -\sin(x)/y$. At the point $(\pi/2, 1)$ for example we have $y' = -1$.



What is the difference between related rates and implicit differentiation?

Implicit differentiation is the **special case** of related rates where one of the variables is time.

Derivatives of inverse functions

Implicit differentiation has an important application: it allows us to compute the derivatives of inverse functions. It is good that we review this, because we can use these derivatives to find anti-derivatives. We have seen this already. Lets do it again.

5 Find the derivative of $\log(x)$ by differentiating $\exp(\log(x)) = x$.
Solution:

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{dx}\exp(\log(x)) \\ &= \exp(\log(x))\frac{d}{dx}\log(x) = x\log'(x) \ . \end{aligned}$$

Solve for $\log'(x) = 1/x$. Since the derivative of $\log(x)$ is $1/x$. The anti-derivative of $1/x$ is $\log(x) + C$.

6 Find the derivative of $\arccos(x)$ by differentiating $\cos(\arccos(x)) = x$.
Solution:

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{dx}\cos(\arccos(x)) \\ &= -\sin(\arccos(x))\arccos'(x) = -\sqrt{1-\cos^2(\arccos(x))}\arccos'(x) \\ &= -\sqrt{1-x^2}\arccos'(x) \ . \end{aligned}$$

Solving for $\arccos'(x) = -1/\sqrt{1-x^2}$. The anti-derivative of $\arccos(x)$ is $-1/\sqrt{1-x^2}$.

7 Find the derivative of $\arctan(x)$ by differentiating $\tan(\arctan(x)) = x$.
Solution: This is a derivative which we have seen several times by now. We use the identity $1/\cos^2(x) = \tan^2(x) + 1$ to get

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{dx}\tan(\arctan(x)) \\ &= \frac{1}{\cos^2(\arctan(x))}\arctan'(x) \\ &= (1+\tan^2(\arctan(x)))\arctan'(x) \ . \end{aligned}$$

Solve for $\arctan'(x) = 1/(1+x^2)$. The anti-derivative of $\arctan(x)$ is $1/(1+x^2)$.

8 Find the derivative of $f(x) = \sqrt{x}$ by differentiating $(\sqrt{x})^2 = x$.
Solution:

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{x}(\sqrt{x})^2 \\ &= 2\sqrt{x}f'(x) \end{aligned}$$

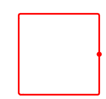
so that $f'(x) = 1/(2\sqrt{x})$.

Homework

1 The equation $y^2 = x^2 - x$ defines the graph of the function $f(x) = \sqrt{x^2 - x}$. Find the slope of the graph at $x = 2$ directly by differentiating f. Then use the implicit differentiation method and differentiate $y^2 = x^2 - x$ assuming $y(x)$ is a function of x and solving for y'.

2 The equation $x^2 + 4y^2 = 5$ defines an ellipse. Find the slope of the tangent at $(1, 1)$.

3 The equation $x^{100} + y^{100} = 1 + 2^{100}$ defines a curve which looks close to a square. Find the slope of the curve at $(2, 1)$.



4 Derive the derivative of $\text{arctanh}(x)$ by using the identity $\tanh(\text{arctanh}(x)) = 1$. You can use $\cosh^2(x) - \sinh^2(x) = 1$ which implies that $1 - \tanh^2(x) = 1/\cosh^2(x)$.

5 The relation $\sin(x - y) - 2\cos((\pi/2)xy) = 0$ relates $x(t)$ and $y(t)$. What is $y' = 2$ at $(1, 1)$ what is x' at this point?