

# Lecture 14: Newton's method

Recall that a point  $a$  is called a **root** of a function  $f$  if  $f(a) = 0$ . We were able to find the roots of functions using a “divide and conquer” technique: start with an interval  $[a, b]$  for which  $f(a) < 0$  and  $f(b) > 0$ . If  $f((a+b)/2)$  is positive, then use the interval  $[a, (a+b)/2]$  otherwise  $[(a+b)/2, b]$ . After  $n$  steps, we are  $(b-a)/2^n$  close to the root.

If the function  $f$  is differentiable, we can do much better. We can use the value of the derivative to get closer to a point  $y = T(x)$ . Lets find  $y$ . If we draw a tangent at  $(x, f(x))$ , then

$$f'(x) = \frac{f(x)}{x - y}.$$

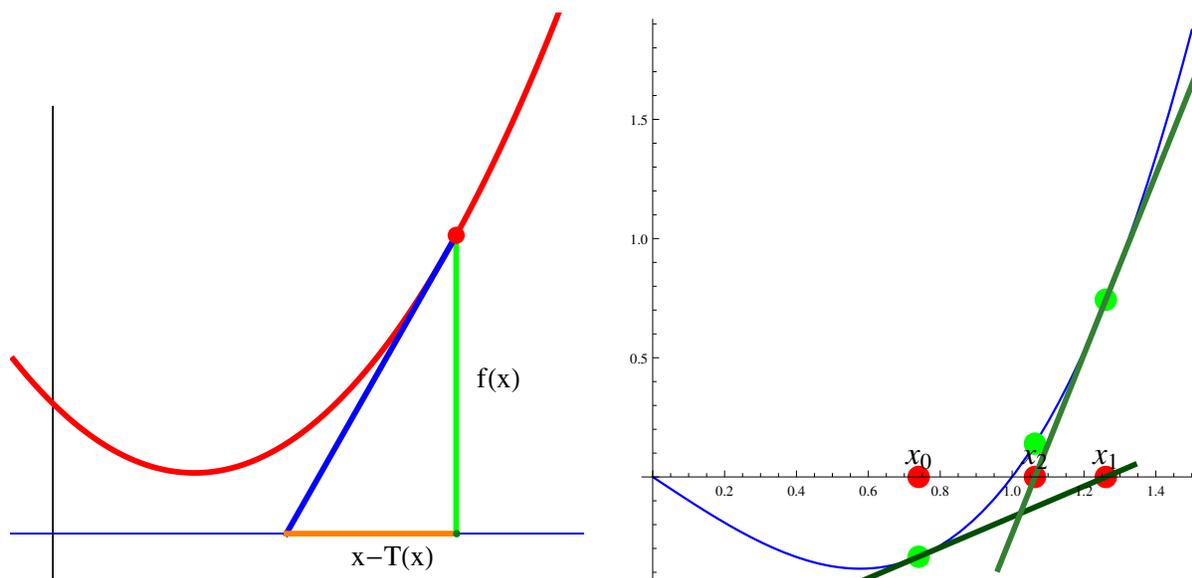
because  $f'(x)$  is the slope of the tangent and the right hand side is ”rise” over ”run”. If we solve for  $y$  we get

The **Newton map** is defined as

$$y = T(x) = x - \frac{f(x)}{f'(x)}.$$

**Newton's method** is the process of applying this map again and again until we are sufficiently close to the root. It is an extremely fast method to find the root of a function. Start with a point  $x$ , then compute a new point  $x_1 = T(x)$ , then  $x_2 = T(x_1)$  etc.

If  $p$  is a root such that  $f'(p) \neq 0$ , and  $x_0$  is close enough to  $p$ , then  $x_1 = T(x_0), x_2 = T^2(x_0)$  converges to the root  $p$ .



- 1 If  $f(x) = ax + b$ , we reach the root in one step.
- 2 If  $f(x) = x^2$  then  $T(x) = x - x^2/(2x) = x/2$ . We get exponentially fast to the root 0 but not as fast as the method promises. Indeed, the root 0 is also a critical point of  $f$ . This slows us down.
- 3 The Newton map brings us to infinity if we start at a critical point.

Newton used this method to find the roots of polynomials. It is amazingly fast: Starting 0.1 close to the point, we have after one step 0.01 after 2 steps 0.0001 after 3 steps 0.00000001 and after 4 steps 0.0000000000000001.

The Newton method converges extremely fast to a root  $f(p) = 0$  if  $f'(p) \neq 0$  if we start sufficiently close to the root.

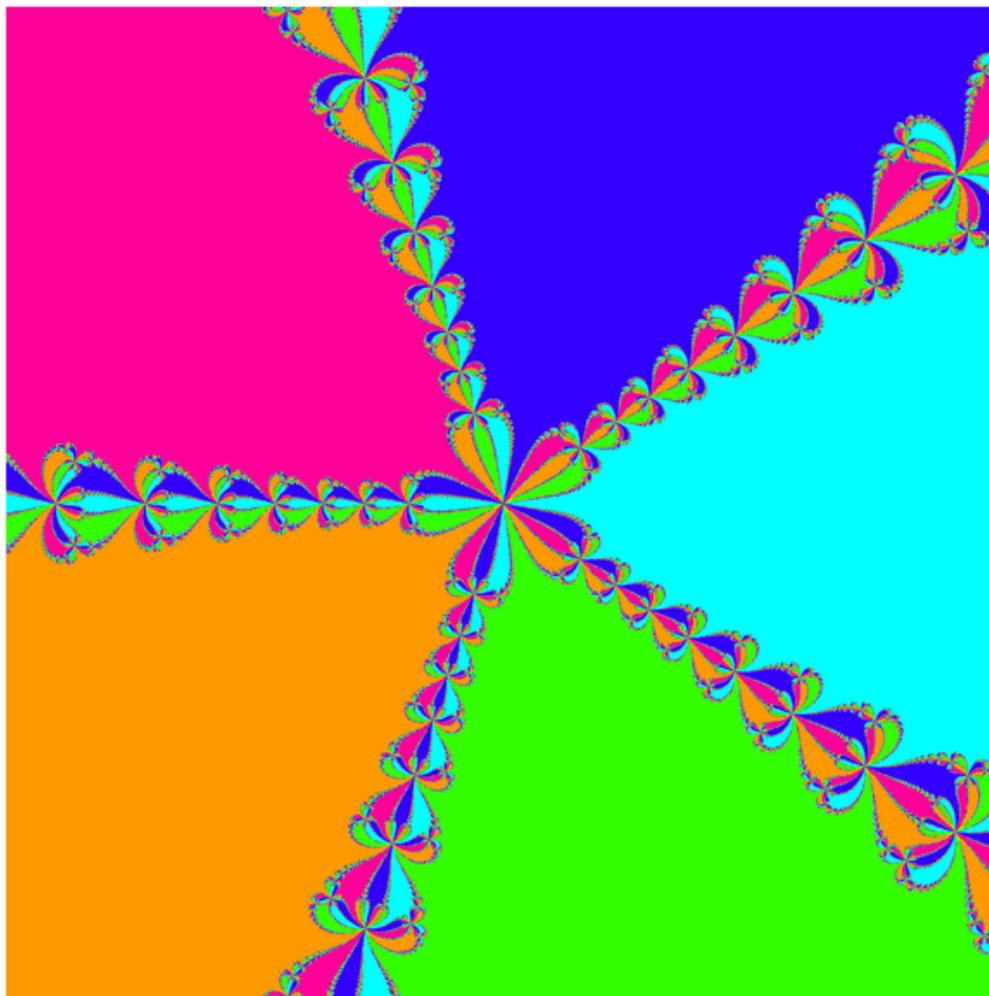
In 10 steps we can get a  $2^{10} = 1024$  digits accuracy. Having a fast method to compute roots is useful. For example in computer graphics, where things can not be fast enough. Also in number theory, when working with integers having thousands of digits the Newton method can help. There is much theoretical use of the method. It goes so far as to explain stability of planetary motion or stability of plasma in fusion reactors.

- 4 Verify that the Newton map  $T(x)$  in the case  $f(x) = (x - 1)^3$  has the property that we approach the root  $x = 1$ . **Solution.** You see that the approach is not that fast: we get  $T(x) = x + (1 - x)/3 = (1 + 2x)/3$ . It converges exponentially fast, but not super exponential. The reason is that the derivative at  $x - 1$  is not zero. That slows us down.

If we have several roots, and we start at some point, to which root will the Newton method converge? Does it at all converge? This is an interesting question. It is also historically intriguing because it is one of the first cases, where "chaos" was observed at the end of the 19'th century.

- 5 Find the Newton map in the case  $f(x) = x^5 - 1$ . **Solution**  $T(x) = x - (x^5 - 1)/(5x^4)$ .

If we look for roots in the complex like for  $f(x) = x^5 - 1$  which has 5 roots in the complex plane, the "basin of attraction" of each of the roots is a complicated set which we call the **Newton fractal**. Here is a picture:



- 6 Lets compute  $\sqrt{2}$  to 12 digits accuracy. We want to find a root  $f(x) = x^2 - 2$ . The Newton map is  $T(x) = x - (x^2 - 2)/(2x)$ . Lets start with  $x = 1$ .

$$T(1) = 1 - (1 - 2)/2 = 3/2$$

$$T(3/2) = 3/2 - ((3/2)^2 - 2)/3 = 17/12$$

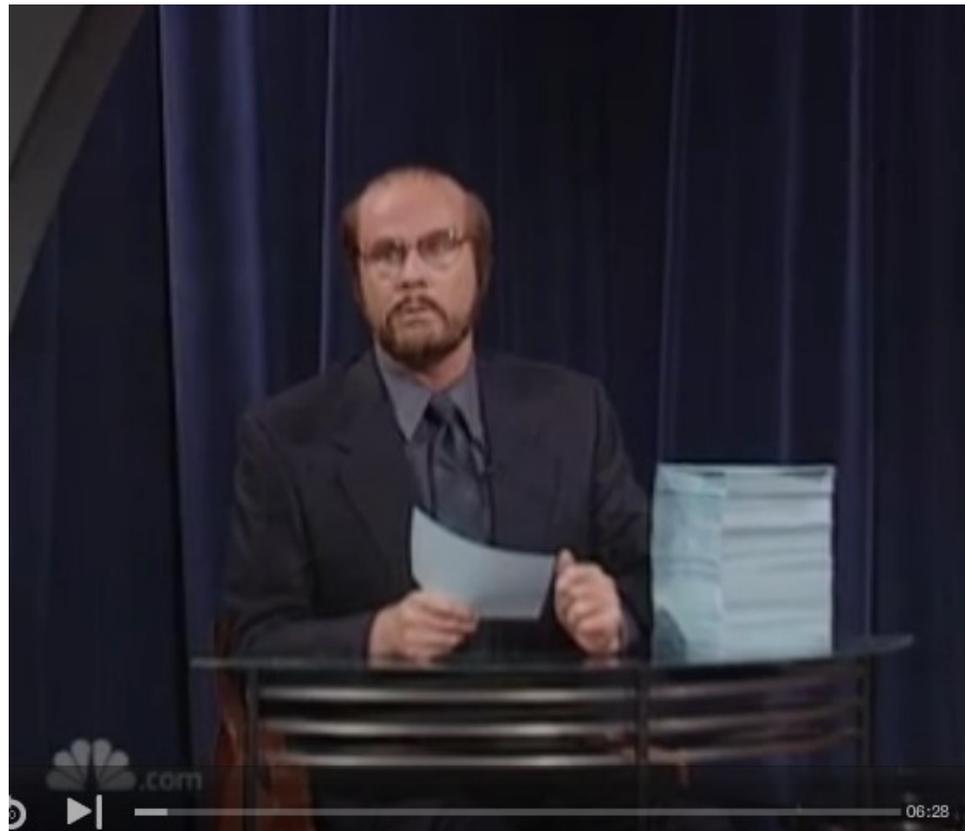
$$T(17/12) = 577/408$$

$$T(577/408) = 665857/470832 .$$

This is already  $1.6 \cdot 10^{-12}$  close to the real root! 12 digits, by hand! My grandfather Sigbert Bader (who was a work-safety officer at the SIG industries, and competed at the world championship in gymnastics and later owned and operated a bicycle shop) mastered this technique very well and continued to ask me while I studied mathematics: "How accurately can you compute square roots?" - I can now proudly answer: 12 digits, Grand pa! He would probably laugh and get 20 digits.

- 7 To find the cube root of 10 we have to find a root of  $f(x) = x^3 - 10$ . The Newton map is  $T(x) = x - (x^3 - 10)/(3x^2)$ . If we start with  $x = 2$ , we get the following steps 2, 13/6, 3277/1521, 105569067476/49000820427. After three steps we have a result which is already  $2.2 \cdot 10^{-9}$  close to the root.

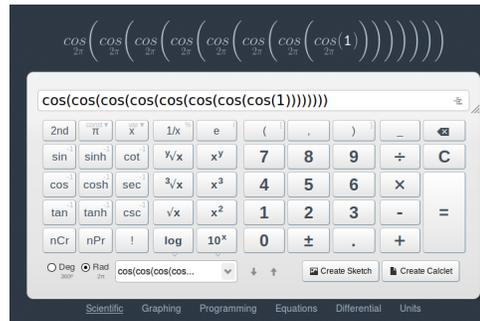
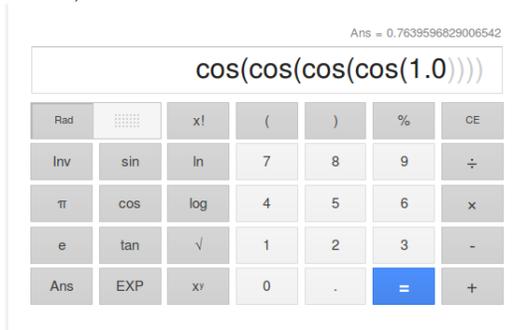
The Newton method is scrumtrulescent!



Will Ferrell's sketch: "Inside the Actors Studio" at Saturday Night live created the term "scrumtrulescent". See <http://www.hulu.com/watch/3524>

# Homework

- 1 Find a formula for the Newton map  $T(x) = x - f(x)/f'(x)$  in the following cases
- $f(x) = (x - 2)^2$
  - $f(x) = e^{5x}$
  - $f(x) = 2e^{-x^2}$
  - $f(x) = \cot(x)$ .
- 2 The function  $f(x) = \cos(x) - x$  has a root between 0 and 1. We get closer to the root by doing one Newton step starting with  $x = 1$ . (Compare with the actual root  $x = 0.739085\dots$ , which you can get on any calculator by entering 1, then punching the key "cos" a lot of times).



- 3 We want to find the square root of 102. We have to solve  $\sqrt{102} = x$  or  $f(x) = x^2 - 102 = 0$ . Perform two Newton steps starting at  $x = 10$ .
- 4 Find the Newton step  $T(x) = x - f(x)/f'(x)$  in the case  $f(x) = 1/x$ . What happens if you apply the Newton step again and again?
- 5 Verify that the Newton map in the case  $f(x) = (4 - 3/x)^{1/3}$  is the quadratic map  $T(x) = 4x(1 - x)$ .

This is an example of a chaotic map. The Newton step does not converge.

