Lecture 6: Fundamental theorem

Calculus is the theory of **differentiation** and **integration**. We explore this concept in a simple setup and practice differentiation and integration without taking limits. We fix a positive constant h and take differences and sums. Without taking limits, we prove a fundamental theorem of calculus. We can so differentiate and integrate polynomials, exponentials and trigonometric functions. Later, we will do the same with actual derivatives and integrals. But now, we can work with arbitrary continuous functions. The constant h might confuse you a bit at first. Just think of it as something small. If you like, take $h = 1$ everywhere. One of the advantages of the calculus we are going to learn is that h will no more appear.

Given a function $f(x)$, define the **difference quotient**

$$Df(x) = (f(x+h) - f(x)) \tfrac{1}{h}$$

If f is continuous then Df is a continuous. For shorthand, we call it simply the "derivative". It will in the limit $h \to 0$ become the derivative we are going to define later in the course but we keep h constant and positive here.

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1 Lets take the constant function $f(x) = 5$. We get $Df(x) = (f(x+h)-f(x))/h = (5-5)/h = 0$ everywhere. You can see that in general, if f is a constant function, then $Df(x) = 0$.

2 $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$. You see in general that if f is a linear function $f(x) = ax + b$, then $Df(x) = a$ is constant.

3 If $f(x) = ax + b$, then $Df(x) = \boxed{a}$.

For constant functions, the derivative is zero. For linear functions, the derivative is the slope.

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4 For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h$ which is $\boxed{2x + h}$.

Given a function f, define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \le jh < x$. That is, if k is such that $(k-1)h$ is the largest below x, then

$$Sf(x) = h[\, f(0) + f(h) + f(2h) + \cdots + f((k-1)h)\,]$$

In short hand, we call Sf also the "integral" or "antiderivative" of f. It will become the integral in the limit $h \to 0$ later in the course.

**5** Compute $Sf(x)$ for $f(x) = 1$. **Solution**. We have $Sf(x) = 0$ for $x \le h$, and $Sf(x) = h$ for $h \le x < 2h$ and $Sf(x) = 2h$ for $2h \le x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but grows in quantized steps.

The difference $Df(x)$ will become the **derivative** $f'(x)$.

The sum $Sf(x)$ will become the **integral** $\int_0^x f(t)\, dt$.

Df means rise over run and is close to the slope of the graph of f.

Sf means areas of rectangles and is close to the area under the graph of f.
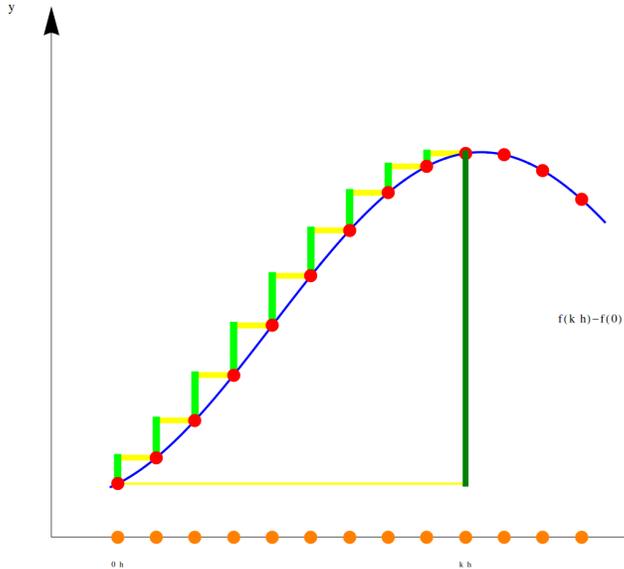
Theorem: Sum the differences and get
$$SDf(kh) = f(kh) - f(0)$$
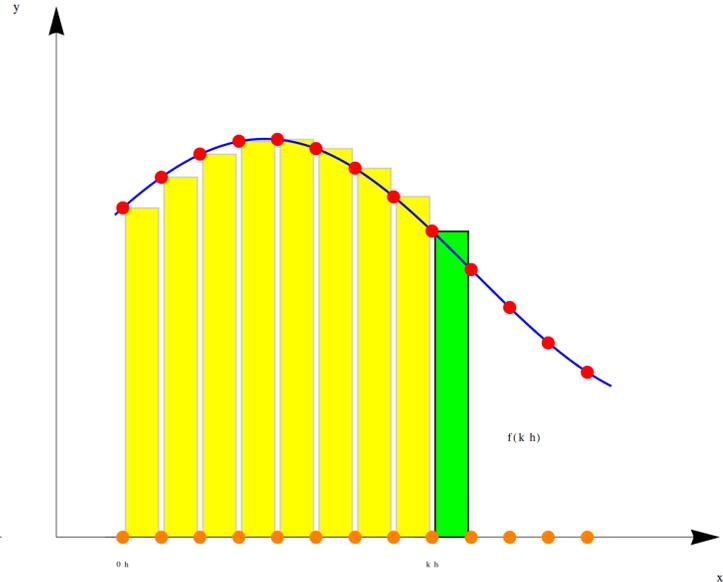
Theorem: Difference the sum and get
$$DSf(kh) = f(kh)$$

6 For $f(x) = [x]_h^m = x(x-h)(x-2h)...(x-mh+h)$ we have

$$f(x+h) - f(x) = (x(x-h)(x-2h)...(x-kh+2h))\,((x+h) - (x-mh+h)) = [x]^{m-1}hm$$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. We have obtained the important formula $\boxed{D[x]^m = m[x]^{m-1}}$

We can establish from this differentiation formulas for **polynomials**. We will leave away the square brackets often and do the calculus we will do later on.

7 If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$.

The fundamental theorem allows us to integrate and get the right values at the points k/n:

8 Find Sf for the same function. The answer is $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero. As a rescaled exponential, it is continuous and monotone. Indeed, using rules of the logarithm we can see $\exp_h(x) = e^{x(\log(1+h)/h)}$. We see that it agrees with the exponential function after a rescaling of x.

9 You have already computed the derivative in a homework. Lets do it again. The function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. **Solution**: $\exp_h(x+h) = (1+h)\exp(x)$ shows that. $\boxed{D\exp_h(x) = \exp_h(x)}$

10 Define $\exp(a \cdot x) = (1 + ah)^{x/h}$. It satisfies $\boxed{D\exp_h(a \cdot x) = a\exp_h(a \cdot x)}$ We write a dot because $\exp_h(ax)$ is not equal to $\exp_h(a \cdot x)$. For now, only the differentiation rule for this function is important.

11 This is example which uses complex numbers $i = \sqrt{-1}$. If we allow a to become complex, we get $\exp(1 + ia)(1 + aih)^{x/h}$. We still have $D\exp_h^{ai}(x) = ai\exp_h^{ai}(x)$. Taking real and imaginary parts define new functions $\exp_h^{ai}(x) = \cos_h(a \cdot x) + i\sin_h(a \cdot x)$. These functions are real and morph into the familiar cos and sin functions for $h \to 0$. For any $h > 0$ and any a, we have now $\boxed{D\cos_h(a \cdot x) = -a\sin_h(a \cdot x) \text{ and } D\sin_h(a \cdot x) = a\cos_h(a \cdot x).}$ No worries, we will derive these identities for the usual trig functions.

Homework

We leave the h away in this homework. To have more fun, also define $\log_h$ as the inverse of $\exp_h$ and define $1/[x]_h = D\log_h(x)$ for $x > 0$. If we start integrating from 1 instead of 0 as usual we write $S_1 f$ and get $S_1 1/[x]_h = \log_h(x)$. We also write simply x^n for $[x]_h^n$ and write $\exp(a \cdot x) = e^{a \cdot x}$ instead of $\exp_h^a(x)$ and $\log(x)$ instead of $\log_h(x)$. Use the differentiation and integration rules on the right to find derivatives and integrals.

1 Find the derivatives $Df(x)$ of the following functions:

a) $f(x) = x^{11} - 3x^4 + 5x + 1$
a) $f(x) = -x^4 + 8\log(x)$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Df(0)$?

2 Find the integrals $Sf(x)$ of the following functions assuming $Sf(0) = 0$:

a) $f(x) = x^9$.
b) $f(x) = x^2 + 6x^7 - x$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Sf(1)$ in the case $h = 1$?

3 Find the derivatives $Df(x)$ of the following functions

a) $f(x) = \exp(9 \cdot x) + x^6$
b) $f(x) = 8\exp(-3 \cdot x) + 9x^6$
c) $f(x) = -\exp(5 \cdot x) + x^6$

4 Find the integrals $Sf(x)$ of the following functions

a) $f(x) = \exp(6 \cdot x) - 3x^6$
b) $f(x) = \exp(8 \cdot x) + x^6$
c) $f(x) = -\exp(5 \cdot x) + x^6$

5 Define $f(x) = \sin(4 \cdot x) - \exp(2 \cdot x) + x^5$.
a) Find $Df(x)$
b) Find $g(x) = Sf(x)$
c) Verify that $Dg(x) = f(x)$.

All calculus on 1/3 page

Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

Differentiation rules
$Dx^n = nx^{n-1}$
$De^{a \cdot x} = ae^{a \cdot x}$
$D\cos(a \cdot x) = -a\sin(a \cdot x)$
$D\sin(a \cdot x) = a\cos(a \cdot x)$
$D\log(x) = 1/x$

Integration rules (for $x = kh$)
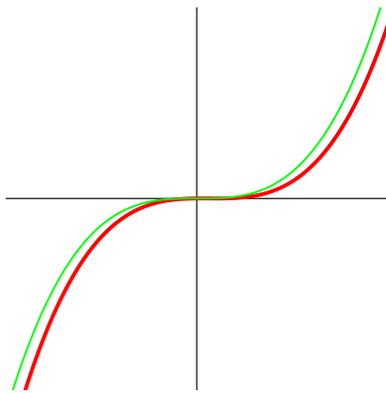$Sx^n = x^{n+1}/(n+1)$
$Se^{a \cdot x} = e^{a \cdot x}/a$
$S\cos(a \cdot x) = \sin(a \cdot x)/a$
$S\sin(a \cdot x) = -\cos(a \cdot x)/a$
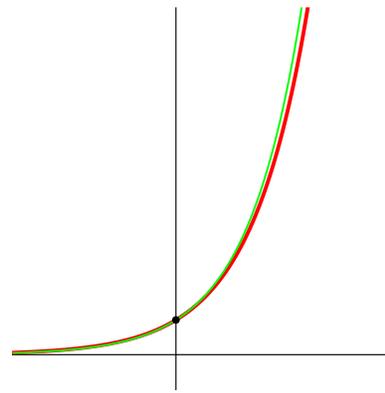$S\frac{1}{x} = \log(x)$

Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x + h)$.
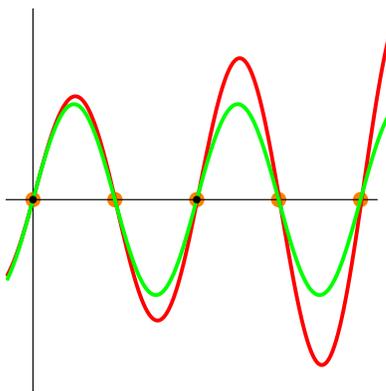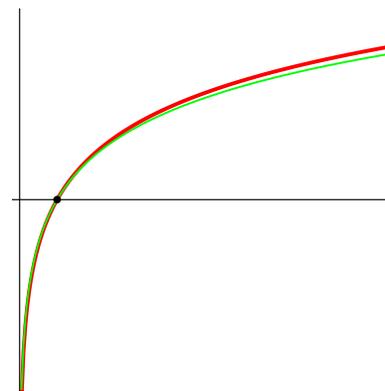
Pictures

$[x]_h^3$ for $h = 0.1$

$\exp_h(x)$ for $h = 0.1$

$\sin_h(x)$ for $h = 0.1$

$\log_h(x)$ for $h = 0.1$