4/11/2013: Second midterm exam

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F** If f is a continuous function then $\int_0^x f(t)\ dt$ is an area and therefore positive.

Solution:
Take $f(x) = -x$ then $\int_0^x -t\ dt = -x^2/2$ is negative. Parts under the x axes are negative.

2) T **F** The anti-derivative of $\text{arccot}(x)$ is $-\log(\sin(x)) + C$.

Solution:
Differentiate the right hand side to check.

3) T **F** The fundamental theorem of calculus implies that $\int_0^3 f''(x)\ dx = f'(3) - f(0)$.

Solution:
Yes this is a special case of the fundamental theorem, if there were not the missing prime at the end. It had been noted on the blackboard that there is no typo in this problem.

4) **T** F The volume of a cylinder of height 3 and radius 5 is given by the integral $\int_0^3 \pi 5^2\ dx$.

Solution:
Yes the area of a slice is $r^2\pi$.

5) T **F** The antiderivative of $\tan(x)$ is $1/\cos^2(x)$.

Solution:
The derivative of $\tan(x)$ is $1/\cos(x)^2$.

6) **T** F The mean value theorem implies that the derivative of $\sin(x)$ in the interval $[0, \pi/2]$ is $2/\pi$ somewhere.

Solution:
This is a typical application of the mean value theorem.

7) **T** F The function $F(x) = \int_0^x \sin(t^2)\ dt$ has the derivative $\sin(x^2)$.

Solution:
The first derivative of F is f.

8) T **F** The level of wine in a parabolic glass changes with a constant rate if the volume decreases in a constant rate.

Solution:
This is a related rates problem. The balloon radius grows slower for large volumes.

9) T **F** The identity $\frac{d}{dx}\int_0^1 \sin(x)\,dx = \sin(1)$ holds.

Solution:
We differentiate a constant. This was the most commonly wrongly checked problem.

10) **T** F If a solid is scaled by a factor 2 in all directions then its volume increases by a factor 8.

Solution:
Yes, the volume goes cubic.

11) T **F** If $x^2 - y^2 = 3$ and $x'(t) = 1$ at $(2, 1)$ then $y' = 1$.

Solution:
This is a simple example of related rates. But the result is off by a factor.

12) **T** F If $f(x)$ is smaller than $g(x)$ for all x, then $\int_0^1 f(x) - g(x)\,dx$ is negative.

Solution:
Yes, we can take the 7 constant outside the integral.

13) T **F** Every improper integral defines an infinite area.

Solution:
No, it can be finite.

14) **T** F The anti derivative of $f'(x)$ is equal to $f(x) + c$.

Solution:
Yes, taking anti derivatives cancels taking derivatives, up to a constant.

15) **T** F Catastrophes can explain why minima can change discontinuously.

Solution:
This is a definition.

16) T **F** If f is discontinuous at 0, then $\int_{-1}^1 f(x)\,dx$ is infinite.

Solution:
Think about the sign function. The integral is finite.

17) T **F** If $f(-\infty) = 0$ and $f(\infty) = 1$ then $f' = 1$ somewhere on $(-\infty, \infty)$.

Solution:
No, the slope can be arbitrarily small. We have only to increase the value by 1 and all the space of the world to do that. This was probably the second most commonly wrongly checked TF problem in this exam.

18) **T** F The anti-derivative of $1/x$ is $\log(x) + C$, where log is the natural log.

Solution:
Yes, we know that, and should never,never,never,never forget!

19) T **F** A catastrophe is defined as a critical point of f which is a minimum.

Solution:
No, it deals with changes of critical points.

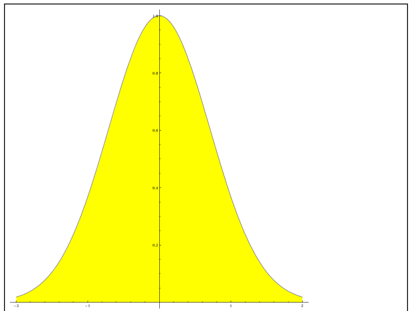
20) T **F** The integral $\int_0^\infty 1/x^2\,dx$ represents a finite area.

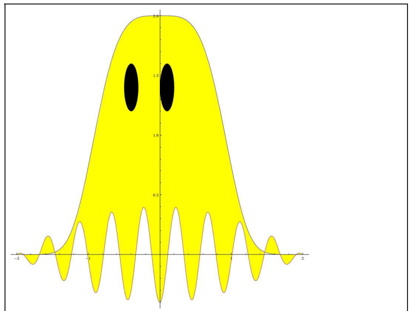
Solution:
We have no problem at infinity but a problem at $x = 0$.

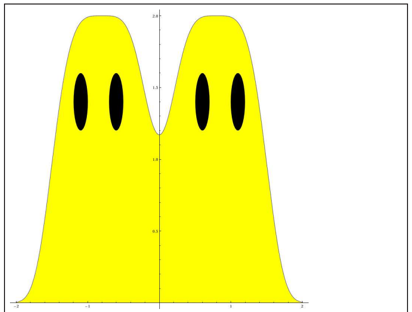
Problem 2) Matching problem (10 points) No justifications are needed.

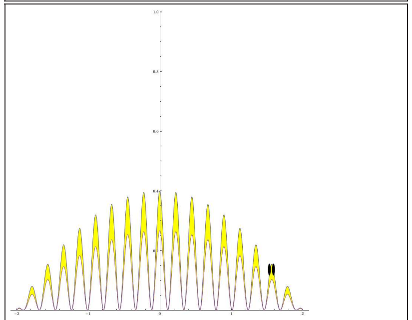
a) (6 points) Match the following integrals with the regions. Graphs 1) and 2) are inspired by a cartoon by Matthew Freeman (J Epidemiol. Community Health. 2006 January; 60(1): 6)

Integral	Fill in 1-4
$\int_{-2}^{2} (4-x^2)\cos^2(14x)/10 - (4-x^2)\cos(14x)/15 dx$	
$\int_{-2}^{2} 2\exp(-3(x+0.8)^4) + 2\exp(-3(x-0.8)^4\ dx$	
$\int_{-2}^{2} \exp(-x^2)\ dx$	
$\int_{-2}^{2} 2\exp(-x^4) - (x^2-4)\cos(14x)/10\ dx$	


1) Normal distribution


2) Paranormal distribution


3) Abnormal distribution


4) Wormal distribution

b) (4 points) Which of the following statements follows from Rolle's theorem? Check only one.

Result	Check
If $f(0) = -1$ and $f(1) = 1$ then there is x with $0 \leq x \leq 1$ with $f'(x) = 2$	
If $f(0) = 1$ and $f(1) = 1$ then there is a critical point x of f in $(0, 1)$	
If $f(0) = 1$ and $f(1) = 1$ then there is point where $f(x) = 2$ in $(0, 1)$	
If $f(0) = 1$ and $f(1) = 1$ then there is point where $f''(p) = 0$ in $(0, 1)$	

Solution:
a) 4,3,1,2.
b) second choice.

Problem 3) (10 points)

a) (4 points) Having seen some applications of integration and differentiation, complete the table:

Function f	Antiderivative F
Probability density function	
	Total cost
	Mass
Area	
	Velocity
Power	
Velocity	

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Disection method	
Newton method	

c) (2 points) Which is more general? In each row, check one box.

	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	

d) (2 points) Which integral is finite? Chose one!

Integral	finite	infinite	
$\int_1^\infty 1/\sqrt{x}\, dx$			
$\int_1^\infty 1/x^2\, dx$			

Solution:

Function f	Antiderivative F
Probability density function	CDF
Marginal cost	Total cost
Density	Mass
Area	Volume
Position	Velocity
Power	Work or Energy
Velocity	Position

b) (2 points) We have seen two methods to find roots $f(x) = 0$ of equations. Both methods need some assumptions on the functions: Choose from the following: "differentiability", "continuity", "positivity".

Method	Assumption which f has to satisfy
Disection method	Continuity
Newton method	Differnentiability

c) (2 points) Which is more general? In each row, check one box.

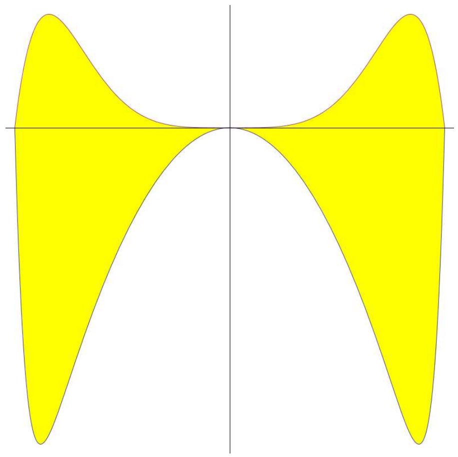
X	Related rates	Implicit differentiation	
	Rolles theorem	Intermediate value theorem	X

d) (2 points) Which integral is finite? Chose one!

Integral	finite	infinite
$\int_1^\infty 1/\sqrt{x}\, dx$		X
$\int_1^\infty 1/x^2\, dx$	X	

Problem 4) Area computation (10 points)

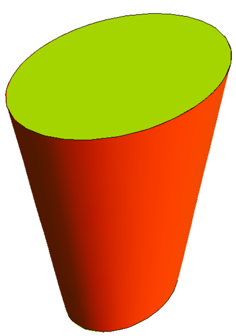
The region enclosed by the graphs of $f(x) = x^{20} - x^2$ and $g(x) = x^4 - x^8$ is a cross section for a catamaran sailing boat. Find the area.



Solution:
Note that $x^4 - x^8$ is positive on $[-1,1]$ an d $x^{20} - x^2$ is negative:

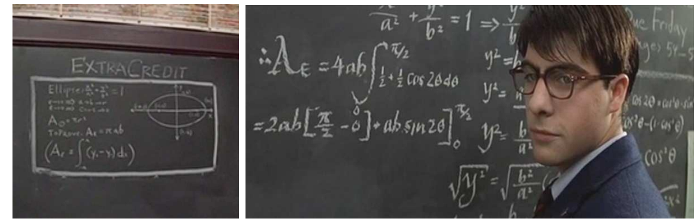
$$\int_{-1}^{1} x^4 - x^8 - (x^{20} - x^2)\ dx = 8/45 + 4/7 = 236/315\ .$$

Problem 5) Volume computation (10 points)



An ellipse with diameters $2b$ and $2a$ has area πab. Find the volume of part of a cone whose height is between $z = 3$ and $z = 5$ for which the cross section at height z is an ellipse with parameters $a = 2z$ and $b = 3z$.

Remark. We will see later the area formula. In the movie "Rushmore", the teacher tells about the problem: "I put that up as a joke. It's probably the hardest geometry equation in the world".



Screen shots from the movie Rushmore shows a blackboard where the formula for the ellipse is computed using trig substitution. You might spot a double angle formula. We will come to that.

Solution:
The area at height z is $\pi 6z^2$. The answer is $\int_3^5 \pi 6z^2\ dx = 6\pi z^3|_3^5 = 196\pi$.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 (x-1)^4\ dx$

b) (2 points) $\int_0^1 x^{1/3}\ dx$.

c) (2 points) $\int_0^{\sqrt{3}} \frac{6}{1+x^2}\ dx$

d) (2 points) $\int_{-2}^{e-3} \frac{5}{3+x}\ dx$

e) (2 points) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx$.

Solution:
a) 5
b) 3/4
c) 2π
d) 5
e) $\pi/2$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int e^{7x} - \sqrt{x}\ dx$

b) (2 points) $\int \frac{5}{x+1} + 7\cos^2(x)\ dx$

c) (2 points) $\int \frac{11}{1+x^2} + 9\tan(x)\ dx$

d) (2 points) $\int \frac{4}{\cos^2(x)} + \frac{2}{\sin^2(x)}\ dx$

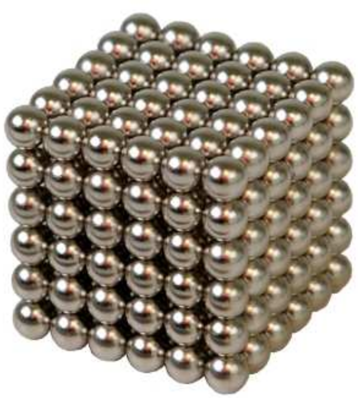
e) (2 points) $\int 2x\cos(x^2)\ dx$

Solution:
a) $\exp(7x)/7 - 2x^{3/2}/3 + C$.
b) $7x/2 + 5\log(1+x) + 7\sin(2x)/4 + C$.
c) $11\arctan(x) + 9\tan(x) + C$.
d) $\tan(x) - 2\cot(x) + C$.
e) $\sin(x^2) + C$.

Problem 8) Implicit differentiation and related rates (10 points)

a) (5 points) Find the slope y' of the curve $x^2y = \sin(xy) + (y-1)$ at $x = \pi/2, y = 1$.

b) (5 points) A magnetic Neodym metal cube of length x is heated and changes the volume in time at a rate $V' = 1$. At which rate does the length $x(t)$ of the cube change, when the volume is $V = 27$?



Neodym magnets. Soon outlawed since kids can swallow them, leading to a change of topology of their intestines. Dangerous stuff! Gun bullets can be obtained more easily, naturally because they can not be swallowed ...

Solution:
a) $2xy + x^2y' = \cos(xy)(x'y + xy') + 1$ gives $\pi + \pi/4y' = y'$ so that $y' = \pi/(\pi^2/4 - 1)$ which is equivelent to $y' = 4\pi/(4-\pi^2)$.
b) Since $V(x(t)) = x(t)^3$ we have $1 = V' = 3x^2x'$ so that $x' = 1/(3x^2) = 1/27$.
P.S. There is now even a recall of some magnets:
en.wikipedia.org/wiki/Neodymium_magnet_toys.

Problem 9) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = x^6 - cx^4 - cx^2$, where c is a parameter.

a) (4 points) Verify that f has a critical point 0 for all c.

b) (3 points) Determine whether 0 is a minimum or maximum depending on c.

c) (3 points) For which c does a catastrophe occur?

Solution:
a) Differentiate to see that $f' = 6x^5 - 4x^3c - 2xc$ has critical points at $x = 0$.
b) The second derivative is $f''(x) = 30x^4 - 12x^2c - 2c$ which is $f''(0) = -2c$. We see that for $c < 0$ we have a minimum and for $c > 0$ we have a maximum.
c) $c = 0$ is the catastrophe, because for this parameter the number a minimum becomes a maximum.