3/7/2013: First hourly Practice B

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F** The function arcsin(x) is defined as $1/\sin(x)$.

Solution:
The arcsin function is the inverse not the reciprocal of $\sin(x)$.

2) T **F** The function $f(x) = \sin(1/x^2)$ can be defined at 0 so that it becomes a continuous everywhere on the real line.

Solution:
This is the prototype oscillatory singularity.

3) T **F** The function $x/\sin(x)$ can be defined at $x = 0$ so that it becomes a continuous function on the real line.

Solution:
The value is 1 at 0 by the fundamental theorem of trigonometry. It can not be made continuous on the entire real line because at $\pi, 2\pi$ etc the function can not be saved.

4) **T** F The function $f(x) = \sin^2(x)/x^2$ has the limit 1 at $x = 0$.

Solution:
Yes, it is the square or the *sinc* function.

5) T **F** The function $f(x) = 1/\log|x|$ has the limit 1 at $x = 0$.

Solution:
l'Hopital gives 0, not 1.

6) **T** F The function $f(x) = (1 + h)^{x/h}$ has the property that $Df(x) = [f(x + h) - f(x)]/h = f(x)$.

Solution:
We have seen that several time in this course and done in the homework.

7) T **F** $\cos(3\pi/2) = 1$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees. The cosine is the x value and so zero.

8) **T** F If a function f is continuous on the interval $[3, 10]$, then it has a global maximum on this interval.

Solution:
This is a consequence of the extreme value theorem.

9) T **F** The reciprocal rule assures that $d/dx(1/g(x)) = 1/g(x)^2$.

Solution:
The minus sign is missing as well as the factor $g'(x)$.

10) **T** F If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x\to 0}(f(x)/g(x)) = 1$

Solution:
This is a consequence of l'Hopital's rule when applied twice.

11) **T** F An inflection point is a point where the function $f''(x)$ changes sign.

Solution:
This is a definition.

12) **T** F If $f''(x) > 0$ then f is concave up at x.

Solution:
The slope of the tangent increases which produces a concave up graph. One can define concave up with the property $f''(x) > 0$

13) T **F** The chain rule assures that $d/dx f(g(x)) = f'(x)g'(x)$.

Solution:
This is not true. We have $f'(g(x))$ in the first factor.

14) **T** F The function $f(x) = 1/x + \log(x)$ is continuous on the interval $[1, 2]$.

Solution:
While there is a problem at 0, everything is nice and dandy at $[1, 2]$.

15) **T** F If we perform the Newton step for the function $\exp(x)$, we get the map $T(x) = x - 1$.

Solution:
The exponential function is its own derivative so that $f(x)/f'(x) = 1$.

16) **T** F The graph of the function $f(x) = x/(1 + x^2)$ has slope 1 at 0.

Solution:
$f'(x) = 1/(1 + x^2) - 2x^2/(1 + x^2)^2$. This is 1 for $x = 0$.

17) **T** F There is a differentiable function for which $f'(0) = 0$ but for which 0 is not a local extremum.

Solution:
Take $f(x) = x^3$.

18) T **F** The second derivative test assures that $x = p$ is a local minimum if $f'(p) = 0$ and $f''(p) < 0$.

Solution:
It is $f''(x) > 0$.

19) **T** F The identity $(x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ holds for all $x \neq 1$.

Solution:
Multiply out to see it.

20) T **F** The slope of the tangent at a point $(x, f(x))$ of the graph of a differentiable function f is equal to $1/f'(x)$.

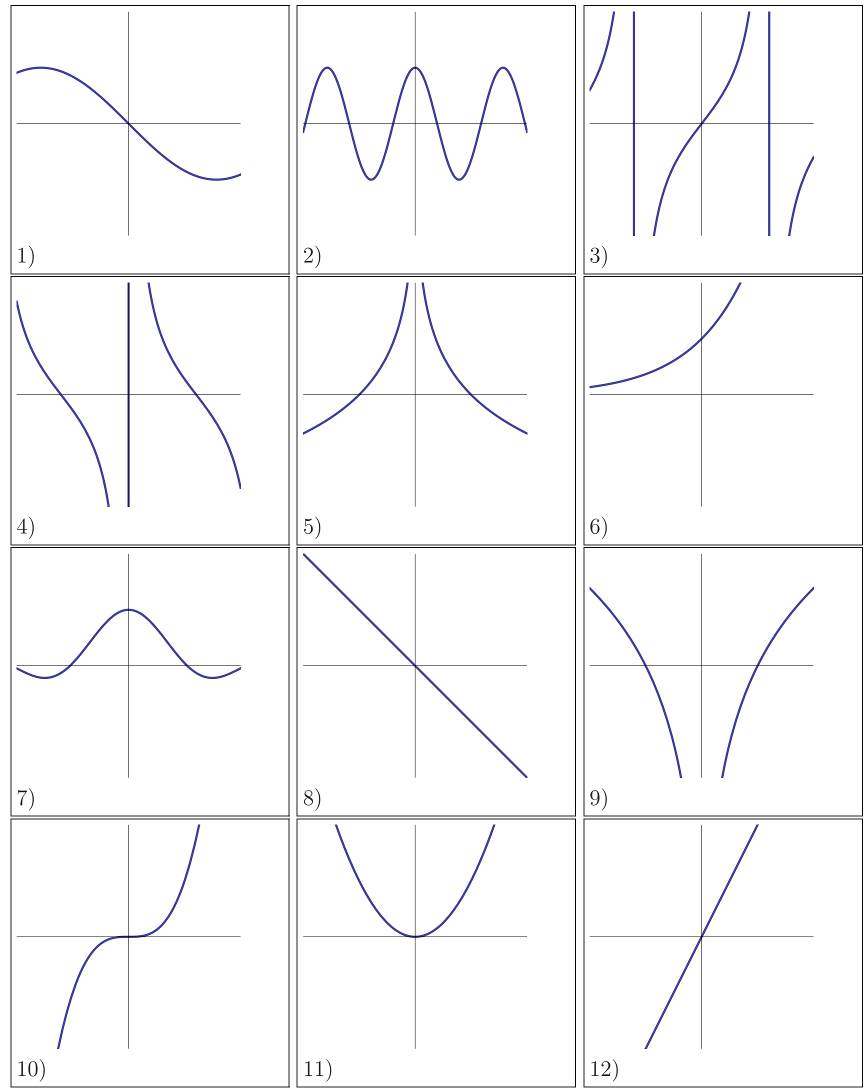
Solution:
The slope is $f'(x)$ not $1/f'(x)$.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs. Naturally, only 10 of the 12 graphs will appear.

Function	Enter 1-12
$\cot(x)$	
$\cos(2x)$	
$2x$	
$\tan(x)$	
$\log(1/\|x\|)$	

Function	Enter 1-12
x^2	
$\exp(x)$	
$-\sin(x)$	
x^3	
$\mathrm{sinc}(x)$	



Solution:

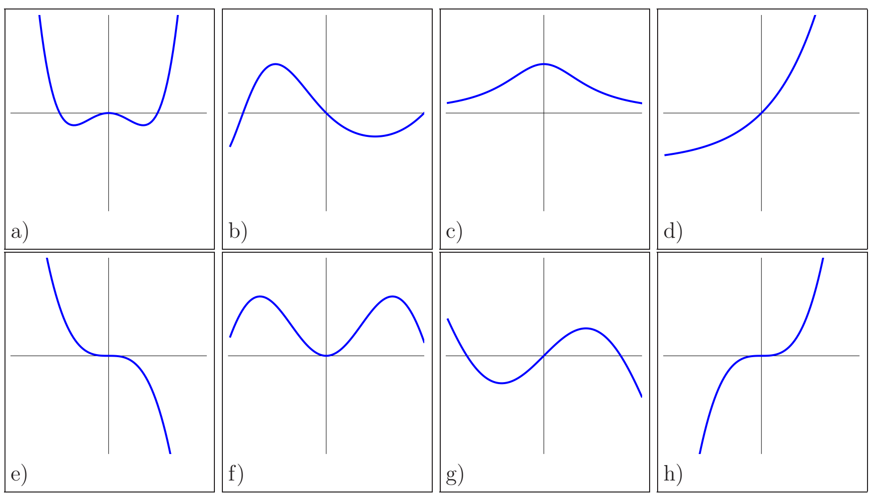
Function	Enter 1-12
$\cot(x)$	4
$\cos(2x)$	2
$2x$	12
$\tan(x)$	3
$\log(1/\|x\|)$	5

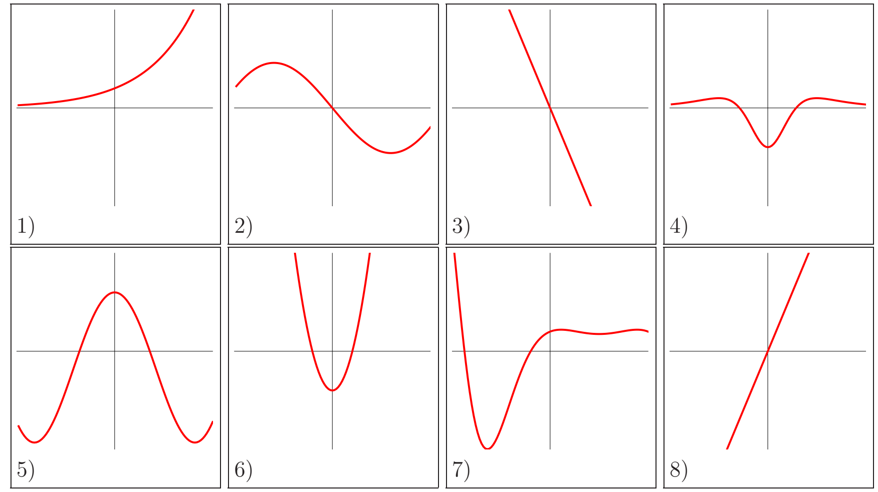
Function	Enter 1-12
x^2	11
$\exp(x)$	6
$-\sin(x)$	1
x^3	10
$\text{sinc}(x)$	7

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).

Function	Second derivative (Enter 1- 8 here)
a)	
b)	
c)	
d)	
e)	
f)	
g)	
h)	





Solution:

Function	Solution
a)	6 or 4
b)	7
c)	4 or 6
d)	1
e)	3
f)	5
g)	2
h)	8

Both 6,7,4 and 4,7,6 are possible.

Problem 4) Continuity (10 points)

Some of the following functions might a priori not be defined yet at the point a. In each case, decide whether f can be made a continuous function by assigning a value $f(a)$ at the point a. If no such value exist, state that the function is not continuous.

a) (2 points) $f(x) = \frac{(x^3-1)}{(x-1)}$, at $x = 1$

b) (2 points) $f(x) = \sin(\frac{1}{x}) + \cos(x)$, at $x = 0$

c) (2 points) $f(x) = \sin(\frac{1}{\log(|x|)})$, at $x = 0$

d) (2 points) $f(x) = \log(|\sin(x)|)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Solution:
a) Heal the function by dividing out $(x-1)$. For $x \neq 1$ we get x^2+x+1. At $x=0$ we have $\boxed{3}$.
b) The function contains the prototype $\sin(1/x)$ function, which has $\boxed{\text{no limit}}$ at $x=0$.
c) We had seen in class that $\lim_{x\to 0} 1/\log|x| = 0$ because $\log|x| \to -\infty$. Therefore $\sin(1/\log|x|) \to 0$. Assigning the value $f(0)=0$ makes the function continuous.
d) The function can is hopelessly discontinuous at $x=0$. For $|x|\to 0$ we have $\sin(x)\to 0$ and $\log|\sin(x)| \to -\infty$.
e) We can write this as $1-1/x$. This is a prototype case $1/x$ where the function converges to ∞.

Problem 5) Chain rule (10 points)

a) (2 points) Write $1+\cot^2(x)$ as an expression which only involves the function $\sin(x)$.

b) (3 points) Find the derivative of the function $\text{arccot}(x)$ by using the chain rule for

$$\cot(\text{arccot}(x)) = x \ .$$

c) (2 points) Write $1+\tan^2(x)$ as an expression which only involves the function $\cos(x)$.

d) (3 points) Find the derivative of the function $\arctan(x)$ by using the chain rule for

$$\tan(\arctan(x)) = x \ .$$

Remark: even if you should know the derivatives of arccot or arctan, we want to see the derivations in b) and d).

Solution:
We have done a),b) in homework problem 3) of Lecture 10.
a) $1+\cot^2(x) = 1+\cos^2(x)/\sin^2(x) = (\sin^2(x)+\cos^2(x))/\sin^2(x) = 1/\sin^2(x)$.
b) $\cot'(x) = -1/\sin^2(x) = 1+\cot^2(x)$ implies $(1+\cot^2(\text{arccot}(x)))\text{arccot}'(x) = 1$ and so $\text{arccot}'(x) = -1/(1+x^2)$.
We have done c),d) in class (on the side blackboard).
c) $1+\tan^2(x) = 1+\sin^2(x)/\cos^2(x) = 1/\cos^2(x)$.
d) $d/dx \cot(\arctan(x)) = (1/\cos^2(\arctan(x)))\arctan'(x) = (1+\tan^2(\arctan(x)))\arctan'(x) = 1$. Therefore, $\arctan'(x) = 1/(1+x^2)$.
We will come back to this later in the course.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{\cos(3x)}{\cos(x)}$

b) (2 points) $f(x) = \sin^2(x)\log(1+x^2)$

c) (2 points) $f(x) = 5x^4 - \frac{1}{x^2+1}$

d) (2 points) $f(x) = \tan(x) + \exp(-\sin(x^2))$

e) (2 points) $f(x) = \frac{x^3}{(1+x^2)}$

Solution:
a) Use the quotient rule $[-3\sin(3x)\cos(x)+\sin(x)\cos(3x)]/\cos^2(x)$.
b) Use the chain rule and the product rule $2\sin(x)\cos(x)\log(1+x^2)+2x\sin^2(x)/(1+x^2)$.
c) Use the quotient and chain rule for the second summand $20x^3+(2x)/(x^2+1)^2$.
d) The second sum uses the chain rule twice $1/\cos^2(x)+e^{-\sin(x^2)}(-\cos(x^2))2x$.
e) Use the quotient rule $(3x^2(1+x^2)-x^3(2x))/(1+x^2)^2$.

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x=0$ or state (providing reasoning as usual) that the limit does not exist.

a) (2 points) $f(x) = \frac{\sin(3x)}{\sin(x)}$

b) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

c) (2 points) $f(x) = \sin(\log(|x|))$

d) (2 points) $f(x) = \tan(x)\log(x)$

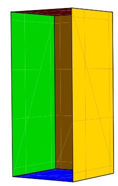
e) (2 points) $f(x) = \frac{(5x^4-1)}{(x^2+1)}$

Solution:
a) Apply l'Hopital once, to get the limit $\boxed{3}$.
b) $\boxed{1}$ since it is the square of $\sin(x)/x$ which has limit 1. One could also use l'Hopital twice.
c) There is $\boxed{\text{no limit}}$ because $log(|x|)$ goes to $-\infty$ and $\sin(\log|x|)$ oscillates indefinitely.
d) $\boxed{0}$ as we have done in class. Write as $[\sin(x)\log(x)]\cos(x)$ and $\cos(x)$ has no problem at $x=0$. The limit $\sin(x)\log(x)$ is the same as $x\log(x)$ which we have done in class.
e) $\boxed{-1}$. There is no problem at this point because the nominator is not zero. We can just plug in $x=0$ and get the value.

Problem 8) Extrema (10 points)

A rectangular shoe-box of width x, length x and height y is of volume 2 so that $x^2y = 2$. The surface area adds up three rectangular parts of size $(x \times y)$ and 2 square parts of size $(x \times x)$ and leads to

$$f = 2x^2 + 3xy \; .$$



a) (2 points) Write down the function $f(x)$ of the single variable x you want to minimize.

b) (6 points) Find the value of x for which the surface area is minimal.

c) (2 points) Check with the second derivative test, whether the point you found is a local minimum.

Solution:
a) Solve for $y = 2/x^2$ and substitute it into f. The function is $f(x) = 2x^2 + 6/x$.
b) $f'(x) = 4x - 6/x^2 = 0$ for $2x^3 = 3$ so that $x = (3/2)^{(1/3)}$.
c) $f''(x) = 4 + 12/x^3$. The second derivative is positive at the critical point. The critical point is a local minimum.

Problem 9) Global extrema (10 points)

In this problem we study the function $f(x) = 3x^5 - 5x^3$ on the interval $[-2, 2]$.

a) (2 points) Find all roots of f.

b) (3 points) Find all local extrema of the function.

c) (3 points) Use the second derivative test to analyze the critical points, where applicable.

d) (2 points) Find the **global** maximum and minimum of f on the interval $[-2, 2]$.

Solution:
a) The roots are $\boxed{0, 0, 0, -\sqrt{5/3}, \sqrt{5/3}}$ as you can see by factoring x^3 out. The root 0 is a triple root.
b) The derivative is $15x^4 - 15x^2 = 15x^2(x^2 - 1)$ which has roots at $\boxed{0 \text{ and } 1 \text{ and } -1}$. These are candidates for local extrema.
c) The second derivative is $30x(2x^2 - 1)$. At $x = 0$, the second derivative is zero. The second derivative test does not apply at this point. At $x = 1$, the second derivative is positive at $x = -1$ it is negative. $\boxed{x = 1 \text{ is a local min, and } x = -1 \text{ is a local max.}}$
d) We compare $(f(2), f(-2), f(1), f(-1), f(0)) = (56, -56, -2, 2, 0)$ to see that the global extrema are located at the boundary. The point $\boxed{2}$ is the global maximum and the point $\boxed{-2}$ is the global minimum.

Problem 10) Newton (10 points)

Perform one Newton step for the function $f(x) = x^5 - x$ starting at the point $x = 3$.

Solution:
The map is $T(x) = x - f(x)/f'(x) = x - (x^5 - z)/(5z^4 - 1)$. We have $T(3) = 3 - (3^5 - 3)/(5 \cdot 3^4 - 1) = 243/101$.