Lecture 29: Integration by parts

If we integrate the product rule $(uv)' = u'v + uv'$ we obtain an integration rule called **integration by parts**. It is a powerful tool, which complements substitution. As a rule of thumb, always try first to simplify a function and integrate directly, then give substitution a first shot before trying integration by parts.

$$\int \text{u(x)}\ \text{v'}\ (x)dx = u(x)v(x) - \int u'(x)v(x)\ dx.$$

1 Find $\int x\sin(x)\ dx$. **Solution.** Lets identify the part which we want to differentiate and call it u and the part to integrate and call it v'. The integration by parts method now proceeds by writing down uv and subtracting a new integral which integrates $u'v$:

$$\int \text{x}\ \sin(x)\ dx = \text{x}\ (-\cos(x)) - \int 1\ (-\cos(x)\ )\ dx = -x\cos(x) + \sin(x) + C\ .$$

2 Find $\int xe^x\ dx$. **Solution.**

$$\int \text{x}\ \exp(x)\ dx = x\exp(x) - \int 1\exp(x)\ dx = x\exp(x) - \exp(x) + C\ .$$

3 Find $\int \log(x)\ dx$. **Solution.** There is only one function here, but we can look at it as $\log(x)\cdot 1$

$$\int \log(x)\ 1\ dx = \log(x)x - \int \frac{1}{x}x\ dx = x\log(x) - x + C\ .$$

4 Find $\int x\log(x)\ dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed like this. We would get through but what if we do not know? Lets differentiate $\log(x)$ and integrate x:

$$\int \log(x)\ \text{x}\ dx = \log(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\ dx$$

which is $\log(x)x^2/2 - x^2/4$.

We see that it is better to differentiate log first.

5 **Marry go round:** Find $I = \int \sin(x)\exp(x)\ dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\ dx\ .$$

Lets do it again:

$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\ dx.$$

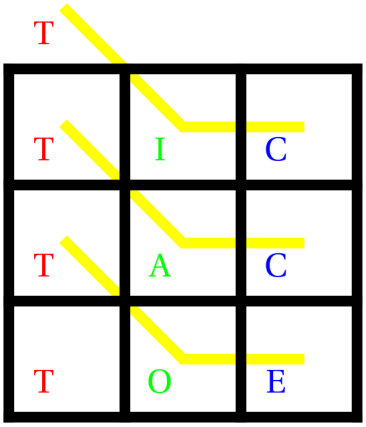
We moved in circles and are stuck! Are we really. We have derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

which we can solve for I and get

$$I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2\ .$$

Tic-Tac-Toe



Integration by parts can bog you down if you do it several times. Keeping the order of the signs can be daunting. This is why a **tabular integration by parts method** is so powerful. It has been called "Tic-Tac-Toe" in the movie Stand and deliver. Lets call it Tic-Tac-Toe therefore.

6 Find the anti-derivative of $x^2 \sin(x)$. **Solution:**

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The antiderivative is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\ .$$

7 Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

$(x-1)^3$	$\exp(2x)$	
$3(x-1)^2$	$\exp(2x)/2$	$\oplus$
$6(x-1)$	$\exp(2x)/4$	$\ominus$
6	$\exp(2x)/8$	$\oplus$
0	$\exp(2x)/16$	$\ominus$

The anti-derivative is

$$(x-1)^3 e^{2x}/2 - 3(x-1)^2 e^{2x}/4 + 6(x-1)e^{2x}/8 - 6e^{2x}/16 + C\ .$$

8 Find the anti-derivative of $x^2\cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	$\oplus$
2	$-\cos(x)$	$\ominus$
0	$-\sin(x)$	$\oplus$

The anti-derivative is

$$x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C\ .$$

Ok, we are now ready for more extreme stuff.

9 Find the anti-derivative of $x^7\cos(x)$. **Solution:**

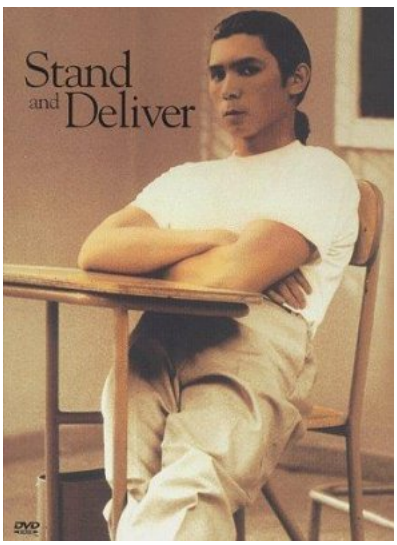
x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	$\oplus$
$42x^5$	$-\cos(x)$	$\ominus$
$120x^4$	$-\sin(x)$	$\oplus$
$840x^3$	$\cos(x)$	$\ominus$
$2520x^2$	$\sin(x)$	$\oplus$
$5040x$	$-\cos(x)$	$\ominus$
5040	$-\sin(x)$	$\oplus$
0	$\cos(x)$	$\ominus$

The anti-derivative is

$$\begin{aligned} F(x) &= x^7\sin(x) \\ &+ 7x^6\cos(x) \\ &- 42x^5\sin(x) \\ &- 210x^4\cos(x) \\ &+ 840x^3\sin(x) \\ &+ 2520x^2\cos(x) \\ &- 5040x\sin(x) \\ &- 5040\cos(x) + C\ . \end{aligned}$$

Do this without this method and you see the value of the method.

[1] [2] [3].



I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to an article of V.N. Murty. The method realizes in a clever way an iterated integration by parts method:

$$\begin{aligned} \int fg\,dx &= fg^{(-1)} - f^{(1)}g^{-2} + f^{(2)}g^{(-3)} - \dots \\ &- (-1)^n \int f^{(n+1)}g^{(-n-1)}\,dx \end{aligned}$$

which can easily shown to be true by induction and justifies the method: the f function is differentiated again and again and the g function is integrated again and again. You see, where the alternating minus signs come from. You see that we always pair a k'th derivative with a $k+1$'th integral and take the sign $(-1)^k$.

Coffee or Tea?

[1] V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, pages 90-94.
[2] David Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, pages 307-311.
[3] K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, pages 542-543

When doing integration by parts, We want to try first to differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

For coffee lovers, there is an equivalent one: **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials which can be remembered as **LIATE** which is close to "latte" (the coffee).

Whether you prefer to remember it as a "coffee latte" or a "lipton tea" is up to you.

There is even a better method, the "opportunistic method":

> Just integrate what you can integrate and differentiate the rest.

An don't forget to consider integrating 1, if nothing else works.

Homework

1. Integrate $\int x^3 \log(x)\, dx$.
2. Integrate $\int x^5 \sin(x)\, dx$
3. Find the anti derivative of $\int 2x^6 \exp(x)\, dx$. (*)
4. Find the anti derivative of $\int \sqrt{x} \log(x)\, dx$.
5. Find the anti derivative of $\int \sin(x) \exp(-x)\, dx$.

(*) If you want to go for the record. Lets see who can integrate the largest $x^n \exp(x)$! It has to be done by hand, not with a computer algebra system although.