Lecture 2: Functions

A **function** is a rule which assigns to a real number a new real number. The function $f(x) = x^2 - 2x$ for example assigns to the number $x = 4$ the value $4^2 - 8 = 8$. A function is given with a **domain** A, the points where f is defined and a **codomain** B, a set of numbers which f can reach. Usually, functions are defined everywhere, like the function $f(x) = x^2 - 2x$. Other functions like $g(x) = 1/x$ can not be evaluated at 0 so that the domain excludes the point 0.

We have some flexibility to define the domain and codomain. Let R_0 be the set of all real numbers which do not contain 0. If we equip $f(x) = 1/x$ with the domain and codomain R_0 then f is a map from R_0 to R_0 and it is its own inverse. Here are a few examples of functions. We will look at them in more detail during the lecture. Very important are polynomials, trigonometric functions, the exponential and logarithmic function. You won't find the h-exponentials and h-logarithms in textbooks. But they will be important for us. They are the exponentials and logarithms in "calculus without limit" and will in the limit $h \to 0$ become the regular exponential and logarithm functions.

constant	$f(x) = 1$	power	$f(x) = 2^x$
identity	$f(x) = x$	exponential	$f(x) = e^x = \exp(x)$
linear	$f(x) = 3x + 1$	logarithm	$f(x) = \log(x) = \exp^{-1}(x)$
quadratic	$f(x) = x^2$	absolute value	$f(x) = \|x\|$
cosine	$f(x) = \cos(x)$	devil comb	$f(x) = \sin(1/x)$
sine	$f(x) = \sin(x)$	bell function	$f(x) = e^{-x^2}$
h-exponentials	$f(x) = \exp_h(x) = (1 + h)^{x/h}$	Agnesi function	$f(x) = \frac{1}{1+x^2}$
h-logarithms	$f(x) = \log_h(x) = \exp_h^{-1}$	sinc function	$\sin(x)/x$

We can build new functions by:

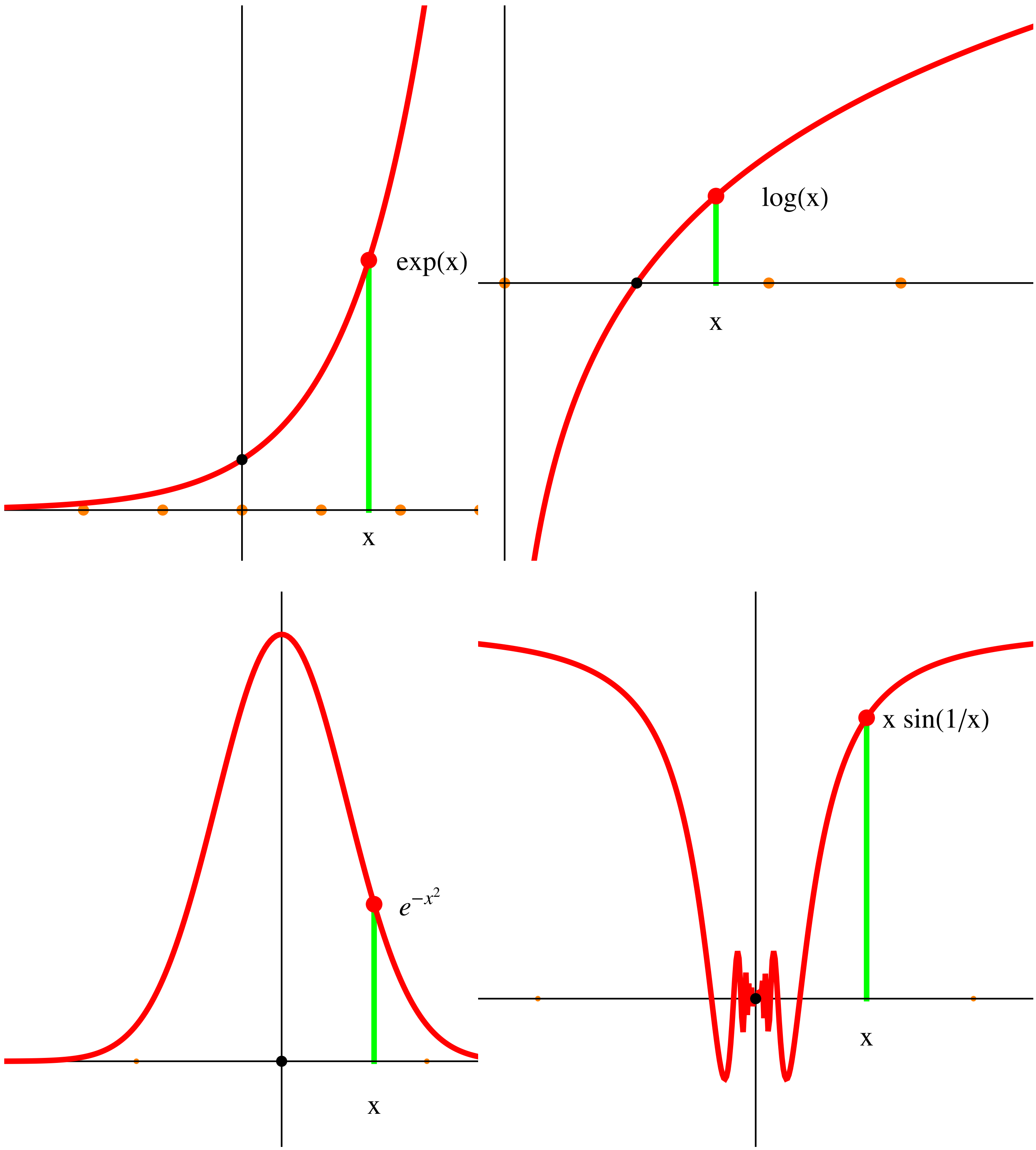
addition	$f(x) + g(x)$
scaling	$2f(x)$
translate	$f(x + 1)$
compose	$f(g(x))$
invert	$f^{-1}(x)$
difference	$f(x + 1) - f(x)$
sum up	$f(x) + f(x + 1) + \dots$

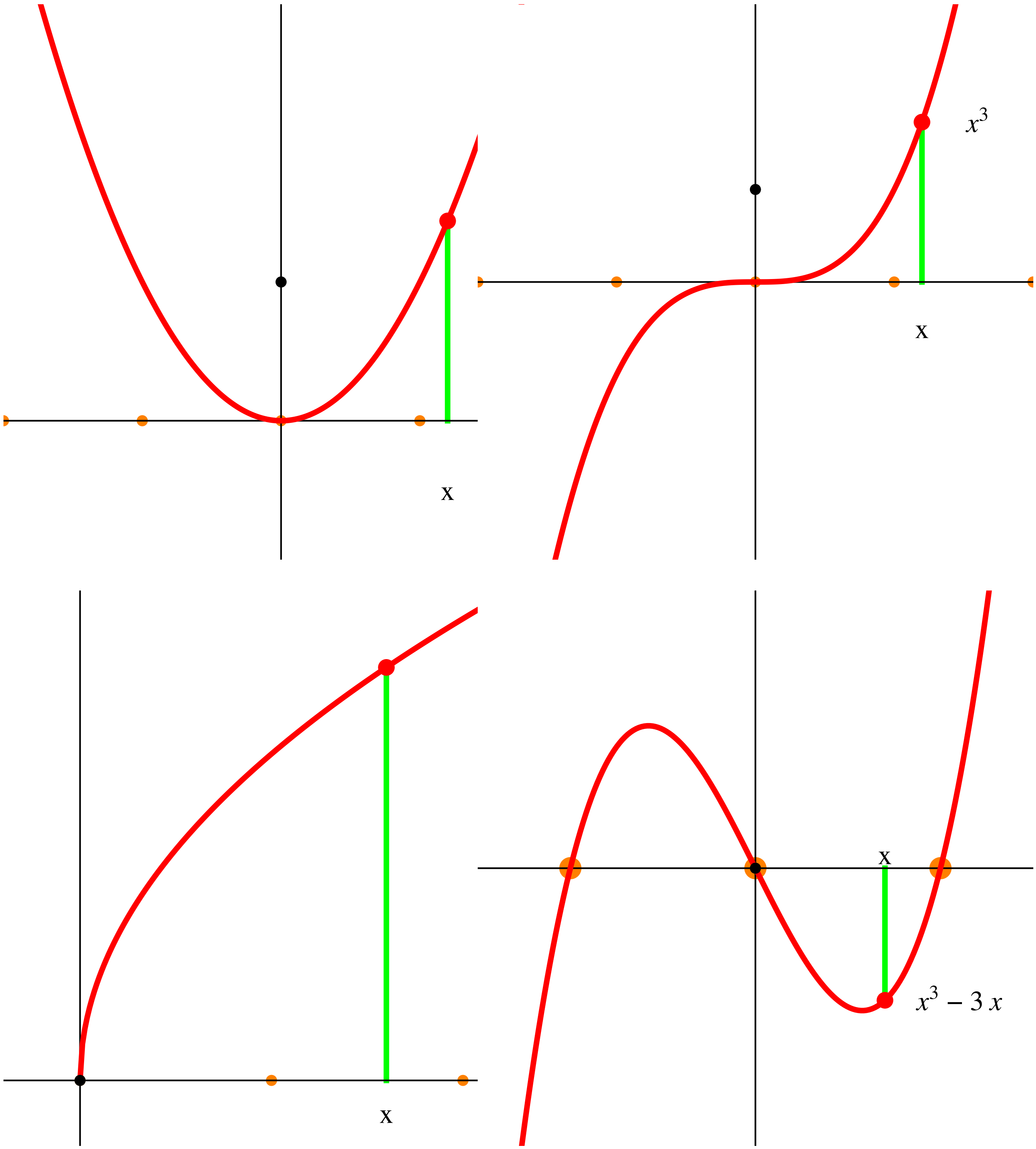
Here are important functions:

polynomials	$x^2 + 3x + 5$
rational functions	$(x + 1)/(x^4 + 1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arcsin^{-1}(x), \arctan(x)$.
roots	$\sqrt{x}, x^{1/3}$

We will look at these functions **a lot** during the semester. The logarithm, exponential and trigonometric functions are especially important. For some functions, we need to restrict the domain, where the function is defined. For the square root function $\sqrt{x}$ or the logarithm $\log(x)$ for example, we have to assume that the number x on which we evaluate the function is positive. We write that the domain is $(0, \infty) = \mathbf{R}^+$. For the function $f(x) = 1/x$, we have to assume that x is different from zero. Keep these three examples in mind.

The **graph** of a function is the set of points $\{(x, y) = (x, f(x))\ \}$ in the plane, where x runs over the domain A of f. Graphs allow us to **visualize** functions. We can "see them", when we draw the graph.




x
x^3
x
x
x
x^3 − 3 x

Homework

1 Draw the function $f(x) = x^3 \cos(x)$. Its graph goes through the origin $(0, 0)$. You can use Wolfram Alpha or a calculator to plot it if you like.
a) A function is called **odd** if $f(-x) = -f(x)$. Is f odd?
b) A function is called **even** if $f(x) = f(-x)$. Is f even?
c) A function is called **monotone increasing** if $f(y) > f(x)$ if $y > x$. Is f monotone increasing on the interval $[-1, 1]$? There is need to decide this yet analytically. Just draw(*) and decide.

2 A function $f : A \to B$ is called **invertible** or **one to one** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$. In that case, the function g is called the **inverse** of f. For example, the function $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0, \infty)$ to $B = [0, \infty)$. Determine from the following functions whether they are invertible. If they are invertible, find the inverse.
a) $f(x) = \cos(x)$ from $A = [0, \pi/2]$ to $B = [0, 1]$
b) $f(x) = \sin(x)$ from $A = [0, \pi]$ to $B = [0, 1]$
c) $f(x) = x^3$ from $A = \mathbf{R}$ to $B = \mathbf{R}$
d) $f(x) = \exp(x)$ from $A = \mathbf{R}$ to $B = R^+ = (0, \infty)$.
e) $f(x) = 1/(1 + x^2)$ from $A = [0, \infty)$ to $B = (0, 1]$.

3 We have defined $\log_h(x)$ as the inverse of $\exp_h(x) = (1 + h)^{x/h}$.
a) Draw the graphs of $\exp_1(x), \exp_{1/2}(x), \exp_{1/10}(x)$ and $\exp(x)$.
b) Draw the graphs of $\log_1(x), \log_{1/2}(x), \log_{1/10}(x)$ and $\log(x)$.
You are welcome to use technology for a). For b), just "flip the graph".

4 Plot the Function $\exp(\exp(\exp(x)))$ on $[0, 1]$. This is a fine function but computer programs do not plot the graph correctly. Describe until where Mathematica or Wolfram plots the function.

5 A function $f(x)$ has a **root** at $x = a$ if $f(a) = 0$. Roots are places, where the function is zero. Find one root for each of the following functions or state that there is none.
a) $f(x) = \cos(x)$
b) $f(x) = 4\exp(-x^4)$
c) $f(x) = x^5 - x^3$
d) $f(x) = \log(x) = \ln(x)$, the inverse of exp
e) $f(x) = \sin(x) - 1$
f) $f(x) = \csc(x) = 1/\sin(x)$

(*) Here is how you can use the Web to plot a function. The example given is $\sin(x)$.

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http://www.wolframalpha.com/input/?i=Plot+sin(x)
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