Lecture 28: Substitution

You should by now also to be able to integrate functions like e^{6x} or $1/(1+x)$. Substitution makes this easier. If we differentiate the function $\sin(x^2)$ and use the chain rule, we get $\cos(x^2)2x$. By the fundamental theorem of calculus, the anti derivative of $\cos(x^2)2x$ is $\sin(x^2)$. We know therefore

$$\int \cos(x^2)2x\; dx = \sin(x^2) + C\;.$$

Spotting the chain rule

How can we see the integral without knowing the result already? Here is a very important case:

If we can spot that $f(x) = g(u(x))u'(x)$, then the anti derivative of f is $G(u(x))$ where G is the anti derivative of g.

1 Find the anti derivative of

$$e^{x^4+x^2}(4x^3+2x)\;.$$

Solution: The derivative of the inner function is to the right.

2 Find

$$\int \sqrt{x^5+1}x^4\; dx\;.$$

Solution. The derivative of x^5+1 is $5x^4$. This is almost what we have there but the constant can be adapted. The answer is $(2/15)(x^5+1)^{3/2}$.

3 Find the anti derivative of

$$\frac{\log(x)}{x}\;.$$

Solution: The derivative of $\log(x)$ is $1/x$. The antiderivative is $\log(x)^2/2$.

4 Find the anti derivative of

$$\cos(\sin(x^2))\cos(x)2x\;.$$

Solution: We see the derivative of $\sin(x^2)$ appear on the right. Therefore, we have $\sin(\sin(x^2))$.

In the next three examples, substitution is actually not necessary. You can just write down the anti derivative, and adjust the constant. It uses the following "speedy rule":

If $\int f(ax+b)\; dx = F(ax+b)/a$ where F is the anti derivative of f.

5 $\int \sqrt{x+1}\,dx$. **Solution:** $(x+1)^{3/2}(2/3)$.

6 $\int \frac{1}{1+(5x+2)^2}\,dx$. **Solution:** $\arctan(5x+2)(1/5)$.

Doing substitution

Spotting things is sometimes not easy. The method of substitution helps to formalize this. To do so, identify a part of the formula to integrate and call it u then replace an occurrence of $u'dx$ with du.

$$\int f(\text{ u(x) })\text{ u'(x) dx} = \int f(\text{ u })\text{ du }.$$

Here is a more detailed description: replace a prominent part of the function with a new variable u, then use $du = u'(x)dx$ to replace dx with du/u'. We aim to end up with an integral $\int g(u)\,du$ which does not involve x anymore. Finally, after integration of this integral, replace the variable u again with the function $u(x)$. The last step is called **back-substitution**.

7 Find the anti-derivative

$$\int \log(\log(x))/x\,dx .$$

Solution: Replace $\log(x)$ with u and replace $u'dx = 1/xdx$ with du. This gives $\int \log(u)\,du = u\log(u) - u = \log(x)\log\log(x)) - \log(x)$.

8 Solve the integral

$$\int x/(1+x^4)\,dx .$$

Solution: Substitute $u = x^2, du = 2xdx$ to get $(1/2)\int du/(1+u^2)\,du = (1/2)\arctan(u) = (1/2)\arctan(x^2)$.

9 Solve the integral

$$\int \sin(\sqrt{x})/\sqrt{x} .$$

Here are some examples which are not so straightforward:

10 Solve the integral

$$\int \sin^3(x)\,dx .$$

Solution:. We replace $\sin^2(x)$ with $1 - \cos^2(x)$ to get

$$\int \sin^3(x)\; dx = \int \sin(x)(1-\cos^2(x))\; dx = -\cos(x) + \cos^3(x)/3 \;.$$

11 Solve the integral

$$\int \frac{x^2+1}{\sqrt{x+1}}\; dx \;.$$

Solution: Substitute $u = \sqrt{x+1}$. This gives $x = u^2 - 1, dx = 2udu$ and we get $\int 2(u^2 - 1)^2 + 1\; du$.

12 Solve the integral

$$\int \frac{x^3}{\sqrt{x^2+1}}\; dx \;.$$

Trying $u = \sqrt{x^2+1}$ but this does not work. Try $u = x^2 + 1$, then $du = 2xdx$ and $dx = du/(2\sqrt{u-1})$. Substitute this in to get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}}\; du = \int \frac{(u-1)}{2\sqrt{u}} = \int u^{1/2}/2 - u^{-1/2}/2\; du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2} \;.$$

Definite integrals

When doing definite integrals, we could find the antiderivative as described and then fill in the boundary points. Substituting the boundaries directly accelerates the process since we do not have to substitute back to the original variables:

$$\int_a^b g(u(x))u'(x)\; dx = \int_{u(a)}^{u(b)} g(u)\; du \;.$$

Proof. This identity follows from the fact that the right hand side is $G(u(b)) - G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has the anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b)) - G(u(a))$.

Top: To keep track which bounds we consider it can help to write $\int_{x=a}^{x=b} f(x)\; dx$.

13 Find the anti derivative of $\int_0^2 \sin(x^3 - 1)x^2\; dx$. **Solution**:

$$\int_{x=0}^{x=2} \sin(x^3+1)x^2\; dx \;.$$

Solution: Use $u = x^3 + 1$ and get $du = 3x^2 dx$. We get

$$\int_{u=1}^{u=9} \sin(u) du/3 = (1/3)\cos(u)|_1^9 = [-\cos(9) + \cos(1)]/3 \;.$$

Also here, we can see the integrals directly

To integrate $f(Ax+B)$ from a to b we get $[F(Ab+B) - F(Aa+B)]/A$, where F is the anti-derivative of f.

14 $\int_0^1 \frac{1}{5x+1}\; dx = [\log(u)]/5|_1^6 = \log(6)/5$.

15 $\int_3^5 \exp(4x-10)\; dx = [\exp(10) - \exp(2)]/4$.

Homework

1 Find the following anti derivatives.
a) $\int 20x\sin(x^2)\;dx$
b) $\int e^{x^6+x}(6x^5+1)\;dx$
c) $\cos(\cos^3(x))\sin(x)\cos^2(x)$
d) $e^{\tan(x)}/\cos^2(x)$.

2 Compute the following definite integrals.
a) $\int_2^5 \sqrt{x^5+x}(x^4+1/5)\;dx$
b) $\int_0^{\sqrt{\pi}} \sin(x^2)x\;dx$.
c) $\int_{1/e}^{e} \frac{\sqrt{\log(x)}}{x}\;dx$.
d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}}\;dx$.

3 a) Find the integral $\int_0^1 x^3\sqrt{1-x^4}\;dx$ using a substitution method.
b) Find the moment of inertia of a rod with density $f(x)=\sqrt{x^3+1}$ between $x=0$ and $x=4$. Remember that the moment of inertia is $\int_0^4 x^2 f(x)\;dx$.

4 a) Integrate

$$\int_0^1 \frac{\arcsin(x)}{\sqrt{1-x^2}}\;dx\;.$$

b) Find the definite integral

$$\int_e^{6e} \frac{dx}{\sqrt{\log(x)}x}\;.$$

5 a) Find the indefinite integral

$$\int \frac{x^5}{\sqrt{x^2+1}}\;dx\;.$$

b) Find the anti-derivative of

$$f(x)=\frac{1}{x(1+\log(x)^2)}\;.$$