Math 1A: introduction to functions and calculus　　　　Oliver Knill, 2013

Lecture 38: Review since second midterm

Look also at the older review sheets.

Related rates

Implicit differentiation and related rates are applications of the chain rule.
A) Related rates: we have an equation $\boxed{F(x,y)=c}$ relating two variables $x(t), y(t)$ which depend on time t. Differentiate the equation with respect to t using the chain rule and solve for y'.
B) Implicit differentiation: we have an equation $\boxed{F(x,y(x))=c}$ relating y with x. Differentiate the equation with respect to x using the chain rule and solve for y'.

Examples:
A) $x^3 + y^3 = 1$, $x(t) = \sin(t)$, then $3x^2 x' + 3y^2 y' = 0$ so that $y' = -x^2 x'/y^2 = -\sin^2(t)\cos(t)/(1 - \sin^3(t))^{1/3}$.
B) Find y' for $x^4 + 3y^3 = x + 3y^2$ at $x = 1, y = 1$: $4x^3 + 9y^2 y' = 1 + 6y'y$ gives $4 + 9y' = 1 + 6y'$ so that $y' = -1$.

Substitution

Substitution replaces $\int f(x)\, dx$ with $\int g(u)\, du$ with $u = u(x), du = u'(x)dx$. Special cases:

A) The antiderivative of $f(x) = g(u(x))u'(x)$, is $G(u(x))$ where G is the anti derivative of g.
B) $\int f(ax+b)\, dx = F(ax+b)/a$ where F is the anti derivative of f.

Examples:
A) $\int \sin(x^5)x^4\, dx = \int \sin(u)\, du/5 = -\cos(u)/5 + C = -\cos(x^5)/5 + C$.
B) $\int \log(5x+7)\, dx = \int \log(u)\, du/5 = (u\log(u) - u)/5 + C = (5x+7)\log(5x+7) - (5x+7) + C$.

Integration by parts

A) Direct:

$$\int \boxed{\text{x}}\ \boxed{\sin(x)}\ dx = \boxed{\text{x}}\ \boxed{(-\cos(x))} - \int \boxed{1}\ \boxed{(-\cos(x))}\ dx = -x\cos(x) + \sin(x) + C\ dx\ .$$

B) Tic-Tac-Toe: To integrate $x^2 \sin(x)$

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The anti-derivative is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\ .$$

C) Merry go round: Example $I = \int \sin(x)e^x\, dx$. Use parts twice and solve for I.

Partial fractions

A) Make a common denominator on the right hand side $\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$. and compare coefficients $1 = Ax - Ab + Bx - Ba$ to get $A + B = 0, Ab - Ba = 1$ and solve for A, B.
B) If $f(x) = p(x)/(x-a)(x-b)$ with different a, b, the coefficients A, B in $\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ can be obtained from

$$A = \lim_{x\to a}(x-a)f(x) = p(a)/(a-b),\ \ B = \lim_{x\to b}(x-b)f(x) = p(b)/(b-a)\ .$$

Examples:
A) $\int \frac{1}{(x+1)(x+2)}\, dx = \int \frac{A}{x+1}\, dx + \int \frac{B}{x+2}\, dx$. Find A, B by multiplying out and comparing coefficients in the nominator.
B) Directly write down $A = 1$ and $B = -1$, by plugging in $x = -2$ after multiplying with $x - 2$. or plugging in $x = -1$ after multiplying with $x - 1$.

Improper integrals

A) Integrate over infinite domain.
B) Integrate over singularity.

Examples:
A) $\int_0^\infty 1/(1+x^2) = \arctan(\infty) - \arctan(0) = \pi/2 - 0 = \pi/2$.
B) $\int_0^1 1/x^{2/3}\, dx = (3/1)x^{1/3}|_0^1 = 3$.

Trig substitution

When integrating function like $\sqrt{1-x^2}$, replace x by $\sin(u)$.
Example:
$\int_{-1}^1 \sqrt{1-x^2}\, dx = \int_{-\pi/2}^{\pi/2} \cos(u)\cos(u)du = \int_{-\pi/2}^{\pi/2}(1+\cos(2u))/2 = \frac{\pi}{2}$.

Applications, Buzz words

Music: hull function, piano function
Economics: average cost, marginal cost and total cost. Strawberry theorem
Operations research: find extrema, critical points, 2. derivative test
Computer science: curvature, random function from given functions, level of complexity of a function
Statistics: PDF, cumulative distribution function, expectation, variance.
Distributions: normal distribution, geometric distribution, Cauchy distribution.
Geometry: area between two curves, volume of solid
Numerical integration: Riemann sum, trapezoid rule, Simpson rule, Monte Carlo
Root finding: Bisection method, Newton algorithm.
Psychology: critical points and catastrophes.
Physics: position, velocity and acceleration, work and power
Gastronomy: turn table to prevent wobbling, bottle calibration.