Lecture 28: Review for second midterm

Major points

The **mean value theorem** assures that there is $x \in (a,b)$ with $f'(x) = (f(b) - f(a))/(b-a)$. A special case is Rolle's theorem, where $f(b) = f(a)$.

Catastrophes are parameter values where a local minimum disappears. To find the parameter look at the second derivative f'' at the critical point and find c which makes this zero.

Definite integrals $F(x) = \int_0^x f(t)\, dt$ are defined as a limit of Riemann sums S_n/n.

A function $F(x)$ satisfying $F' = f$ is called the anti-derivative of f. The general anti-derivative is $F + c$ where c is a constant.

The **fundamental theorem of calculus** tells $d/dx \int_0^x f(x)\ dx = f(x)$ and $\int_0^x f'(x)\, dx = f(x) - f(0)$.

The integral $\int_a^b g(x) - f(x)\, dx$ is the **signed area between the graphs** of f and g. Places, where $f < g$ are counted negative. When area is asked, split things up.

The integral $\int_a^b A(x)\ dx$ is a **volume** if $A(x)$ is the area of a slice of the solid perpendicular to a point x on an axes.

Write **improper integrals** as limits of definite integrals $\int_1^\infty f(x)\ dx = \lim_{R\to\infty} \int_1^R f(x)\, dx$. We similarly treat points, where f is discontinuous.

Besides **area**, **volume**, **total cost**, or **position**, we can compute **averages**, **inertia** or **work** using integrals.

If x, y are related by $F(x(t), y(t)) = 0$ and $x(t)$ is known we can compute $y'(t)$ using the chain rule. This is **related rates**.

If $f(g(t))$ is known we can compute $g'(x)$ using the chain rule. This works for inverse functions. This is **implicit differentiation**.

To determine the **catastrophes** for a family $f_c(x)$ of functions, determine the critical points in dependence of c and find values c, where a critical point changes from a local minimum to a local maximum.

Important integrals

Which one is the derivative which the integral?

$\sin(x)$	$-\cos(x)$.	$\log(x)$	$x\log(x) - x$
$\tan(x)$	$1/\cos^2(x)$.	$1/x$	$\log(x)$
$\arctan(x)$	$1/(1+x^2)$.	$-1/(1+x^2)$	$\text{arccot}(x)$
$1/\sqrt{1-x^2}$	$\arcsin(x)$	$-1/\sqrt{1-x^2}$	$\arccos(x)$

Improper integrals

$\int_1^\infty 1/x^2\, dx$ Prototype of first type improper integral which exists.
$\int_1^\infty 1/x\, dx$ Prototype of first type improper integral which does not exist.
$\int_0^1 1/x\, dx$ Prototype of second type improper integral which does not exist.
$\int_0^1 1/\sqrt{x}\, dx$ Prototype of second type improper integral which does exist.

The fundamental theorem

$$\frac{d}{dx}\int_0^x f(t)\, dt = f(x)$$

$$\int_0^x f'(t)\, dt = f(x) - f(0).$$

This implies

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

Without limits of integration, we call $\int f(x)\, dx$ the **anti derivative**. It is defined up to a constant. For example $\int \sin(x)\, dx = -\cos(x) + C$.

Applications

Calculus applies directly if there are situations where one quantity is the derivative of the other.

function	anti derivative
acceleration	velocity
velocity	position
function	area under the graph
length of cross section	area of region
area of cross section	volume of solid
marginal prize	total prize
power	work
probability density function	cumulative distribution function

Tricks

Make a picture, whenever we deal with an area or volume computation! In related rates problems, we have to understand the variables and the constants.
For volume computations, find the area of the cross section $A(x)$ and integrate.
For area computations find the length of the slice $f(x)$ and integrate.

Most important integrals

The most important integral is the integral

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1}}$$

holds for all n different from 1.

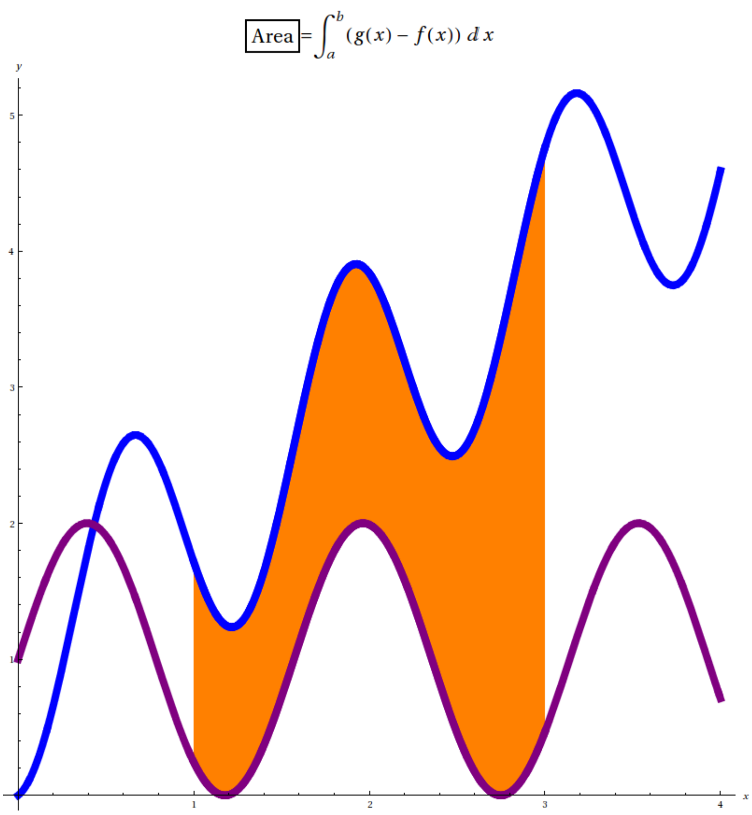
$$\boxed{\int \frac{1}{x} \, dx = \log(x)}$$

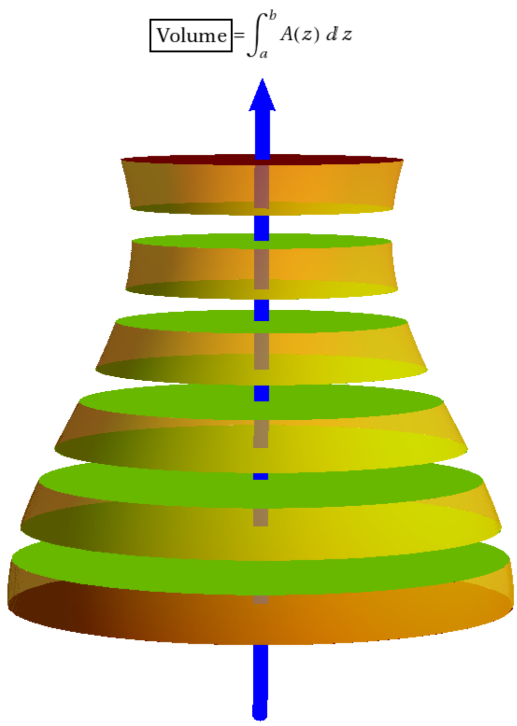
Example: $\int \sqrt{x+7} \, dx = \frac{2}{3}(x+7)^{3/2}$.
Example: $\int \frac{1}{x+5} \, dx = \log(x+5)$
Example: $\int \frac{1}{4x+3} \, dx = \log(4x+3)/4$

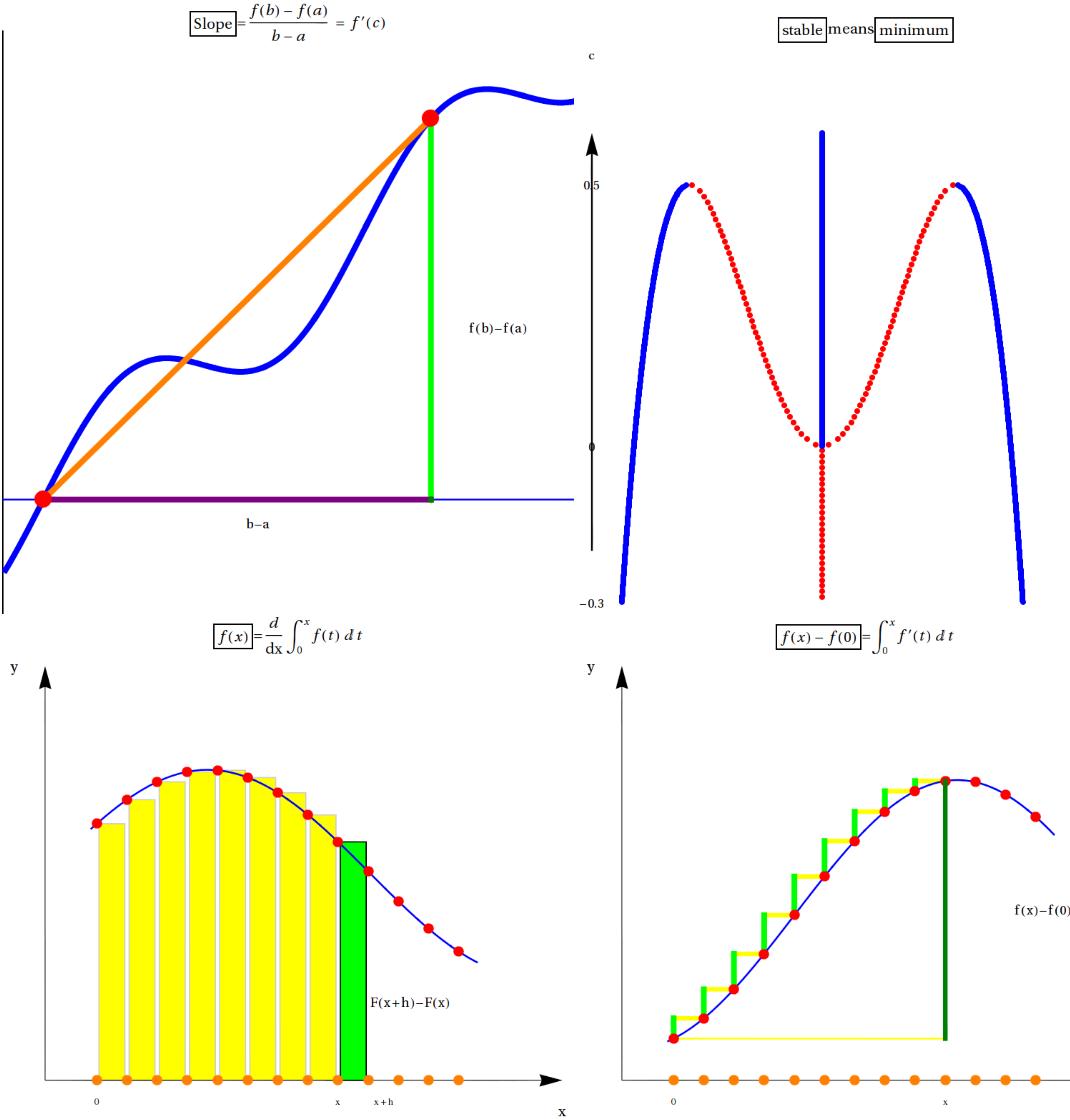
Related rates - Implicit differentiation

Assume $\cos(xy) + y^4 = 2y$, where x, y both change and $x' = 7$. Find y' at $x = 0, y = 1$. Givencos$(xy) + y^4 = 2y$. Find $y'(x)$ at $x = 0$.

Key pictures







Some integration tricks

$\int f(ax+b)\,dx = F(ax+b)/a$. Example: $\int \frac{1}{1+(x+1)^2}\,dx = \arctan(1+x)$. We have learned to deal with this with integration.