

# Lecture 18: Riemann integral

In this lecture, we define the integral  $\int_0^x f(t) dt$  if  $f$  is a differentiable function. We then compute it for some basic functions.

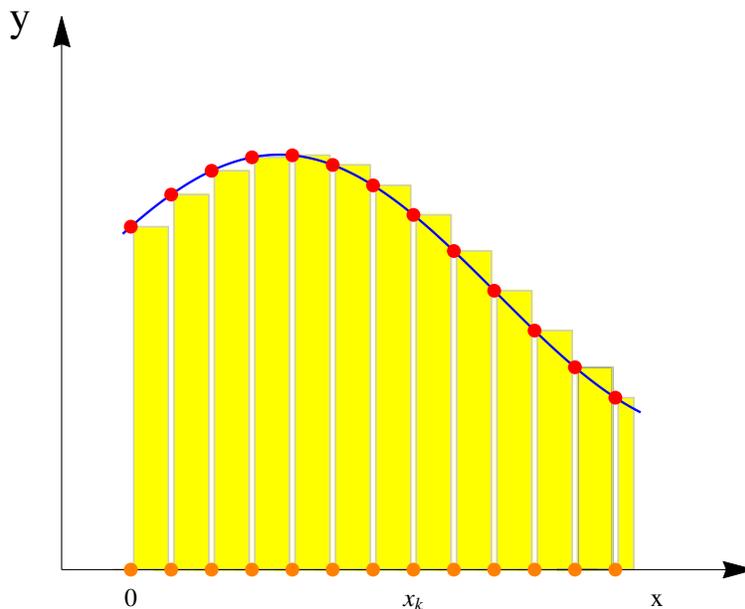
We have previously defined the **Riemann sums**

$$Sf(x) = h[ f(0) + f(h) + f(2h) + \cdots + f(kh) ] ,$$

where  $k$  is the largest integer such that  $kh < x$ . Lets write  $S_n$  if we want to stress that the parameter  $h = 1/n$  was used in the sum. We define the **integral** as the limit of these sums  $S_n f$ , when the **mesh size**  $h = 1/n$  goes to zero.

Define

$$\int_0^x f(t) dt = \lim_{n \rightarrow \infty} S_n f(x) .$$



For any differentiable function, the limit exists

**Proof:** Lets first assume  $f \geq 0$  on  $[0, x]$ . Let  $M$  be such that  $f \leq M$  and  $f' \leq M$  on  $[0, x]$ . The Riemann sum  $S_n f(x)$  is the total area of  $k$  rectangles. Let  $Sf(x)$  denote the area under the curve. If  $M$  is the maximal slope of  $f$  on  $[0, x]$ , then on each interval  $[j/n, (j+1)/n]$ , we have  $|f(x) - f(j/n)| \leq M/n$  so that the area error is smaller than  $M/n^2$ . Additionally, we have a piece above the interval  $[kh, x]$  with area  $\leq M/n$ . If we add all the  $k \leq xn$  "roof area errors" and the "side area" up, we get

$$Sf(x) - S_n f(x) \leq \frac{kM}{n^2} + \frac{M}{n} \leq \frac{xnM}{n^2} + \frac{M}{n} = \frac{xM + M}{n} .$$

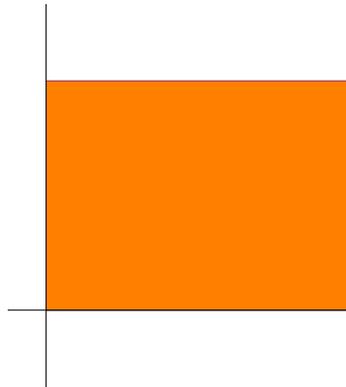
This converges to 0 for  $n \rightarrow \infty$ . The limit is therefore the area  $Sf(x)$ . For a general, not necessarily nonnegative function, we write  $f = g - h$ , where  $g, h$  are nonnegative (see homework) and have  $\int_0^x f(x) dx = \int_0^x g(x) dx - \int_0^x h(x) dx$ .

For nonnegative  $f$ , the value  $\int_0^x f(x) dx$  is the **area between the x-axis and the graph** of  $f$ . For general  $f$ , it is a **signed area**, the difference between two areas.

**Remark:** the Riemann integral is defined here as the limit  $h \sum_{x_k=kh \in [0,x]} f(x_k)$ . It converges to the area under the curve for all **continuous** functions but since we work with differentiable functions in calculus we restricted to that. Not **all** bounded functions can be integrated naturally like this. There are discontinuous functions like the **salt and pepper function** which is defined to be  $f(x) = 1$  if  $x$  is rational and zero else. Now  $Sf(x) = 1$  for rational  $h$  and  $Sf(x) = 0$  if  $h$  is irrational. Therefore, another integral, the **Lebesgue integral** is used too: it can be defined as the limit  $\frac{1}{n} \sum_{k=1}^n f(x_k)$  where  $x_k$  are **random points** in  $[0, x]$ . This **Monte-Carlo integral** definition of the Lebesgue integral gives the integral 0 for the salt and pepper function because rational numbers have zero probability.

Remark: Some calculus books define the Riemann integral with partitions  $x_0 < x_1 < \dots < x_n$  of points of the interval  $[0, x]$  such that the maximal distance  $(x_{k+1} - x_k)$  between neighboring  $x_j$  goes to zero. The Riemann sum is then  $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$ , where  $y_k$  is arbitrarily chosen inside the interval  $(x_k, x_{k+1})$ . For continuous functions, the limiting result is the same the  $Sf(x)$  sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated and keep a wide mesh, where the function does not change much. This leads to **numerical analysis** of integrals.

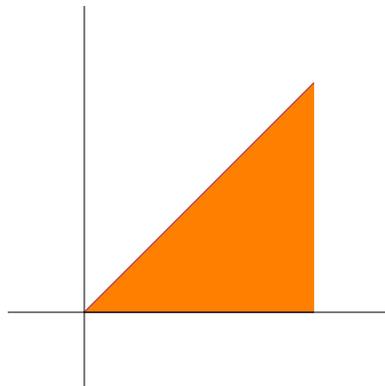
- 1 Let  $f(x) = c$  be constant everywhere. Now  $\int_0^x f(t) dt = cx$ . We can see also that  $cnx/n \leq S_n f(x) \leq c(n+1)x/n$ .



- 2 Let  $f(x) = cx$ . The area is half of a rectangle of width  $x$  and height  $cx$  so that the area is  $cx^2/2$ . Remark: we could also have added up the Riemann sum but that's more painful: for every  $h = 1/n$ , let  $k$  be the largest integer smaller than  $xn = x/h$ . Then (remember Gauss's punishment?)

$$S_n f(x) = \frac{1}{n} \sum_{j=1}^k \frac{cj}{n} = \frac{ck(k+1)/2}{n^2}.$$

Taking the limit  $n \rightarrow \infty$  and using that  $k/n \rightarrow x$  shows that  $\int_0^x f(t) dt = cx^2/2$ .



- 3 Let  $f(x) = x^2$ . In this case, we can not see the numerical value of the area geometrically. But since we have computed  $S[x^2]$  in the first lecture of this course and seen that it is

$[x^3]/3$  and since we have defined  $S_h f(x) \rightarrow \int_0^x f(t) dt$  for  $h \rightarrow 0$  and  $[x^k] \rightarrow x^k$  for  $h \rightarrow 0$ , we know that

$$\int_0^x t^2 dt = \frac{x^3}{3}.$$

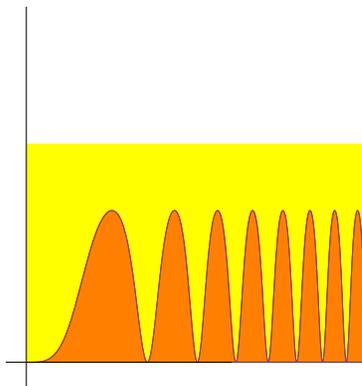
This example actually computes the **volume of a pyramid** which has at distance  $t$  from the top an area  $t^2$  cross section. Think about  $t^2 dt$  as a slice of the pyramid of area  $t^2$  and height  $dt$ . Adding up the volumes of all these slices gives the volume.



**Linearity of the integral** (see homework)  $\int_0^x f(t) + g(t) dt = \int_0^x f(t) dt + \int_0^x g(t) dt$  and  $\int_0^x \lambda f(t) dt = \lambda \int_0^x f(t) dt$ .

**Upper bound:** If  $0 \leq f(x) \leq M$  for all  $x$ , then  $\int_0^x f(t) dt \leq Mx$ .

- 4  $\int_0^x \sin^2(\sin(\sin(t)))/x dt \leq x$ . **Solution.** The function  $f(t)$  inside the interval is nonnegative and smaller or equal to 1. The graph of  $f$  is therefore contained in a rectangle of width  $x$  and height 1.



We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If  $f$  and  $g$  satisfy  $|f(x) - g(x)| \leq c$ , then

$$\int_0^x |f(x) - g(x)| dx \leq cx.$$

We know identities like  $S_n[x]_h^n = \frac{[x]_h^{n+1}}{n+1}$  and  $S_n \exp_h(x) = \exp_h(x)$  already. Since  $[x]_h^k \rightarrow x^k$  we have  $S_n[x]_h^k - S_n[x]^k \rightarrow 0$  and from  $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$ . The other equalities are the same since  $\exp_h(x) = \exp(x) \rightarrow 0$ . This gives us:

$$\int_0^x t^n dt = \frac{x^{n+1}}{n+1}$$

$$\int_0^x e^t dt = e^x - 1$$

$$\int_0^x \cos(t) dt = \sin(x)$$

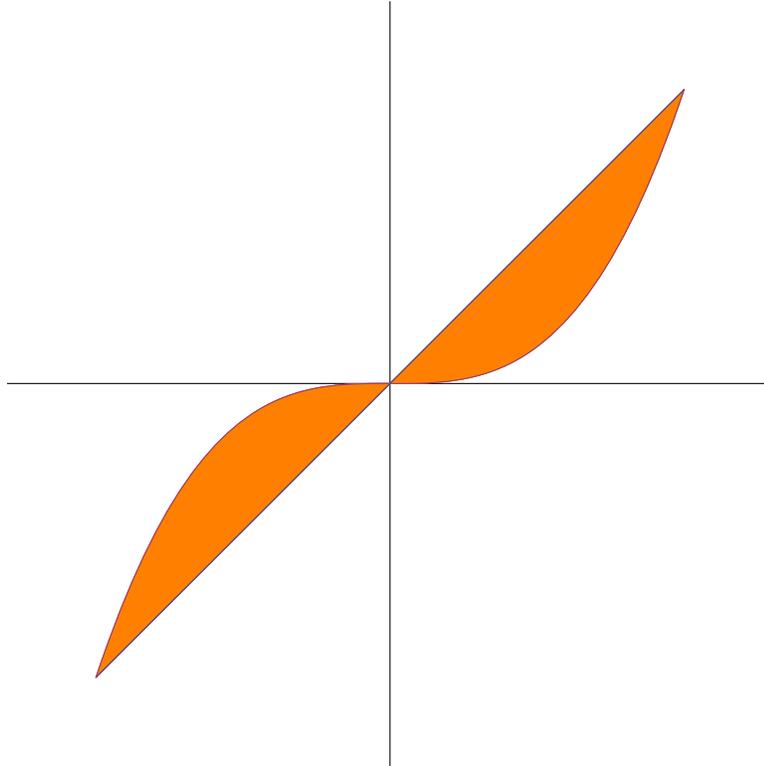
$$\int_0^x \sin(t) dt = 1 - \cos(x)$$

# Homework

In the following homework you can use that  $\int_a^b f(x) dx = F(b) - F(a)$  if  $F$  is a function which satisfies  $F'(x) = f(x)$ . We have already verified it for sums.

- 1 a) What is the integral  $\int_1^2 x^6 dx$ ?  
 b) Find the integral  $\int_0^1 8t^7 + e^t dt$ .  
 c) Calculate  $\int_{-1}^1 \frac{1}{1+x^2} dx$ .  
 d) Find  $\int_0^{\pi/2} \sin^2(t) dt$ .  
 For d), use a double angle formula.

- 2 The region enclosed by the graph of  $x$  and the graph of  $x^3$  has a propeller type shape as seen in the picture. Find its (positive) area.



- 3 Make a geometric picture for each of the following statements:

- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$ .
- $\int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b (f(x) - g(x)) dx$ .
- $\int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx$ .

- 4 Here are some more challenging integrals. Maybe you have to guess:

- a)  $\int_0^1 (3/2)\sqrt{1+x} dx$   
 b)  $\int_0^{\sqrt{\log(2)}} 16xe^{-x^2} dx$   
 c)  $\int_0^{\pi} \sin^4(x) dx$   
 d)  $\int_1^e 5 \log(x)/x dx$  For c), use double angle formulas, twice.

- 5 In this problem, it is crucial that you plot the function first. Split the integral up into parts.

- a) Find  $\int_0^3 |x-1| dx$ . Distinguish cases.  
 b) Find  $\int_0^3 f(x) dx$  for  $f(x) = |x - |x-1||$ . Also here, distinguish cases