

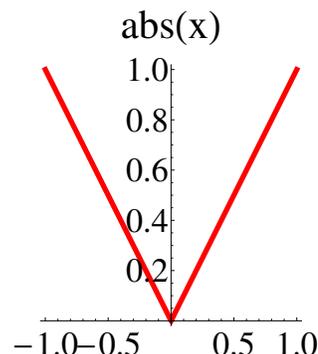
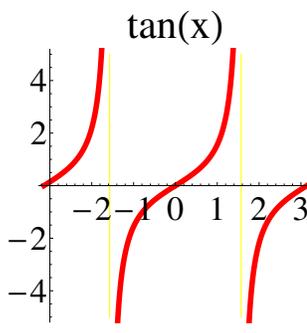
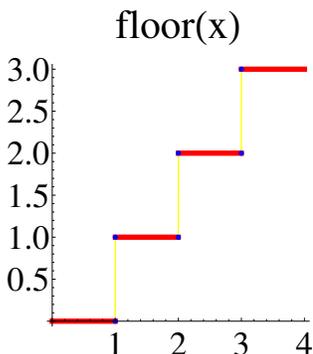
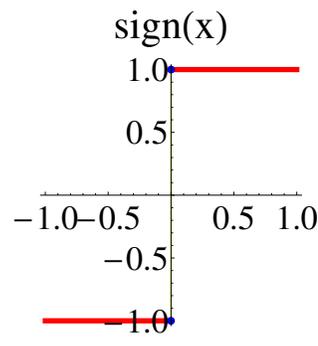
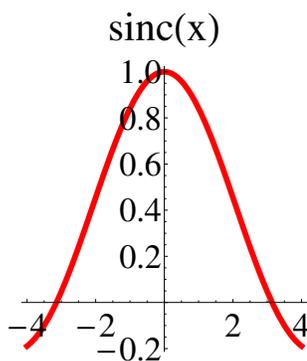
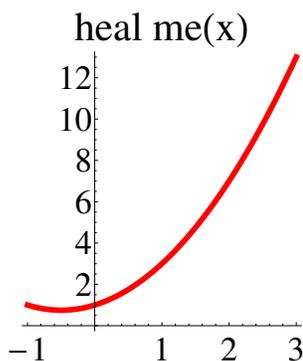
## Lecture 3: Limits

We have seen that functions like  $1/x$  are not defined everywhere. Sometimes, however, functions do not make sense at first at some points but can be fixed. A silly example is  $f(x) = x/x$  which is a priori not defined at  $x = 0$  because we divide by 0 but can be "saved" by noticing that  $f(x) = 1$  for all  $x$  different from 0. Functions often can be continued to "forbidden" places if we write the function differently. This can involve dividing out a common factor. Let's look at examples:

- 1 The function  $f(x) = (x^3 - 1)/(x - 1)$  is at first not defined at  $x = 1$ . However, for  $x$  close to 1, nothing really bad happens. We can evaluate the function at points closer and closer to 1 and get closer and closer to 3. We will say  $\lim_{x \rightarrow 1} f(x) = 3$ . Indeed, as you might have noticed already, we have  $f(x) = x^2 + x + 1$  by factoring out the term  $(x - 1)$ . While the function was initially not defined at  $x = 1$ , we can assign a natural value 3 at the point  $x = 1$  so that the graph continues nicely through that point.

We write  $x \rightarrow a$  to indicate that  $x$  approaches  $a$ . This approach can be from either side. A function  $f(x)$  has a **limit** at a point  $a$  if there exists  $b$  such that  $f(x) \rightarrow b$  for  $x \rightarrow a$ . We write  $\lim_{x \rightarrow a} f(x) = b$ . Again, it should not matter, whether we approach  $a$  from the left or from the right. If the limit exists, we must get the same limiting value  $b$ .

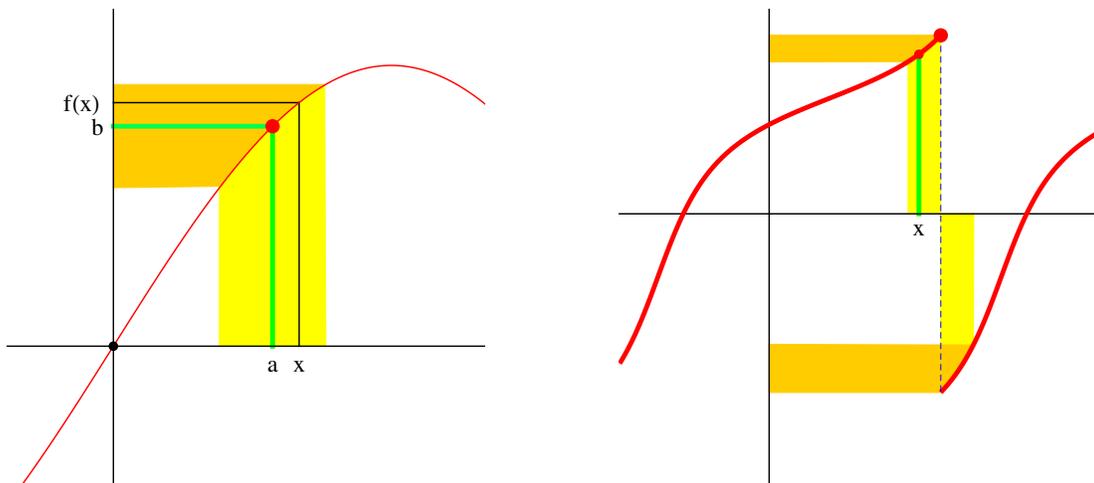
- 2 The function  $f(x) = \sin(x)/x$  is called  $\text{sinc}(x)$ . We see experimentally that it converges to 1 as  $x \rightarrow 0$ . We can see this geometrically by comparing the side  $a = \sin(x)$  of a right angle triangle with a small angle  $\alpha = x$  and hypotenuse 1 with the length of the arc between  $B, C$  of the unit circle centered at  $A$ . The arc has length  $x$  which is close to  $\sin(x)$  for small  $x$ . Keep this example in mind. It is a crucial one. The fact that the limit of  $f(x)$  exists for  $x \rightarrow 0$  is so important that it is sometimes called the **fundamental theorem of trigonometry**. We will see this later.



**Figure:** The graphs  $f(x) = (x^3 - 1)/(x - 1)$ , the sinc function  $\text{sinc}(x) = \sin(x)/x$ , the sign function  $\text{sign}(x) = x/|x|$ , the floor function  $\text{floor}(x)$  giving the largest integer smaller or equal to  $x$ , the tan function and the absolute value function  $\text{abs}(x) = |x|$ .

- 3 The function  $f(x) = x/|x|$  is equal to 1 if  $x > 0$  and equal to  $-1$  if  $x < 0$ . The function is not defined at  $x = 0$  and there is no way to assign a value  $b$  at  $x = 0$  so that  $\lim_{x \rightarrow 0} f(x) = b$ . After defining  $f(0) = 0$  we can call the function the *sign* function.
- 4 The quadratic function  $f(x) = x^2$  has the property that  $f(x)$  approaches 4 if  $x$  approaches 2. To evaluate functions at a point, we do not have to take a limit. The function is already defined there. This is important: most points are "healthy". We do not have to worry about limits. In the overwhelming cases of real applications we only have to worry about limits when the function involves division by 0. For example  $f(x) = (x^4 + x^2 + 1)/x$  needs to be investigated carefully at  $x = 0$ . You see for example that for  $x = 1/1000$ , the function is slightly larger than 1000, for  $x = 1/1000000$  it is larger than one million. There is no rescue. The limit does not exist at 0.
- 5 More generally, for all polynomials, the limit  $\lim_{x \rightarrow a} f(x) = f(a)$  is defined. We do not have to worry about limits, if we deal with polynomials.
- 6 For all trigonometric polynomials involving  $\sin$  and  $\cos$ , the limit  $\lim_{x \rightarrow a} f(x) = f(a)$  is defined. We do not have to worry about limits if we deal with trigonometric polynomials like  $\sin(3x) + \cos(5x)$ . The function  $\tan(x)$  however has no limit at  $x = \pi/2$ . No finite value  $b$  can be found so that  $\tan(\pi/2 + h) \rightarrow b$  for  $h \rightarrow 0$ . This is due to the fact that  $\cos(x)$  is zero at  $\pi/2$ . We have  $\tan(x)$  goes to  $+\infty$  "plus infinity" for  $x \searrow \pi/2$  and  $\tan(x)$  goes to  $-\infty$  for  $x \nearrow \pi/2$ . In the first case, we approach  $\pi/2$  from the right and in the second case from the left.
- 7 The **cube root** function  $f(x) = x^{1/3}$  converges to 0 as  $x \rightarrow 0$ . For  $x = 1/1000$  for example, we have  $f(x) = 1/10$  for  $x = 1/n^3$  the value  $f(x)$  is  $1/n$ . The cube root function is defined everywhere on the real line, like  $f(-8) = -2$  and is continuous everywhere.

Why do we worry about limits at all? One of the main reasons will be that we will define the derivative and integral using limits. But we will also use limits to get numbers like  $\pi = 3.1415926, \dots$ . In the next lecture, we will look at the important concept of continuity, which involves limits too.



**Figure:** To the left we see a case, where the limit exists at  $x = a$ . If  $x$  approaches  $a$  then  $f(x)$  approaches  $b$ . To the right we see the function  $f(x) = \arctan(\tan(x) + 1)$ , where  $\arctan$  is the inverse of  $\tan$ . The limit does not exist for  $a = \pi/2$ . If we approach  $a$  from

the right, we get the limit  $-\pi/2$  From the left, we get the limit  $f(\pi/2) = \pi/2$ . Note that  $f$  is not defined at  $x = \pi/2$  because  $\tan(x)$  becomes infinite there.

**8 Problem:** Determine from the following functions whether the limits  $\lim_{x \rightarrow 0} f(x)$  exist. If it does, find it.

- a)  $f(x) = \cos(x)/\cos(2x)$     b)  $f(x) = \tan(x)/x$   
 c)  $f(x) = (x^2 - x)/(x - 1)$     d)  $f(x) = (x^4 - 1)/(x^2 - 1)$   
 e)  $f(x) = (x + 1)/(x - 1)$     f)  $f(x) = x/\sin(x)$   
 g)  $f(x) = 5x/\sin(6x)$     h)  $f(x) = \sin(x)/x^2$   
 i)  $f(x) = \sin(x)/\sin(2x)$     j)  $f(x) = \exp(x)/x$

**Solutions:**

- a) There is no problem at all at  $x = 0$ . Both, the nominator and denominator converge to 1. The limit is  $\boxed{1}$   
 b) This is  $\text{sinc}(x)/\cos(x)$ . There is no problem at  $x = 0$  for  $\text{sinc}$  nor for  $1/\cos(x)$ . The limit is  $\boxed{1}$ .  
 c) We can heal this function. It is the same as  $x + 1$  everywhere except at  $x = 1$  where it is not defined. But we can continue the simplified function  $x + 1$  through  $x = 1$ . The limit is  $\boxed{0}$ .  
 d) We can heal this function. It is the same as  $x^2 + 1$ . The limit is  $\boxed{1}$ .  
 e) There is no problem at  $x = 0$ . There is mischief at  $x = 1$  although but that is far, far away. At  $x = 0$ , we get  $\boxed{-1}$ .  
 f) This is the prototype, the fundamental theorem of trig! We know that the limit is  $\boxed{1}$ .  
 g) This can be written as  $f(x) = (5/6)6x/\sin(6x) = (5/6)\text{sinc}(6x)$ . The function  $6x/\sin(6x)$  converges to 1 by the fundamental theorem of trigonometry. Therefore the limit is  $\boxed{5/6}$ .  
 h) This limit does not exist. It can be written as  $\text{sinc}(x)/x$ . Because  $\text{sinc}(x)$  converges to 1. we are in trouble when dividing again by  $x$ .  $\boxed{\text{There is no limit.}}$   
 i) We know  $\sin(x)/x \rightarrow 1$  so that also  $\sin(2x)/(2x)$  has the limit 1. If we divide them, see  $\sin(x)/\sin(2x) \rightarrow 1/2$ . The result is  $\boxed{1/2}$ .  
 j) The limit does not exist because  $\exp(x) \rightarrow 1$  but  $1/x$  goes to infinity.

Here are obvious properties which hold for limits:

$$\begin{aligned} \lim_{x \rightarrow a} f(x) = b \text{ and } \lim_{x \rightarrow a} g(x) = c &\text{ implies } \lim_{x \rightarrow a} f(x) + g(x) = b + c. \\ \lim_{x \rightarrow a} f(x) = b \text{ and } \lim_{x \rightarrow a} g(x) = c &\text{ implies } \lim_{x \rightarrow a} f(x) \cdot g(x) = b \cdot c. \\ \lim_{x \rightarrow a} f(x) = b \text{ and } \lim_{x \rightarrow a} g(x) = c \neq 0 &\text{ implies } \lim_{x \rightarrow a} f(x)/g(x) = b/c. \end{aligned}$$

This implies we can sum up and multiply or divide functions which have limits:

Polynomials like  $x^5 - 2x + 6$  have limits everywhere.  
 Trig polynomials like  $\sin(3x) + \cos(5x)$  have limits everywhere.  
 Rational functions have limits except at points where the denominator is zero.  
 Functions like  $\cos^2(x) \tan(x)/\sin(x)$  can be healed by simplification.  
 Prototype functions like  $\sin(x)/x$  have limits everywhere

# Homework

1 Find the limits of each of the following functions at the point  $x \rightarrow 0$ . You can already use the fact that  $\sin(x)/x$  has the limit 1 as  $x \rightarrow 0$ .

- $f(x) = (x^4 - 1)/(x - 1)$
- $f(x) = \sin(5x)/x$
- $f(x) = \sin^2(3x)/x^2$
- $f(x) = \sin(7x)/\sin(11x)$

2 a) Graph of the function

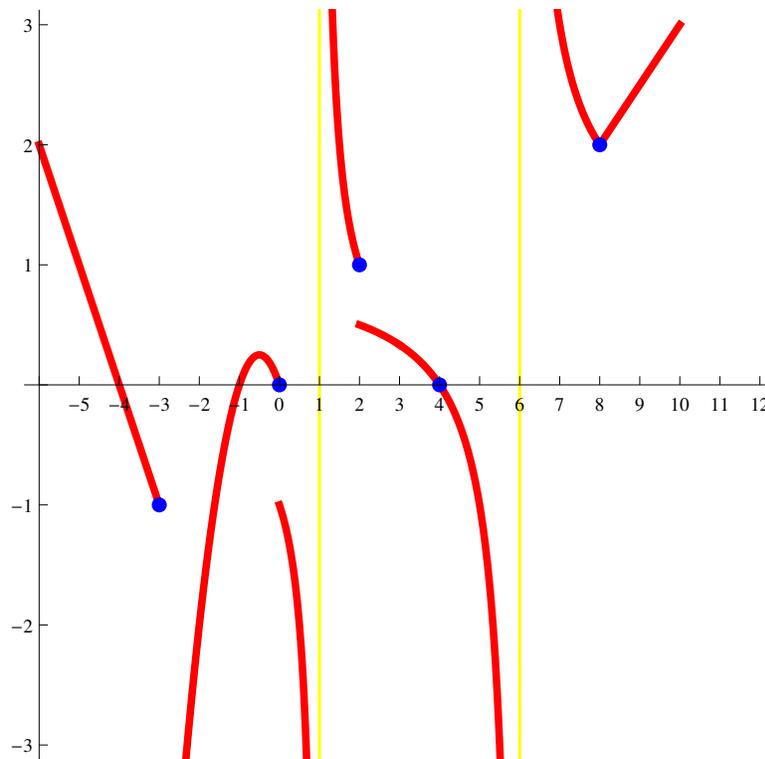
$$f(x) = \frac{(1 - \cos(x))}{x^2}.$$

b) Where is the function  $f$  defined? Can you find the limit at the places, where it is not defined? Hint: use the **double angle formula**  $1 - \cos(x) = 2\sin^2(x/2)$ .

c) Verify that the function  $f(x) = \exp_h(x) = (1+h)^{x/h}$  satisfies  $[f(x+h) - f(x)]/h = f(x)$ .

**Remark.** The exponential function can be defined as  $e^x = \exp(x) = \lim_{h \rightarrow 0} \exp_h(x)$ .

3 Find all points  $a$  at which the function given in the picture has no limits.



4 Find the limits for  $x \rightarrow 0$ :

- $f(x) = (x^2 - 2x + 1)/(x - 1)$ .
- $f(x) = 2^x$ .
- $f(x) = 2^{2^x}$ .
- $f(x) = \sin(\sin(x))/\sin(x)$ .

5 We explore in this problem the limit of the function  $f(x) = x^x$  if  $x \rightarrow 0$ . Can we find a limit? Take a calculator or use Wolfram  $\alpha$  and experiment. What do you see when  $x \rightarrow 0$ ? Only optional: can you find an explanation for your experiments?