Course organization

My name is Oliver Knill ("Oliver"). Office hours are Tuesdays and Thursdays 4-5. Good times for short meetings are also in the morning before class, shortly before noon or Tuesday and Thursday from 3-4.

Lectures

Lectures take place Monday Wednesday and Friday from 10 AM to 11 AM in SC 309. Come to lectures. It will save you time.

Course assistant

Cody Kiechle (ckiechle@college.harvard.edu)
Emma Rausch (erauscha@college.harvard.edu)

Problem Sessions

Weekly problem section will be arranged by the course assistants.

Exam Group

This course is in the exam group 1. You need only to know this if you should need check about possible final exam conflicts.

General Education

When taken for a letter grade, this course meets the General Education requirement for **Empirical and Mathematical Reasoning** or the Core Area requirement for **Quantitative Reasoning**. It can be taken for graduate credit. Talk to me about special requirements in that respect.

Prerequisites

A solid pre-calculus background is required. This course is recommended for students who score 18 or higher on the first part of the Harvard Math Placement Test. You are not expected to have taken calculus in high school. Even if you have seen some calculus, we expect that Math 1a will provide you with a deeper, more conceptual understanding of the subject.

Synopsis

Calculus can be traced back to Archimedes who was born 2300 years ago. It became a powerful tool with work done by Newton and Leibniz. Calculus can be considered one of the biggest achievements of the past two millennia. The core of the course introduces differential and integral calculus. Differential calculus studies "rate of change", integral calculus treats "accumulation". The fundamental theorem of calculus links the two. The subject will be applied to problems from other scientific disciplines in homework as well as in lectures. Calculus is not only important because of its content and applications which currently make billions in medicine (example: tomography), internet (example: complex network analysis with calculus methods), geography (example: google earth, location based services), movie industry (animated or CGI enhanced pictures), game development (also under the hood like in artificial intelligence agents). Your phone uses tons of calculus when recognizing voice commands or faces in images). Often, the ideas of calculus only enter in disguised form. One point I want to make in this course is that calculus can appear in different forms. Actually, in the first lecture we look at calculus on integers and prove the fundamental theorem of calculus in a form which the Egyptians already could have done.

Course Policies

Class attendance is expected. In case of religious holidays, conflicts like a sports competitions or concert, please send me a brief email before.

Computers

The use of computers and computer algebra systems or online tools to experiment with the mathematical structures is encouraged. We do not have a lab component in this course. The use of laptops or tablets in class to take notes is fine. No computer, phone or tablet of any type is permitted however during exams. If you get computer assistance for homework, acknowledge it in the paper. I do recommend that you work out most of the work on paper. The material sticks better when you write things up by hand. It will also prepare you better for exams.

Pass Fail

The course may be taken pass/fail or for graduate credit after talking to Oliver. Note that there is **no** GenEd credit for Pass/Fail.

Textbook

I do not follow a particular book. A popular choice is "Single Variable Calculus: Concepts and Contexts, 4th Edition" by James Stewart (ISBN-10: 0495559725 ISBN-13: 9780495559726). The Cabot library has a desk copy available. It is recommended to read in some book but course material and homework is posted on the website http://www.math.harvard.edu/~knill/teaching/math1a_2012/

Grades

- 20 percent midterm 1
- 20 percent midterm 2
- 20 percent homework
- 40 percent final exam

As in every course, the numerical score needs to be converted to a letter grade. The cutoffs are determined when the final distribution is known. We will say more about this in the first lecture.

Math Question Center

The mathematics question center MQC is open from Sunday through Thursday in SC 309a, 8:30-10:30 PM. The rooms are reserved from 7:30 PM on. This drop-in help service is staffed by calculus course assistants who can answer questions for homework. You may also stop by the MQC to find other students in the course. While staffed from 8:30 on, the room should be available from 7:30 PM on in the spring.

Bureau of Study Counsel BSC

The bureau of Study council at 5 Linden Street is a resource outside the math department. The BSC offers one-on-one peer tutoring for a minimal fee study skills and test-taking workshops, counseling, and many other services. I recommend however to make use first of our resources. The website of the BSC is http://bsc.harvard.edu.

Exams

We have 2 midterm exams and one final exam. You can already mark the calendars for the exam dates:

- 1. Midterm: Tuesday, March 4: 7-8:30, Hall C
- 2. Midterm: Tuesday, April 8: 7-8:30, Hall D

Exams focus on the mathematics done in the course. Calculus is a large area. The syllabus will walk along an efficient and interesting path, which focuses on stuff which is really needed in the sciences. Many calculus books can be overwhelming in this respect.

Homework

Homework is due at the beginning of **every** class. This is easy to remember. Bring every time some homework to class. This course has a "no late homework policy". This makes it possible for the course assistants to return the homework in a timely manner. We will discard the least 3 homework scores.

Academic Integrity

Collaboration policies are the ones established by FAS. Collaboration is permitted or even encouraged for homework but not in exams. Homework needs to be written down individually however. I recommend to attack each homework problem first on your own. This helps you to develop independent thinking and problem solving skills and prepares you for the exams.

Accessible education:

Students who need academic adjustments or accommodations because of a documented disability should provide me with a letter from the Accessible Education Office (AEO). Please talk to me about it also.

Hour by hour syllabus

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1. What is calculus?                         Date   Day
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  1. What is Calculus?                         Jan 27 Mon
  2. Functions                                 Jan 29 Wed
  3. Limits                                    Jan 31 Fri
  4  Continuity                                Feb  3 Mon
  5  Intermediate value theorem                Feb  5 Wed
  6. A fundamental theorem                     Feb  7 Fri
  7. Rate of Change, tangent                   Feb 10 Mon
  8. Derivative as a function                  Feb 12 Wed
  9. Product and Quotient rules                Feb 14 Fri
2. The derivative
-----------------
     Presidents day, no class                  Feb 17 Mon
  1. Chain rule                                Feb 19 Mon
  2. Critical points and extrema               Feb 21 Wed
  3. Optimization problems                     Feb 24 Fri
  4. L'Hopital rule                            Feb 26 Wed
  5. Newton method                             Feb 28 Fri
  6. Review for first midterm                  Mar  3 Mon
  7. Rolles theorem                            Mar  5 Wed
  8. Castastrophe theory                       Mar  7 Fri
3. The integral
---------------
  1. From sums to integrals                    Mar 10 Mon
  2. The fundamental theorem                   Mar 12 Wed
  3. Antiderivatives                           Mar 14 Fri
  4. Computing areas                           Mar 24 Mon
  5. Volume of solids                          Mar 26 Wed
  6. Improper integrals                        Mar 28 Fri
  7. Applications of integration               Apr  1 Mon
4. Calculus Techniques
------------------------
  1. Related rates                             Apr  2 Wed
  2. Implicit differentiation                  Apr  4 Fri
  3. Review for second midterm                 Apr  7 Mon
  4. Substitution                              Apr  9 Wed
  5. Integration by parts                      Apr 11 Fri
  6. Numerical integration                     Apr 14 Mon
  7. Partial fractions                         Apr 16 Wed
  8. Trig substitutions                        Apr 18 Fri
5. Calculus and the world
-------------------------
  1. Calculus and music                        Apr 21 Mon
  2. Calculus and statistics                   Apr 23 Wed
  3. Calculus and economics                    Apr 25 Fri
  1. Calculus and Computing                    Apr 28 Mon
  2. Outlook and review                        Apr 30 Wed
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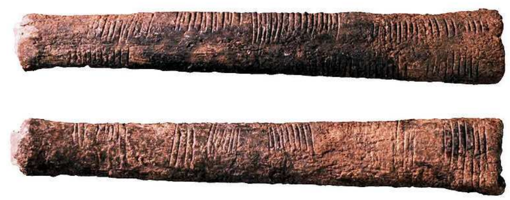
Lecture 1: What is Calculus?



Calculus generalizes the processes of **taking differences** and **performing summation**. Differences measure **change**, sums quantify how things **accumulate**. The process of taking differences has a limit called **derivative**. The process of taking sums has a limit called **integral**. These two operations are related in an intimate way. In this first lecture, we look at calculus in the simplest possible setup. Functions are evaluated on integers and no limits are taken. About 20'000 years ago, numbers were represented by units like

$$1, 1, 1, 1, 1, 1, \dots.$$

Numbers were carved into the fibula bone of a baboon like **Ishango bone**. [1]. It took thousands of years humans figured out to work with numbers using symbols like

$$0, 1, 2, 3, 4, \dots .$$

The concept of function came much, much later and allows to write the counting process as $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$. We mean **function** f assigns to an input like 1001 an output like $f(1001) = 1001$. Lets look at $Df(n) = f(n+1) - f(n)$, the **difference** between two function values. It is a function again. The counting function f we have just considered satisfies $Df(n) = 1$ for all n. We can also formalize the summation process. If $g(n) = 1$ is the function which assigns the constant value 1 for all n, then $Sg(n) = g(0) + g(1) + \dots + g(n-1) = 1 + 1 + ... + 1 = n$. We see that $Df = g$ and $Sg = f$. Lets start with $f(n) = n$ and apply the **summation process** on that function:

$$Sf(n) = f(0) + f(1) + f(2) + \dots + f(n-1) .$$

We get the following values:

$$0, 1, 3, 6, 10, 15, 21, \dots .$$

The new function g satisfies $g(1) = 1, g(2) = 3, g(3) = 6$, etc. These numbers are called **triangular numbers**. From the function g we can get f back by taking difference:

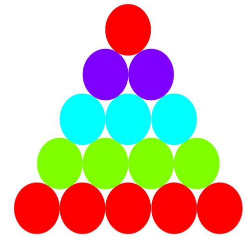
$$Dg(n) = g(n+1) - g(n) = f(n) .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$. And indeed this is $f(5)$. Finding a formula for the sum $Sf(n)$ is not so easy. Can you do it?

Legend tells that when **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, Mr. Büttner gave him the task to sum up the first 100 numbers $1 + 2 + \dots + 100$. Gauss discovered that pairing the numbers up would simplify the summation He would write the sum as $(1 + 100) + (2 + 99) + \dots + (50 + 51)$ so that the answer is $g(n) = n(n-1)/2 = 5050$. We have now an explicit expression for the sum function. Lets apply the difference function again: $Dg(n) = n(n+1)/2 - n(n-1)/2 = n = f(n)$.

[1]The lengths of the marks might have had astronomical significance but this is not relevant here

Since this went so well, lets add up the new sequence again and compute $h = Sg$. We get the sequence

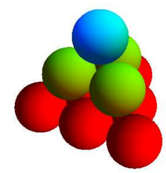


$$0, 1, 4, 10, 20, 35, ...$$

These numbers are called the **tetrahedral numbers**. The reason is that one can use $h(n)$ balls to build a tetrahedron of side length n. For example, we need $h(4) = 20$ golf balls to build a tetrahedron of side length 4. The formula which holds for h is $\boxed{h(n) = n(n-1)(n-2)/6}$. In the worksheet we will check that summing the differences gives the function back. What we have seen in special cases is a general theorem:

$$\boxed{SDf(n) = f(n) - f(0), \qquad DSf(n) = f(n)}$$

It is an arithmetic version of the **fundamental theorem of calculus**. Our goal is to see how this works when we take limits. We will see the **integral** $\boxed{\int_0^x f(x)\, dx}$. and the **derivative** $\boxed{\frac{d}{dx} f(x)}$ and verify the **fundamental theorem of calculus**



$$\boxed{\int_0^x \frac{d}{dt} f(t)\, dt = f(x) - f(0), \qquad \frac{d}{dx} \int_0^x f(t)\, dt = f(x)}$$

It is a fantastic result. A major goal of this course will be to understand it result and see how to apply it. We will revisit the theorem at least twice again. [2] The above version will lead us. Note that if we define $[n]^0 = 1, [n]^1 = n, [n]^2 = n(n-1)/2, [n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1], D[n]^2 = 2[n], D[n]^3 = 3[n]^2$ and in general

$$\boxed{\frac{d}{dx}[x]^n = n[x]^{n-1}}$$

We will generalize this from $h = 1$ to general $h > 0$ and then see that it also holds in the limit $h \to 0$, where it becomes the familiar formula $(d/dx)x^n = nx^{n-1}$ you might know already. The calculus you have just seen, contains the essence of single variable calculus. This core idea will become more powerful and natural if we use it together with the concept of limit.

1

Problem: The sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ satisfies the rule $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$? **Solution:** We take the difference between successive numbers and get the sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, ...$$

which is the same sequence again. We can deduce from this recursion that f has the property that $\boxed{Df(x) = f(x-1)}$. It is called the **Fibonnacci sequence**, a sequence of great fame.

2 **Problem:** Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, ...$ What sequence is that?
Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ is subtracted. It has been relatively easy to find the sum, because we knew what the difference operation did. This example shows:

[2]Many textbooks need hundreds of pages until the fundamental theorem is reached.

We can study differences to understand sums.

The next problem illustrates this too:

3 **Problem:** Find the next term in the sequence
2 6 12 20 30 42 56 72 90 110 132 . **Solution:** Take differences

2	6	12	20	30	42	56	72	90	110	132
2	4	6	8	10	12	14	16	18	20	22
2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0

.

Now we can add an additional number, starting from the bottom and working us up.

2	6	12	20	30	42	56	72	90	110	132	156
2	4	6	8	10	12	14	16	18	20	22	24
2	2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0	0

4 **Problem:** The function $f(n) = 2^n$ is called the **exponential function**. We have for example $f(0) = 1, f(1) = 2, f(2) = 4, \dots$. It leads to the sequence of numbers

n=	0	1	2	3	4	5	6	7	8	...
f(n)=	1	2	4	8	16	32	64	128	256	...

We can verify that f satisfies the equation $\boxed{Df(x) = f(x)}$. because $Df(x) = 2^{x+1} - 2^x = (2-1)2^x = 2^x$.
This is an important special case of the fact that

The derivative of the exponential function is the exponential function itself.

The function 2^x is a special case of the exponential function when the Planck constant is equal to 1. We will see that the relation will hold for any $h > 0$ and also in the limit $h \to 0$, where it becomes the classical exponential function e^x which plays an important role in science.

5 **Problem:** Look at the function $f(n)$ which gives the n'th prime number. Lets look at the derivatives $D^k f$ but take the absolute value $|D^k(f)|$. In other words, we study $T(f)(n) = |f(n+1) - f(n)|$. We know for example that $f(n) = 2^n$ satisfies $Tf = f$. Lets look at the prime function and apply this differences:

n=	1	2	3	4	5	6	7	8	9	...
$f(n) =$	2	3	5	7	11	13	17	23	29	...
$Tf(n) =$	1	2	2	4	2	4	2	4	6	...
$T^2f(n) =$	1	0	2	2	2	2	2	2	4	...
$T^3f(n) =$	1	2	0	0	0	0	0	2	0	...

The **Gilbreath conjecture** of 1959 claims that the first entry remains 1 for ever when applying this absolute differentiation process. The problem is still open.

Homework

1 Predict the future and find the next term in the sequence

$$2, 10, 30, 68, 130, 222, 350, 520, 738, 1010, 1342, ...$$

by taking "derivatives" and then "integrating".

2 Look at the odd numbers $f(n) = 2n + 1$. The sequence starts with

$$1, 3, 5, 7, 9, 11, 13, \dots .$$

Play around and compute $Sf(1) = 1, Sf(2) = 1+3, Sf(3) = 1+3+5$. Guess a formula for $Sf(n) = f(0)+f(1)+f(2)+\dots+f(n-1)$. Verify that $g(n+1)-g(n) = f(n)$ by plugging in your formula for g and simplifying.

3 We have defined $Sf(n) = f(0) + f(1) + f(2) + ... + f(n-1)$ and $Df(n) = f(n+1) - f(n)$ and seen the first two lines of the following table:
(The notation $\boxed{[n]^k}$ is defined on the left of the following table. It is notation so that $D[n]^k = k[n]^{k-1}$ holds.)

$f(n) = [n]^0 = 1$	we have	$g(n) = Sf(n) = n$.
$f(n) = [n]^1 = n$	we have	$g(n) = Sf(n) = n(n-1)/2$.
$f(n) = [n]^2/2 = n(n-1)/2$	we have	$g(n) = Sf(n) = n(n-1)(n-2)/6$.

Verify the third line. That is, show $g(n) = [n]^3/6 = n(n-1)(n-2)/6$ satisfies $Dg(n) = f(n) = [n]^2/2 = n(n-1)/2$.

4 Find $F(n) = Sf(n)$ for the function $f(n) = n^2$. This means we want to find a formula such that $F(1) = 1, F(2) = 5, F(3) = 14$ leading to the sequence of numbers

$$0, 1, 5, 14, 30, 55, 91, 140, 204, 285, \dots .$$

We have already have computed Sg for $g(n) = n$ as well as Sh for $h(n) = n(n-1)/2$ in a previous problem. Try to write f as a combination of g and h and use the **addition rule** $D(g+h) = Dg + Dh$.

5 Find a formula for $g(n) = Df(n)$ for the function $f(n) = 3^n$.
Extra credit (you do not have to turn this in). Can you use this to find a formula for $1 + 3 + 9 + 27 + ...$ Hint: use the fact that D and S are inverses of each other.

General remarks about homework

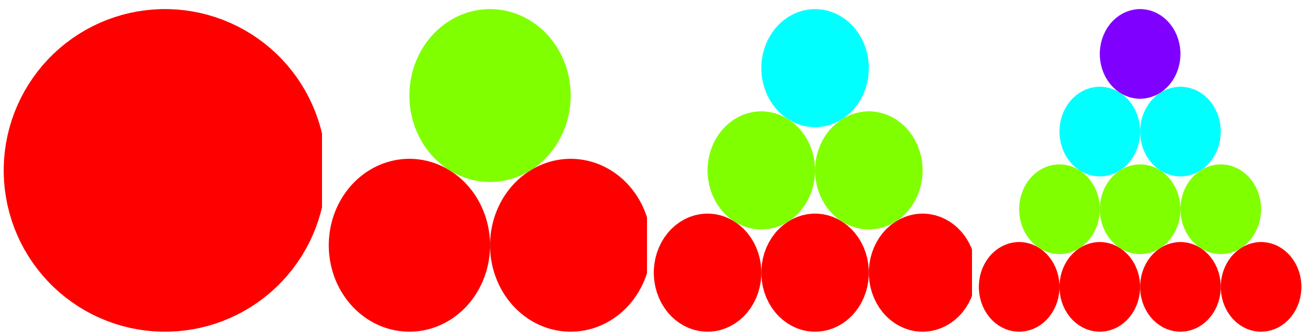
- Make sure to think about the problem yourself first before discussing it with others.
- The time you spend on homework is valuable. Especially the exploration time before you know how to solve it.
- If you do not know how to get started, don't hesitate to ask.

Lecture 1: Worksheet

Triangular numbers

When adding the first integers we get the so called **triangular numbers**.

1	3	6	10	15	21	36	45	...



n=1 n=2 n=3 n=4

This sequence defines a **function** on the natural numbers. For example, $f(4) = 10$.

1 Verify that

$$f(n) = \frac{n(n+1)}{2}$$

gives the above numbers. Check this by algebraically evaluating

$$f(n) - f(n-1) = n \; .$$

Carl-Friedrich Gauss, 1777-1855

Find the next number

How does the following sequence

$$0, 6, 24, 60, 120, 210, 336, 504...$$

continue? Again we can look at this as a function $f(1) = 0$, $f(2) = 6, \dots, f(8) = 504$. Now compute differences. Then use this to go backwards to find the next term $f(9)$.

Tetrahedral numbers

If we build stack oranges onto each other we are led to **tetrahedral numbers**.

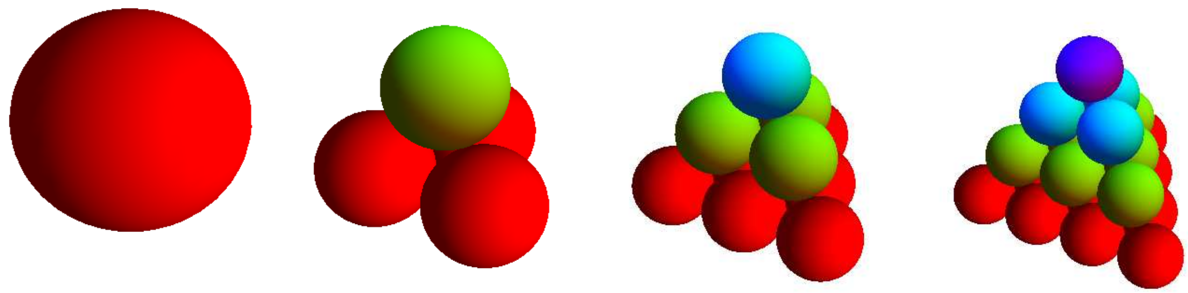
1	4	10	20	35	56	84	120	...

Also this sequence defines a **function**.

2 Verify that

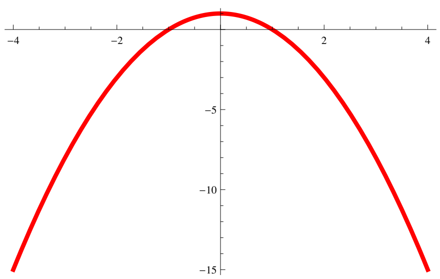
$$g(n) = \frac{n(n+1)(n+2)}{6}$$

satisfies $g(n) - g(n-1) = n(n+1)/2$. We have $g(1) = 1, g(2) = 4, g(3) = 10$.



n=1 n=2 n=3 n=4

May the odds always be in your favor!



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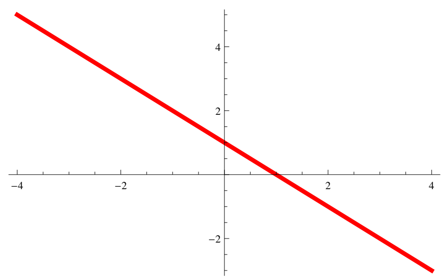

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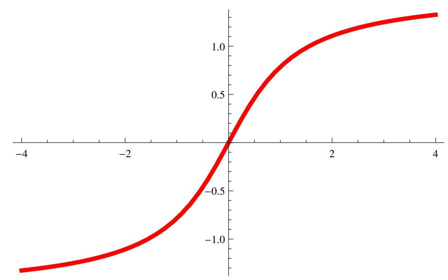

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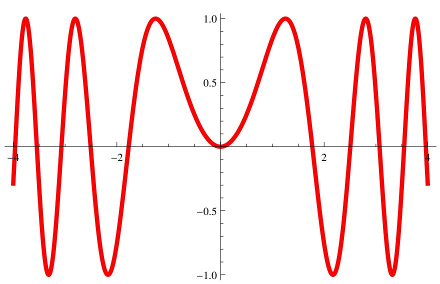

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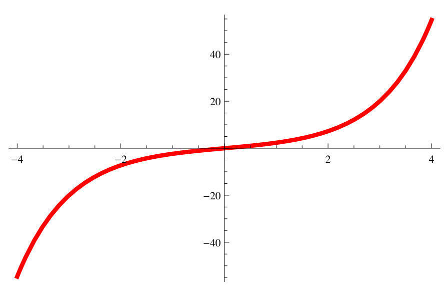

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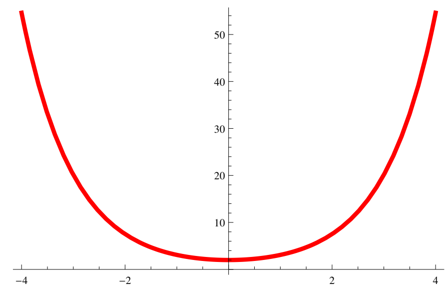

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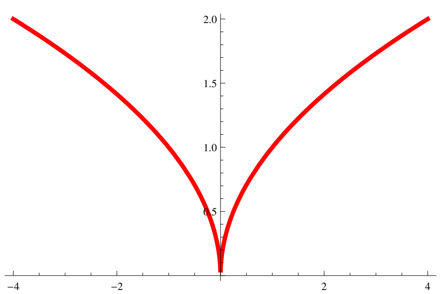

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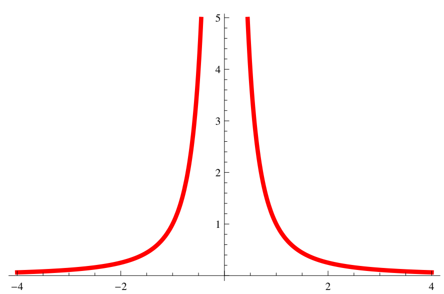

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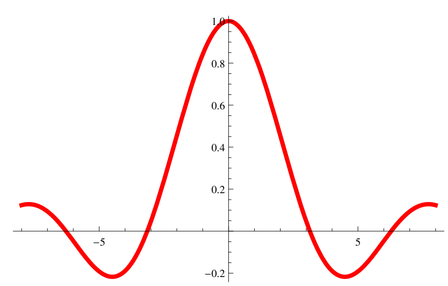

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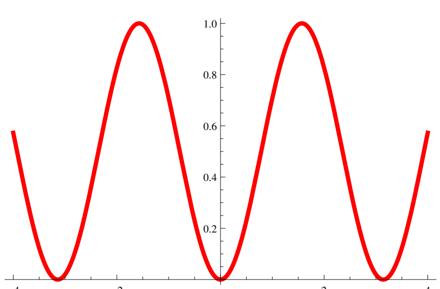

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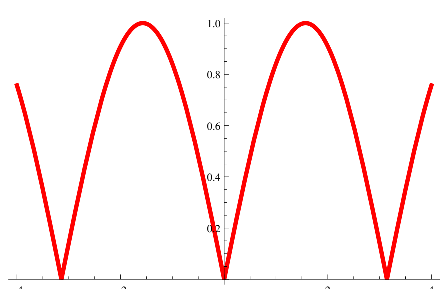

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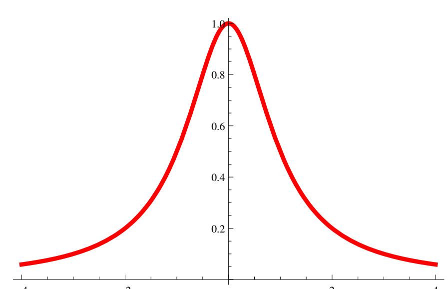


G

$\sin(x^2)$	
$1 - x$	
$1 - x^2$	
$e^x + e^{-x}$	
$e^x - e^{-x}$	
$\sqrt{\lvert x\rvert}$	
$\sin^2(x)$	
$1/x^2$	
$\arctan(x)$	
$\lvert\sin(x)\rvert$	
$\sin(x)/x$	
$1/(1+x^2)$	

Lecture 2: Functions

A **function** is a rule which assigns to a real number a new real number. The function $f(x) = x^2 - 2x$ for example assigns to the number $x = 4$ the value $4^2 - 8 = 8$. A function is given with a **domain** A, the points where f is defined and a **codomain** B a set of numbers which f can reach. Usually, functions are defined everywhere, like the function $f(x) = x^2 - 2x$. Other functions like $g(x) = 1/x$ can not be evaluated at 0 so that the domain excludes the point 0.

We have some flexibility to define the domain and codomain. Let R_0 be the set of all real numbers which do not contain 0. If we equip $f(x) = 1/x$ with the domain and codomain R_0 then f is a map from R_0 to R_0 and it is its own inverse. Here are a few examples of functions. We will look at them in more detail during the lecture. Very important are polynomials, trigonometric functions, the exponential and logarithmic function. You won't find the h-exponentials and h-logarithms in textbooks. But they will be important for us. They are the exponentials and logarithms in "calculus without limit" and will in the limit $h \to 0$ become the regular exponential and logarithm functions.

constant	$f(x) = 1$	power	$f(x) = 2^x$
identity	$f(x) = x$	exponential	$f(x) = e^x = \exp(x)$
linear	$f(x) = 3x + 1$	logarithm	$f(x) = \log(x) = \exp^{-1}(x)$
quadratic	$f(x) = x^2$	absolute value	$f(x) = \|x\|$
cosine	$f(x) = \cos(x)$	devil comb	$f(x) = \sin(1/x)$
sine	$f(x) = \sin(x)$	bell function	$f(x) = e^{-x^2}$
h-exponentials	$f(x) = \exp_h(x) = (1 + h)^{x/h}$	witch of Agnesi	$f(x) = \frac{1}{1+x^2}$
h-logarithms	$f(x) = \log_h(x) = \exp_h^{-1}$	sinc	$\sin(x)/x$

We can build new functions by:

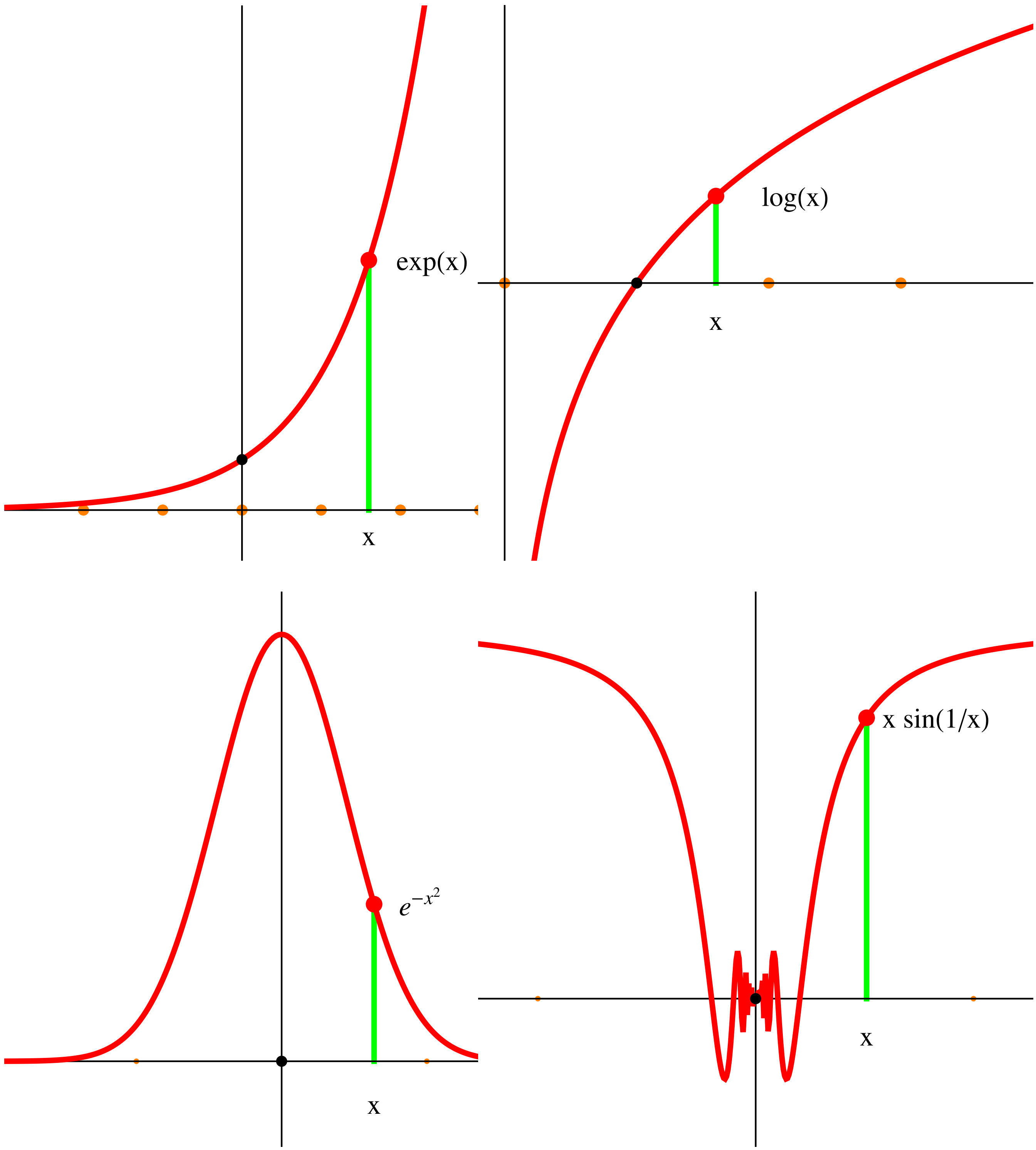
addition	$f(x) + g(x)$
scaling	$2f(x)$
translate	$f(x + 1)$
compose	$f(g(x))$
invert	$f^{-1}(x)$
difference	$f(x + 1) - f(x)$
sum up	$f(x) + f(x + 1) + \dots$

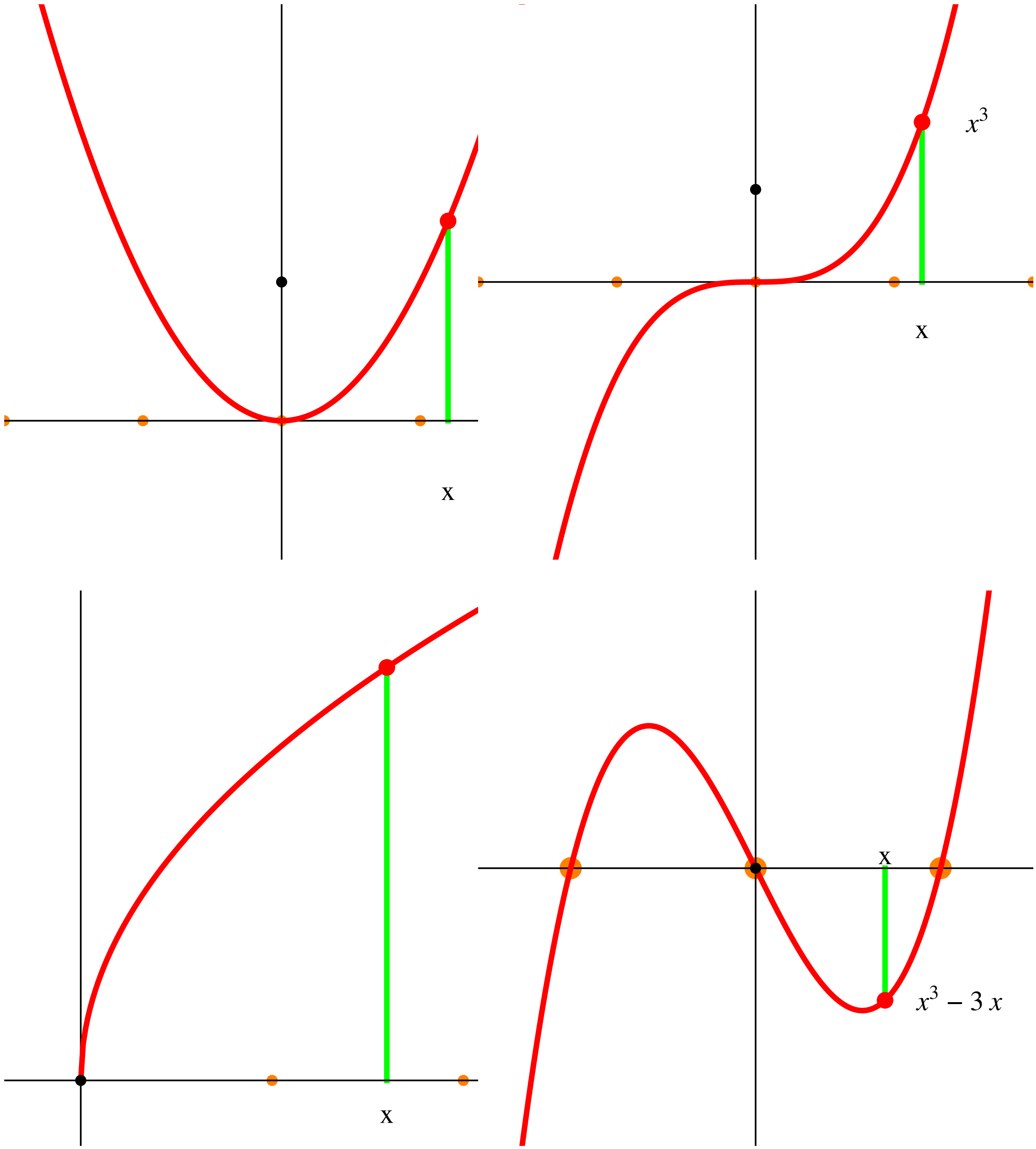
Here are important functions:

polynomials	$x^2 + 3x + 5$
rational functions	$(x + 1)/(x^4 + 1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arcsin^{-1}(x), \arctan(x)$.
roots	$\sqrt{x}, x^{1/3}$

We will look at these functions **a lot** during this course. The logarithm, exponential and trigonometric functions are especially important. For some functions, we need to restrict the domain, where the function is defined. For the square root function $\sqrt{x}$ or the logarithm $\log(x)$ for example, we have to assume that the number x on which we evaluate the function is positive. We write that the domain is $(0, \infty) = \mathbf{R}^+$. For the function $f(x) = 1/x$, we have to assume that x is different from zero. Keep these three examples in mind.

The **graph** of a function is the set of points $\{(x, y) = (x, f(x))\ \}$ in the plane, where x runs over the domain A of f. Graphs allow us to **visualize** functions. We can "see them", when we draw the graph.





x
x^3
x
x
x
$x^3 - 3x$

Homework

1 Draw the function $f(x) = x^3 \cos(x)$. Its graph goes through the origin $(0, 0)$. You can use Wolfram Alpha or a calculator to plot it if you like.
a) A function is called **odd** if $f(-x) = -f(x)$. Is f odd?
b) A function is called **even** if $f(x) = f(-x)$. Is f even?
c) A function is called **monotone increasing** if $f(y) > f(x)$ if $y > x$. Is f monotone increasing on the interval $[-1, 1]$? There is need to decide this yet analytically. Just draw(*) and decide.

2 A function $f : A \to B$ is called **invertible** or **one to one** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$. In that case, the function g is called the **inverse** of f. For example, the function $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0, \infty)$ to $B = [0, \infty)$. Determine from the following functions whether they are invertible. If they are invertible, find the inverse.
a) $f(x) = \cos(x)$ from $A = [0, \pi/2]$ to $B = [0, 1]$
b) $f(x) = \sin(x)$ from $A = [0, \pi]$ to $B = [0, 1]$
c) $f(x) = x^3$ from $A = \mathbf{R}$ to $B = \mathbf{R}$
d) $f(x) = \exp(x)$ from $A = \mathbf{R}$ to $B = R^+ = (0, \infty)$.
e) $f(x) = 1/(1 + x^2)$ from $A = [0, \infty)$ to $B = (0, 1]$.

3 We have defined $\log_h(x)$ as the inverse of $\exp_h(x) = (1 + h)^{x/h}$.
a) Draw the graphs of $\exp_1(x), \exp_{1/2}(x), \exp_{1/10}(x)$ and $\exp(x)$.
b) Draw the graphs of $\log_1(x), \log_{1/2}(x), \log_{1/10}(x)$ and $\log(x)$.
You are welcome to use technology for a). For b), just "flip the graph".

4 Plot the Function $\exp(\exp(\exp(x)))$ on $[0, 1]$. This is a fine function but computer programs do not plot the graph correctly. Describe until where Mathematica or Wolfram plots the function.

5 A function $f(x)$ has a **root** at $x = a$ if $f(a) = 0$. Roots are places, where the function is zero. Find one root for each of the following functions or state that there is none.
a) $f(x) = \cos(x)$
b) $f(x) = 4\exp(-x^4)$
c) $f(x) = x^5 - x^3$
d) $f(x) = \log(x) = \ln(x)$, the inverse of exp
e) $f(x) = \sin(x) - 1$
f) $f(x) = \csc(x) = 1/\sin(x)$

(*) Here is how you can use the Web to plot a function. The example given is $\sin(x)$.

```
http://www.wolframalpha.com/input/?i=Plot+sin(x)
```

Lecture 2: Worksheet

In this lecture, we get acquainted with the most important functions.

Trigonometric functions

The cosine and sine functions can be defined geometrically by the coordinates $(\cos(x), \sin(x))$ of a point on the unit circle. The tangent function is defined as $\tan(x) = \sin(x)/\cos(x)$.

$$\begin{aligned} \cos(x) &= \text{ adjacent side/hypothenuse} \\ \sin(x) &= \text{ opposite side/hypothenuse} \\ \tan(x) &= \text{ opposite side/adjadcent side} \end{aligned}$$

Pythagoras theorem gives us the important identity

$$\cos^2(x) + \sin^2(x) = 1$$

Define also $\cot(x) = 1/\tan(x)$. Less important but sometimes used are $\sec(x) = 1/\cos(x), \csc(x) = 1/\sin(x)$.

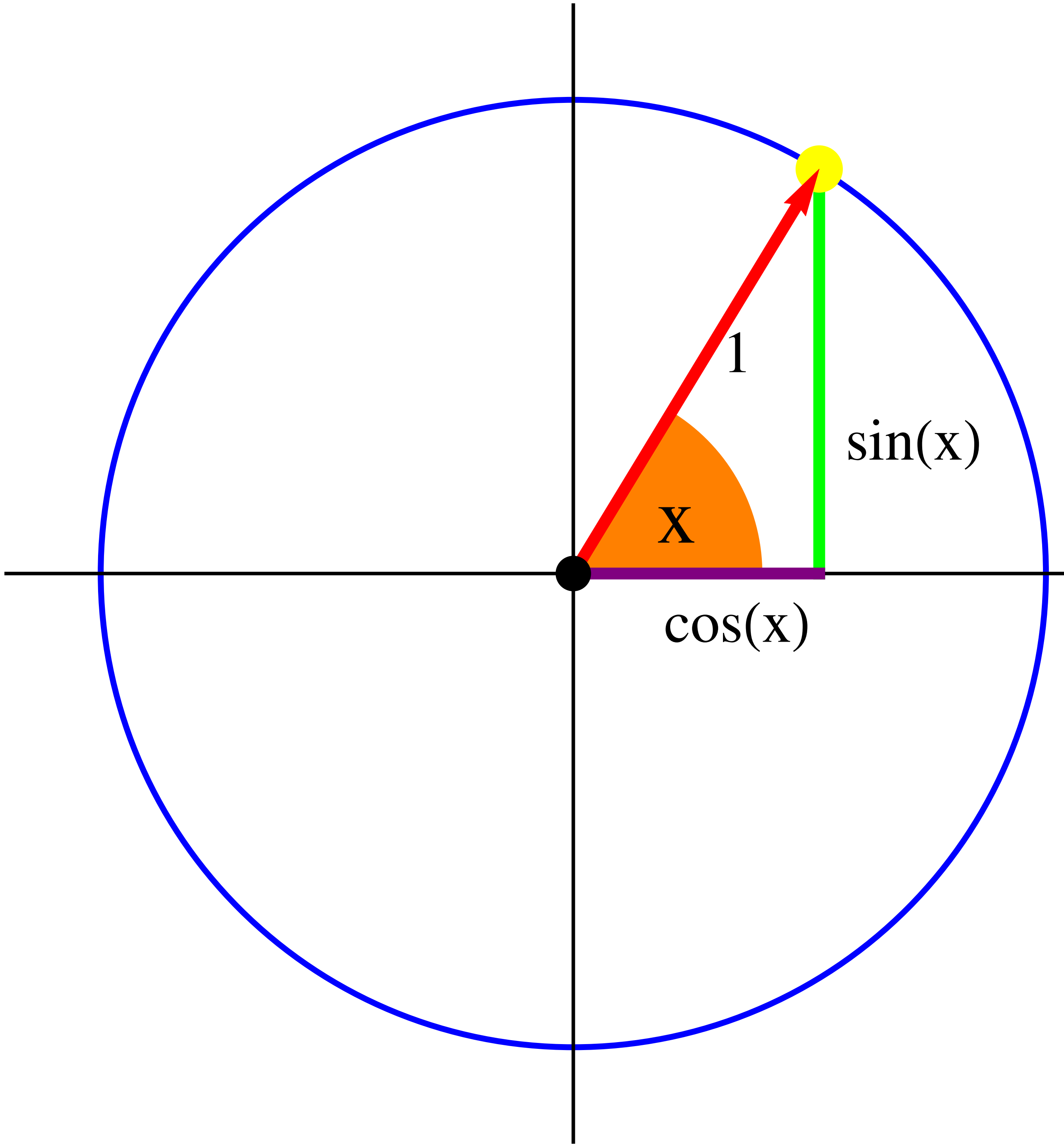
1 Find $\cos(\pi/3)$, $\sin(\pi/3)$.

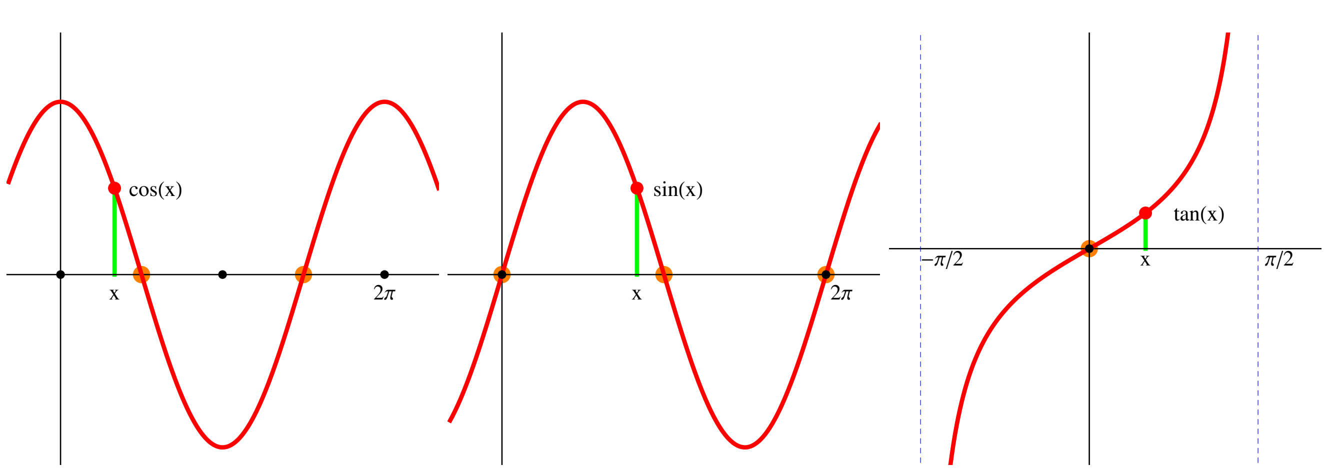
2 Where are the roots os cos and sin?

3 Find $\tan(3\pi/2)$ and $\cot(3\pi/2)$.

4 Find $\cos(3\pi/2)$ and $\sin(3\pi/2)$.

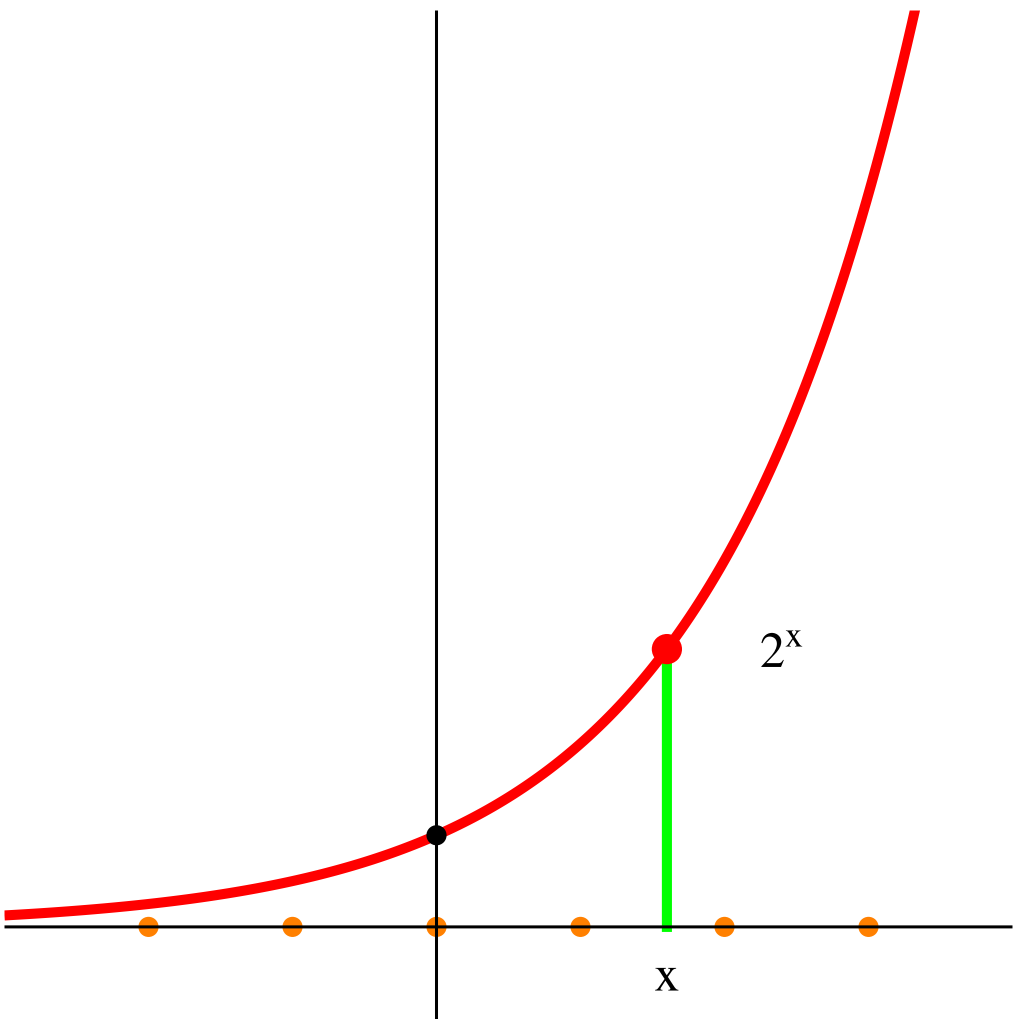
5 Find $\tan(\pi/4)$ and $\cot(\pi/4)$.


1
sin(x)
x
cos(x)


cos(x)
x
2π
sin(x)
x
2π
tan(x)
−π/2
x
π/2

The exponential function

The function $f(x) = 2^x$ is first defined for positive integers like $2^{10} = 1024$, then for all integers with $f(0) = 1, f(-n) = 1/f(n)$. Using roots, it can be defined for rational numbers like $2^{3/2} = 8^{1/2} = \sqrt{8} = 2.828...$. Since the function 2^x is monotonone on the set of rationals, we can fill the gaps and define $f(x)$ for any real x. By taking square roots again and again for example, we see $2^{1/2}, 2^{1/4}, 2^{1/8}, ...$ we approach $2^0 = 1$.



There is nothing special about 2 and we can take any positive base a and define the exponential a^x. It satisfies $a^0 = 1$ and the remarkable rule:

$$\boxed{a^{x+y} = a^x \cdot a^y}$$

It is spectacular because it provides a link between addition and multiplication.

We will especially consider the exponential $\boxed{\exp_h(x) = (1+h)^{x/h}}$, where h is a positive parameter. This is a super cool exponential because it satisfies $\exp_h(x+h) = (1+h)\exp_h(x)$ so that

$$[\exp_h(x+h) - \exp_h(x)]/h = \exp_h(x) \; .$$

We will see this relation again. For cocktail party conversation say that "the quantum derivative of the quantum exponential is the function itself for any Planck constant h".

For $h = 1$, we have the function 2^x we have started with. In the limit $h \to 0$, we get the important exponential function $\exp(x)$ which we also call e^x. For $x = 1$, we get the **Euler number** $e = e^1 = 2.71828...$.

1 What is 2^{-5}?

2 Find $2^{1/2}$.

3 Find $27^{1/3}$.

4 Why is $A = 2^{3/4}$ smaller than $B = 2^{4/5}$? Take the 20th power!

5 Assume $h = 2$ find $\exp_h(4)$.

Lecture 3: Limits

We have seen that functions like $1/x$ are not defined everywhere. Sometimes, however, functions do not make sense at first at some points but can be fixed. A silly example is $f(x) = x/x$ which is a priori not defined at $x = 0$ because we divide by x but can be "saved" by noticing that $f(x) = 1$ for all x different from 0. Functions often can be continued to "forbidden" places if we write the function differently. This can involves dividing out a common factor. Lets look at examples:

1 The function $f(x) = (x^3 - 1)/(x - 1)$ is at first not defined at $x = 1$. However, for x close to 1, nothing really bad happens. We can evaluate the function at points closer and closer to 1 and get closer and closer to 3. We will say $\lim_{x\to 1} f(x) = 3$. Indeed, as you might have noticed already, we have $f(x) = x^2 + x + 1$ by factoring out the term $(x - 1)$. While the function was initially not defined at $x = 1$, we can assign a natural value 3 at the point $x = 1$ so that the graph continues nicely through that point.

We write $x \to a$ to indicate that x approaches a. This approach can be from either side. A function $f(x)$ has a **limit** at a point a if there exists b such that $f(x) \to b$ for $x \to a$. We write $\lim_{x\to a} f(x) = b$. Again, it should not matter, whether we approach a from the left or from the right. If the limit exists, we must get the same limiting value b.

2 The function $f(x) = \sin(x)/x$ is called $\text{sinc}(x)$. We see experimentally that it converges to 1 as $x \to 0$. We can see this geometrically by comparing the side $a = \sin(x)$ of a right angle triangle with a small angle $\alpha = x$ and hypotenuse 1 with the length of the arc between B, C of the unit circle centered at A. The arc has length x which is close to $\sin(x)$ for small x. Keep this example in mind. It is a crucial one. The fact that the limit of $f(x)$ exists for $x \to 0$ is some important that it is sometimes called the **fundamental theorem of trigonometry**. We will see this later.

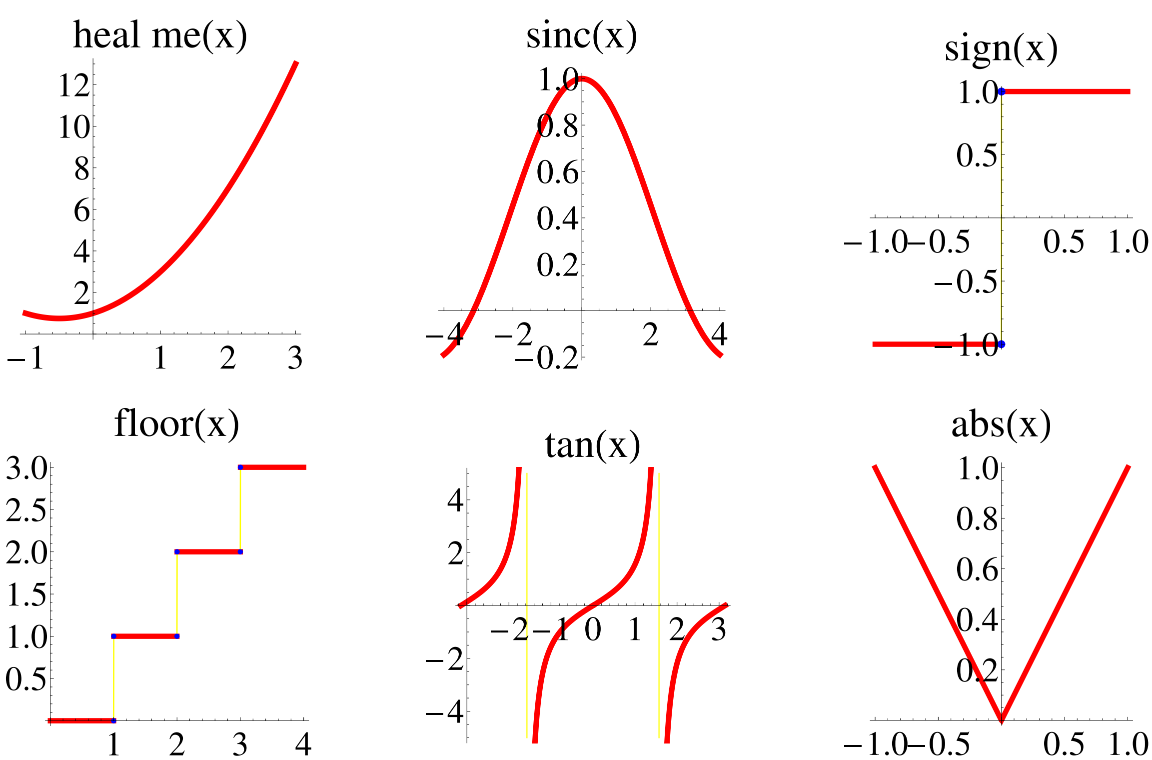


Figure: The graphs $f(x) = (x^3 - 1)/(x - 1)$, the sinc function $\text{sinc}(x) = \sin(x)/x$, the sign function $\text{sign}(x) = x/|x|$, the floor function floor(x) giving the largest integer smaller or equal to x, the tan function and the absolute value function $\text{abs}(x) = |x|$.

3 The function $f(x) = x/|x|$ is equal to 1 if $x > 0$ and equal to -1 if $x < 0$. The function is not defined at $x = 0$ and there is no way to assign a value b at $x = 0$ so that $\lim_{x\to 0} f(x) = b$. After defining $f(0) = 0$ we can call the function the *sign* function.

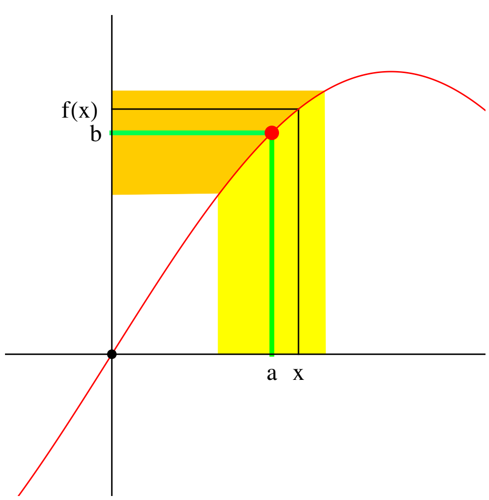
4 The quadratic function $f(x) = x^2$ has the property that $f(x)$ approaches 4 if x approaches 2. To evaluate functions at a point, we do not have to take a limit. The function is already defined there. This is important: most points are "healthy". We do not have to worry about limits. In the overwhelming cases of real applications we only have to worry about limits when the function involves division by 0. For example $f(x) = (x^4 + x^2 + 1)/x$ needs to be investigated carefully at $x = 0$. You see for example that for $x = 1/1000$, the function is slightly larger than 1000, for $x = 1/1000000$ it is larger than one million. There is no rescue. The limit does not exist at 0.

5 More generally, for all polynomials, the limit $\lim_{x\to a} f(x) = f(a)$ is defined. We do not have to worry about limits, if we deal with polynomials.

6 For all trigonometric polynomials involving sin and cos, the limit $\lim_{x\to a} f(x) = f(a)$ is defined. We do not have to worry about limits if we deal with trigonometric polynomials like $\sin(3x) + \cos(5x)$. The function $\tan(x)$ however has no limit at $x = \pi/2$. No finite value b can be found so that $\tan(\pi/2 + h) \to b$ for $h \to 0$. This is due to the fact that $\cos(x)$ is zero at $\pi/2$. We have $\tan(x)$ goes to $+\infty$ "plus infinity" for $x \searrow \pi/2$ and $\tan(x)$ goes to $-\infty$ for $x \nearrow \pi/2$. In the first case, we approach $\pi/2$ from the right and in the second case from the left.

7 The **cube root** function $f(x) = x^{1/3}$ converges to 0 as $x \to 0$. For $x = 1/1000$ for example, we have $f(x) = 1/10$ for $x = 1/n^3$ the value $f(x)$ is $1/n$. The cube root function is defined everywhere on the real line, like $f(-8) = -2$ and is continuous everywhere.

Why do we worry about limits at all? One of the main reasons will is that we will define the derivative and integral using limits. But we will also use limits to get numbers like $\pi = 3.1415926, ...$ In the next lecture, we will look at the important concept of continuity, which involves limits too.



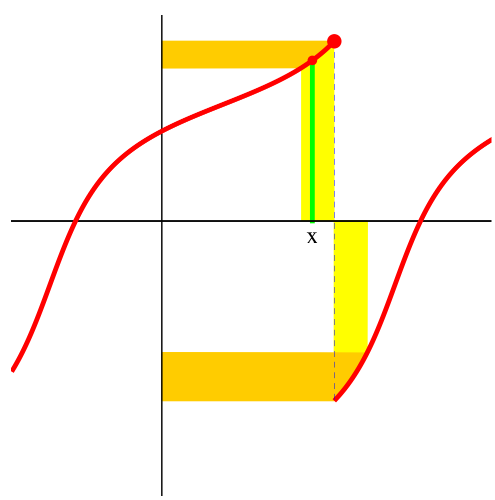


Figure: To the left we see a case, where the limit exists at $x = a$. If x approaches a then $f(x)$ approaches b. To the right we see the function $f(x) = \arctan(\tan(x) + 1)$, where arctan is the inverse of tan. The limit does not exist for $a = \pi/2$. If we approach a from the right, we get the limit $-\pi/2$ From the left, we get the limit $f(\pi/2) = \pi/2$. Note that f is not defined at $x = \pi/2$ because $\tan(x)$ becomes infinite there.

8 **Problem:** Determine from the following functions whether the limits $\lim_{x \to 0} f(x)$ exist. If it does, find it.

a) $f(x) = \cos(x)/\cos(2x)$	b) $f(x) = \tan(x)/x$
c) $f(x) = (x^2 - x)/(x - 1)$	d) $f(x) = (x^4 - 1)/(x^2 - 1)$
e) $f(x) = (x + 1)/(x - 1)$	f) $f(x) = x/\sin(x)$
g) $f(x) = 5x/\sin(6x)$	h) $f(x) = \sin(x)/x^2$
i) $f(x) = \sin(x)/\sin(2x)$	j) $f(x) = \exp(x)/x$

Solutions:
a) There is no problem at all at $x = 0$. Both, the nominator and denominator converge to 1. The limit is $\boxed{1}$
b) This is $\text{sinc}(x)/\cos(x)$. There is no problem at $x = 0$ for sinc nor for $1/\cos(x)$. The limit is $\boxed{1}$.
c) We can heal this function. It is the same as $x + 1$ everywhere except at $x = 1$ where it is not defined. But we can continue the simplified function $x + 1$ through $x = 1$. The limit is $\boxed{0}$.
d) We can heal this function. It is the same as $x^2 + 1$. The limit is $\boxed{1}$.
e) There is no problem at $x = 0$. There is mischief at $x = 1$ although but that is far, far away. At $x = 0$, we get $\boxed{-1}$.
f) This is the prototype, the fundamental theorem of trig! We know that the limit is $\boxed{1}$.
g) This can be written as $f(x) = (5/6)6x/\sin(6x) = (5/6)\text{sinc}(6x)$. The function $6x/\sin(6x)$ converges to 1 by the fundamental theorem of trigonometry. Therefore the limit is $\boxed{5/6}$.
h) This limit does not exist. It can be written as $\text{sinc}(x)/x$. Because $\text{sinc}(x)$ converges to 1. we are in trouble when dividing again by x. $\boxed{\text{There is no limit.}}$
i) We know $\sin(x)/x \to 1$ so that also $\sin(2x)/(2x)$ has the limit 1. If we divide them, see $\sin(x)/\sin(2x) \to 1/2$. The result is $\boxed{1/2}$.
j) The limit does not exist because $\exp(x) \to 1$ but $1/x$ goes to infinity.

Here are obvious properties which hold for limits:

$\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$ implies $\lim_{x \to a} f(x) + g(x) = b + c$.
$\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$ implies $\lim_{x \to a} f(x) \cdot g(x) = b \cdot c$.
$\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c \neq 0$ implies $\lim_{x \to a} f(x)/g(x) = b/c$.

This implies we can sum up and multiply or divide functions which have limits:

Polynomials like $x^5 - 2x + 6$ have limits everywhere.
Trig polynomials like $\sin(3x) + \cos(5x)$ have limits everywhere.
Rational functions have limits except at points where the denominator is zero.
Functions like $\cos^2(x)\tan(x)/\sin(x)$ can be healed by simplification.
Prototype functions like $\sin(x)/x$ have limits everywhere

Homework

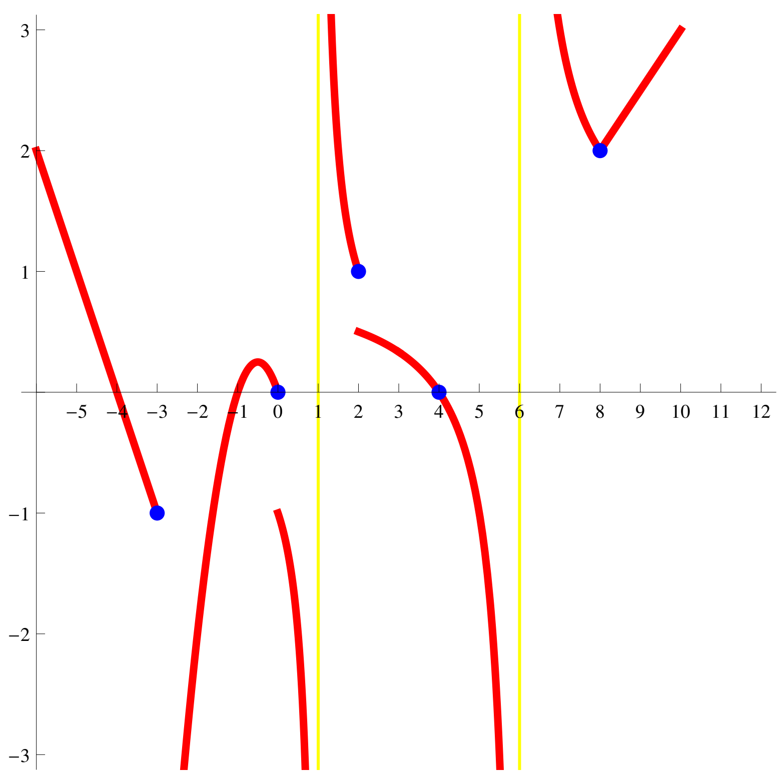
1 Find the limits of each of the following functions at the point $x \to 0$. You can already use the fact that $\sin(x)/x$ has the limit 1 as $x \to 0$.
a) $f(x) = (x^4 - 1)/(x - 1)$
b) $f(x) = \sin(5x)/x$
c) $f(x) = \sin^2(3x)/x^2$
d) $f(x) = \sin(7x)/\sin(11x)$

2 a) Graph of the function

$$f(x) = \frac{(1 - \cos(x))}{x^2} .$$

b) Where is the function f defined? Can you find the limit at the places, where it is not defined? Hint: use the **double angle formula** $1 - \cos(x) = 2\sin^2(x/2)$.
c) Verify that the function $f(x) = \exp_h(x) = (1+h)^{x/h}$ satisfies $[f(x+h) - f(x)]/h = f(x)$.
Remark. The exponential function can be defined as $e^x = \exp(x) = \lim_{h \to 0} \exp_h(x)$.

3 Find all points a at which the function given in the picture has no limits.



4 Find the limits for $x \to 0$:
a) $f(x) = (x^2 - 2x + 1)/(x - 1)$.
b) $f(x) = 2^x$.
c) $f(x) = 2^{2^x}$.
d) $f(x) = \sin(\sin(x))/\sin(x)$.

5 We explore in this problem the limit of the function $f(x) = x^x$ if $x \to 0$. Can we find a limit? Take a calculator or use Wolfram α and experiment. What do you see when $x \to 0$? Only optional: can you find a explanation for your experiments?

Lecture 3: Worksheet

We study a few limits.

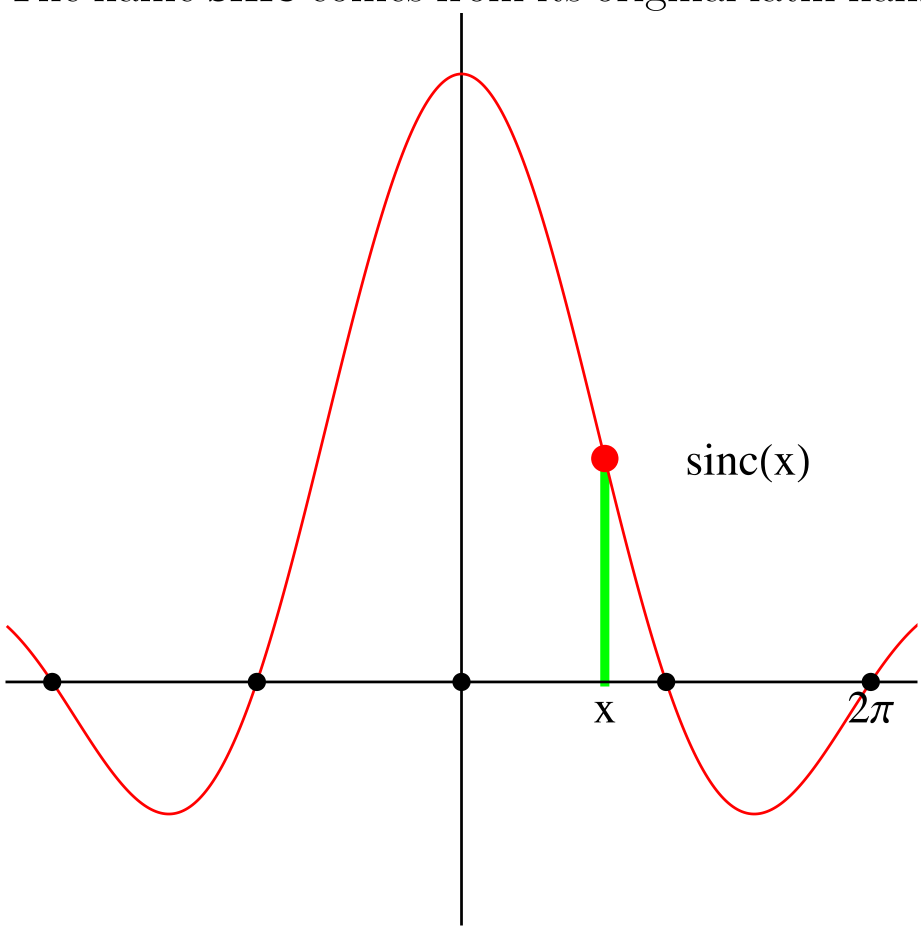
The Sinc function

A prototype function for studying limits is the sinc function

$$f(x) = \frac{\sin(x)}{x} .$$

It is an important function and appears in many applications like in the study of waves or signal processing (it is used in low pass filters).

The name **sinc** comes from its original latin name **sinus cardinalis**.



1 Does the function $\frac{\cos(x)}{x}$ have a limit at $x \to 0$?

2 Does the function $\frac{\sin(x^2)}{x^2}$ have a limit for $x \to 0$?

3 Does the function $\frac{\sin(x^2)}{x}$ have a limit for $x \to 0$?

4 Does the function $\frac{\sin(x)}{x^2}$ have a limit for $x \to 0$?

5 Does the function $\frac{x}{\sin(x)}$ have a limit for $x \to 0$?

6 Does the function $\frac{\sin(x)}{|x|}$ have a limit for $x \to 0$?

Lecture 4: Continuity

A **function** f is called **continuous** at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \to f(x_0)$ for $x \to x_0$. A function f is called **continuous on** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$.

In the interior (a, b), the limit needs to exist both from the right and from the left. At the boundary a, only the right limit needs to exist and at b, only the left limit. Intuitively, a function is continuous if you can **draw the graph of the function without lifting the pencil**. Continuity means that small changes in x results in small changes of $f(x)$.

1 Any polynomial as well as $\cos(x), \sin(x), \exp(x)$ are continuous everywhere. Also the sum and product of such functions is continuous. For example

$$\sin(x^3 + x) - \cos(x^5 + x^3)$$

is continuous everywhere. We can also compose functions like $\exp(\sin(x))$ and still have a continuous function.

2 The function $f(x) = 1/x$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous. This discontinuity is called a **pole**. The **division by zero** kills continuity. Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere. We consider the function healed at 0 even so it was at first not defined at $x = 0$.

3 The function $f(x) = \log|x|$ is continuous for x different from 0. It is not continuous at 0 because $f(x) \to -\infty$ for $|x| \to 0$. Keep the two examples, $1/x$ and $\log|x|$ in mind.

4 The function $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi, x = 2\pi$ and any multiple of π. It has poles there because $\sin(x)$ is zero there and because we would divide by zero at such points. The function $\cot(x) = \cos(x)/\sin(x)$ shares the same discontinuity points as $\csc(x)$.

5 The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just chose $x_n = 2/(4k + 1)$ or $z_n = 2/(4k - 1)$.

6 The **signum function** $f(x) = \mathrm{sign}(x) = \left[\begin{array}{ll} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{array} \right.$ is not continuous at 0. It is a prototype of a function with a **jump** discontinuity at 0. It is impossible to heal the gap.

We can refine the notion of continuity and say that a function is **continuous from the right**, if there exists a limit from the right $\lim_{x \downarrow a} f(x) = b$. Similarly a function f can be continuous from the left only. Most of the time we mean with "continuous"= "continuous everywhere on the real line".

Rules:
a) If f and g are continuous, then $f+g$ is continuous.
b) If f and g are continuous, then $f*g$ is continuous.
c) If f and g are continuous and if $g>0$ then f/g is continuous.
d) If f and g are continuous, then $f \circ g$ is continuous.

7 $\sqrt{x^2+1}$ is continuous everywhere on the real line.

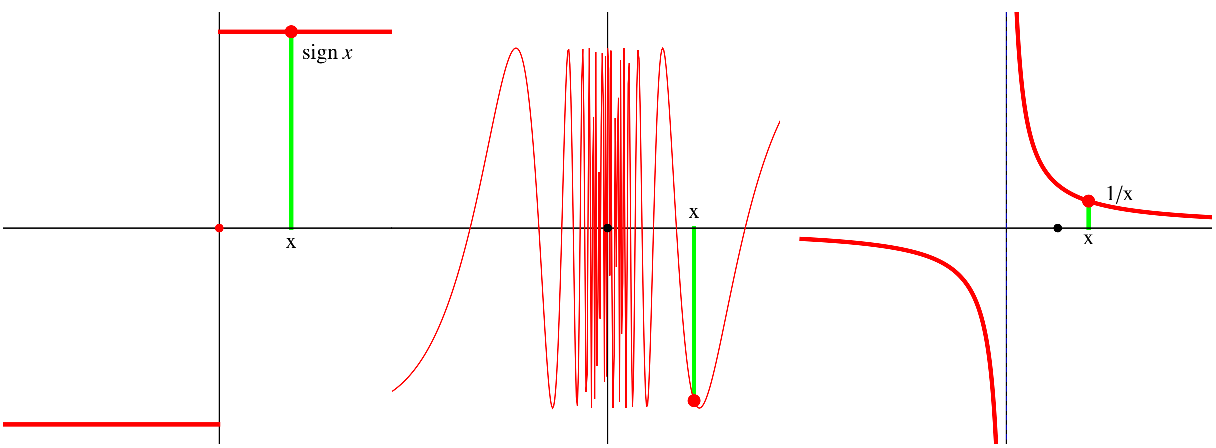
8 $\cos(x)+\sin(x)$ is continuous everywhere.

9 The function $f(x)=\log(|x|)$ (we write $\log = \ln$ for the natural log) is continuous everywhere except at 0. Indeed since for every integer n, we have $f(e^{-n}) = -n$, this can become arbitrarily large for $n \to \infty$ even so e^{-n} converges to 0 for n running to infinity.

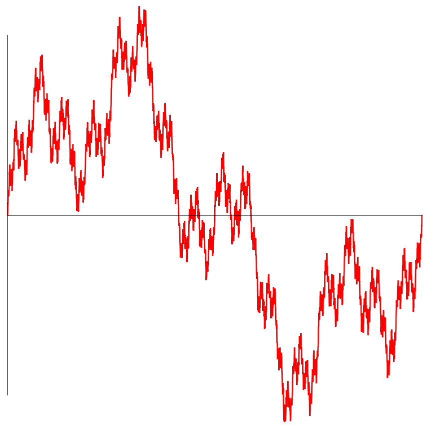
10 While $\log(|x|)$ is not continuous at $x=0$, the function $1/\log|x|$ is continuous at $x=0$. Is it continuous everywhere?

11 The function $f(x)=[\sin(x+h)-\sin(x)]/h$ is continuous for every $h>0$. We will see next week that nothing bad happens when h becomes smaller and smaller and that the continuity will not deteriorate. Indeed, we will see that we get closer and closer to the cos function.

There are three major reasons, why a function is not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity. Here are the prototype examples. We will look at more during the lecture.

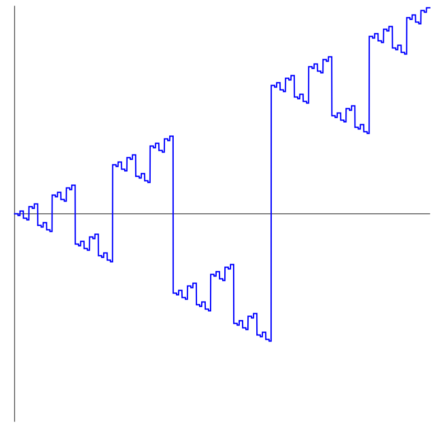


Why do we like continuity? We will see many reasons during this course but for now lets just say that:



A wild continuous function. This Weierstrass function is believed to be a fractal.

"Continuous functions can be pretty wild, but not too crazy."



A crazy discontinuous function. It is discontinuous at every point and known to be a fractal.

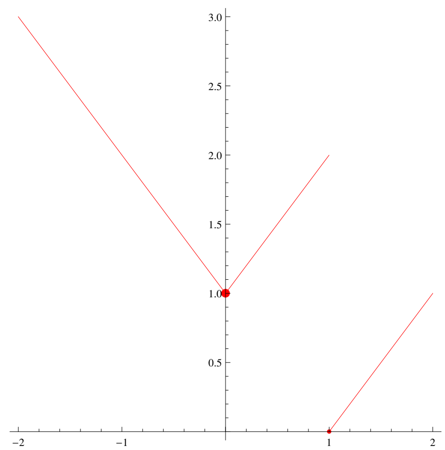
Continuity will be useful when finding maxima and minima. A continuous function on an interval $[a, b]$ has a maximum and minimum. We will see in the next hour that if a continuous function is negative at some place and positive at an other, there is a point between, where it is zero. Being able to find solutions to equations $f(x) = 0$ is important and much more difficult, if f not continuous.

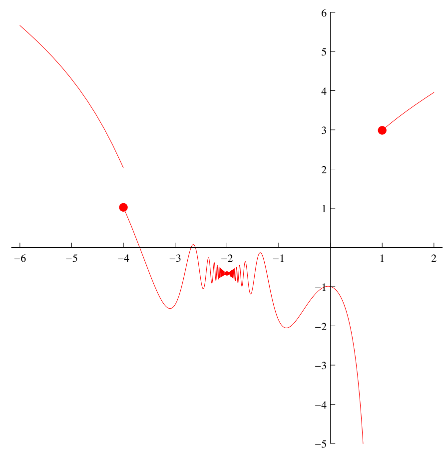
12 **Problem** Determine for each of the following functions, where discontinuities appear:
a) $f(x) = \log(|x^2 - 1|)$
b) $f(x) = \sin(\cos(\pi/x))$
c) $f(x) = \cot(x) + \tan(x) + x^4$
d) $f(x) = x^4 + 5x^2 - 3x + 4$
e) $f(x) = \frac{x^2-4x}{x}$
Solution.
a) $\log(|x|)$ is continuous everywhere except at $x = 0$. Since $x^2 - 1 = 0$ for $x = 1$ or $x = -1$, the function $f(x)$ is continuous everywhere except at $x = 1$ and $x = -1$.
b) The function π/x is continuous everywhere except at $x = 0$. Therefore $\cos(\cos(\pi/x)$ is continuous everywhere except possibly at $x = 0$. We have still to investigate the point $x = 0$ but there, the function $\cos(\pi/x)$ takes values between -1 and 1 for points arbitrarily close to $x = 0$. The function $f(x)$ takes values between $\sin(-1)$ and $\sin(1)$ arbitrarily close to $x = 0$. It is not continuous there.
c) The function x^4 is continuous everywhere. The function $\tan(x)$ is continuous everywhere except at the points $k\pi$. The function $\cot(x)$ is continuous everywhere except at points $\pi/2 + k\pi$. The function f is therefore continuous everywhere except at the point $x = k\pi/2$, multiples of $\pi/2$.
d) The function is a polynomial. We know that polynomials are continuous everywhere.
e) The function is continuous everywhere except at $x = 0$, where we have to look at the function more closely. But we can heal the function by dividing nominator and denominator by x which is possible for x different from 0. The healed function is $f(x) = x - 4$.

Homework

1 For the following functions, determine the points, where f is not continuous.
a) $\mathrm{sinc}(x) + 1/\cos(x)$
b) $\sin(\tan(x))$
c) $f(x) = \cot(2 - x)$
d) $\mathrm{sign}(x)/x$
e) $\frac{x^2+5x+x^4}{x-3}$
State which kind of discontinuity appears.

2 On which intervals are the following functions continuous?


a)


b)

3 Either do the following three problems a),b),c):
a) Construct a function which has a jump discontinuity and an escape to infinity.
b) Find a function which has an oscillatory discontinuity and an escape to infinity.
c) Find a function which has a jump discontinuity as well as an oscillatory discontinuity.
<u>or shoot down the problem with one strike:</u>
Find a function which has a jump discontinuity, a pole and an oscillatory discontinuity all at the same time.

4 Heal the following functions to make them continuous
a) $(x^3 - 8)/(x - 2)$
b) $(x^5 + x^3)/(x^2 + 3)$
c) $((\sin(x))^3 - \sin(x))/(\cos(x)\sin(x))$.
d) $(x^4 + 4x^3 + 6x^2 + 4x + 1)/(x^3 + 3x^2 + 3x + 1)$
e) $(x^{70} - 1)/(x^{10} - 1)$

5 Are the following function continuous? Break the functions up into simpler functions and analyze each. If you are not sure, experiment by plotting the functions.
a) $\sin(\frac{1}{3+\sin(x)\cos(x)}) + |\cos(x)| + \frac{\sin(x)}{x} + x^5 + x^3 + 1 + \frac{7}{\exp(x)}$.
b) $\frac{2}{\log|x|} + x^7 - \cos(\sin(\cos(x))) - \exp(\log(\exp(x)))$

Lecture 4: Worksheet

Good, the happy, bad and ugly

Jumps, poles and Oscillations are the perils for continuity. We do not have to worry about continuity in most cases, most functions are good. But there are very few mechanisms which bring you in peril. A function can either happily jump, badly rush to infinity or have an ugly oscillation. All cases come from division by zero somewhere.

Good Guys	**Bad Guys**
$x^2 + 4x + 6$	$1/x$ at 0
$\sin(x), \cos(x)$	$\tan(x)$ at $\pi/2$
$\exp(x)$	$\log\|x\|$ at 0
$\text{sinc}(x) = \frac{\sin(x)}{x}$	$\sec(x) = \frac{1}{\cos(x)}$ at $\pi/2$

Surprises

$\sin(x)/x$	is continuous at 0
$1/\log\|x\|$	is continuous at 0
$x\sin(1/x)$	is continuous at 0

Which functions are continuous?

Which of the following functions are continuous?

1 Is $f(x) = \log 1 + |x|$ continuous at $x = 0$?

2 Is $f(x) = \sqrt{|x|}$ continuous at $x = 0$?

3 Is $f(x) = \frac{1}{\sqrt{|x|}}$ continuous at $x = 0$?

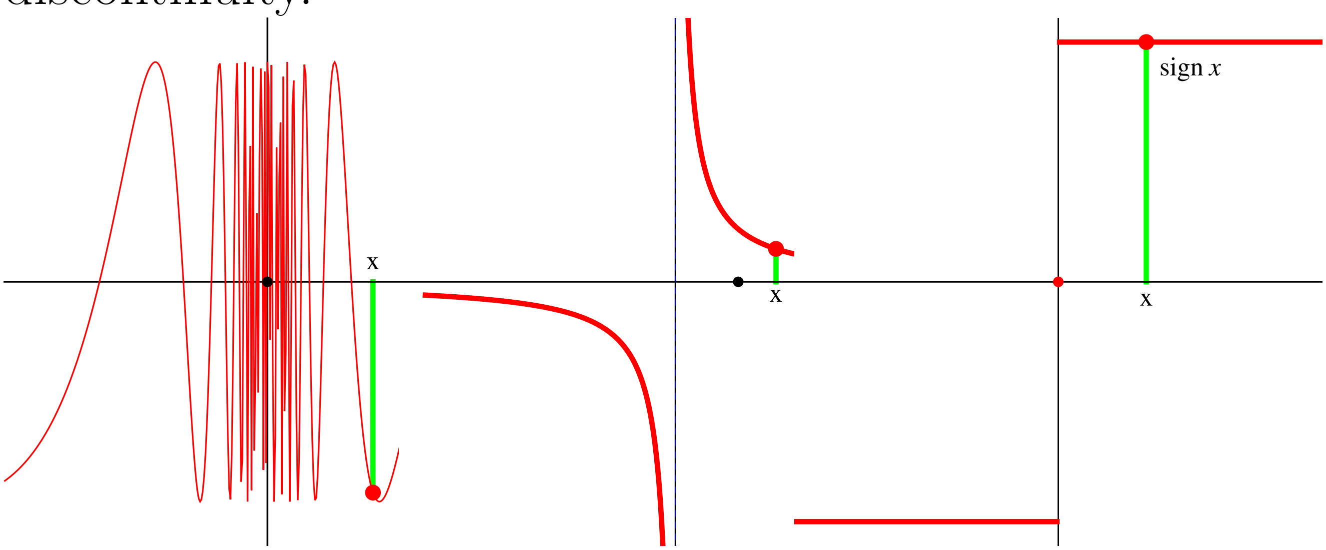
4 Is $\frac{1}{\log |1/x|}$ continuous at $x = 0$?

5 Is $\log(\log |x|)$ continuous everywhere?

6 Is $1/(1 + |x|)$ continuous everywhere?

Enemy of continuity

Oscillations, escape to infinity and jumps are reasons for discontinuity.



Lecture 5: Intermediate Value Theorem

If $f(a) = 0$, then a is called a **root** of f. For $f(x) = \cos(2x)$ for example, there are roots at $x = \pi$ or $x = 3\pi$ or $x = -\pi$.

1 Find the roots of $f(x) = 4x + 6$. **Answer**: we set $f(x) = 0$ and solve for x. In this case $4x + 6 = 0$ and so $x = -3/2$.

2 Find the roots of $f(x) = x^2 + 2x + 1$. **Answer**: Because $f(x) = (x + 1)^2$ the function has a root at $x = -1$.

3 Find the roots of $f(x) = (x - 2)(x + 6)(x + 3)$. **Answer:** Since the polynomial is factored already, it is easy to see the roots $x = 2, x = -6, x = -3$.

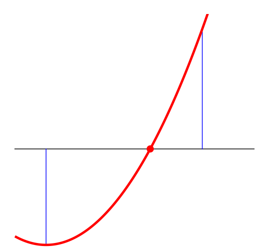
4 $f(x) = 12 + x - 13x^2 - x^3 + x^4$. Find the roots of f. While we do not have a formula for this, but we can try. Indeed, we see that $x = 1, x = -3, x = 4, x = -1$ are the roots.

5 The function $f(x) = \exp(x)$ does not have any root.

6 The function $f(x) = \log|x| = \ln|x|$ has roots $x = 1$ and $x = -1$.

7 $f(x) = 2^x - 16$ has the root $x = 2$.

Intermediate value theorem of Bolzano. If f is continuous on the interval $[a, b]$ and $f(a), f(b)$ have different signs, then there is a root of f in (a, b).

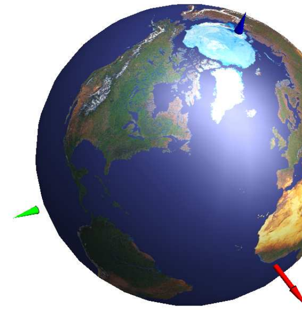


The proof is constructive: we can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at the point $c = (a + b)/2$. If $f(c) < 0$, then look take $[c, b]$ as your new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of length $2^{-n}(b - a)$. Continuity assures that $f(u_n) - f(v_n) \to 0$ and $f(u_n), f(v_n)$ have different signs. Both u_n, v_n converge to a root of f.

8 Verify that the function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root. **Solution.** The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. The intermediate value theorem assures there is a point where $f(x) = 0$.

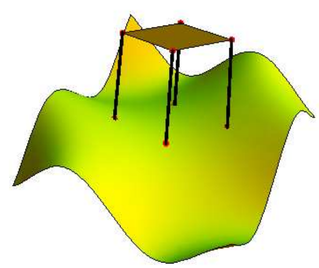
9 There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem.

10 **Earth Theorem.** There is a point on the earth, where temperature and pressure agrees with the temperature and pressure on the antipode.



Proof. Lets draw a meridian through the north and south pole and let $f(x)$ be the temperature on that circle. Define $g(x) = f(x) - f(x+\pi)$. If this function is zero on the north pole, we have found our point. If not, $g(x)$ has different signs on the north and south pole. There exists therefore an x, here the temperature is the same. Now, for every meridian, we have a latitude value $l(x)$ for which the temperature works. Now define $h(x) = l(x) - l(x+pi)$. This function is continuous. Start with meridian 0. If $h(0) = 0$ we have found our point. If not, then $h(0)$ and $h(\pi)$ take different signs. By the intermediate value theorem again, we have a root of h. At this point both temperature and pressure are the same than on the antipode. Remark: this argument in the second part is not yet complete. Do you see where the problem is?

11 **Wobbly Table Theorem.** On an arbitrary floor, a square table can be turned so that it does not wobble any more.



Proof. The 4 legs ABCD are on a square. Let x be the angle of the line AC with with some coordinate axes if we look from above. Given x, we can position the table **uniquely** as follows: the center of ABCD is on the z-axes, the legs ABC are on the floor and AC points in the direction x. Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. (If it is negative, the argument is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h. This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.

Lets call $Df(x) = (f(x+h) - f(x))/h$ the **discrete derivative** of f for the constant h. We will study it more in the next lecture. You have in a homework already verified that $D \exp_h(x) = \exp_h(x)$.

Lets call a point p, where $Df(x) = 0$ a **discrete critical point** for h. Lets call a point a a **local maximum** if $f(a) \geq f(x)$ in an open interval containing a. Define similarly a **local minimum** as a point where $f(a) \leq f(x)$.

12 The function $f(x) = x(x-h)(x-2h)$ has the derivative $Df(x) = 3x(x-h)$ as you have verified in the case $h = 1$ in the first lecture of this course in a worksheet. We will write $[x]^3 = x(x-h)(x-2h)$ and $[x]^2 = x(x-h)$. The computation just done tells that $D[x]^3 = 3[x]^2$. Since $[x]^2$ has exactly two roots $0, h$, the function $[x]^3$ has exactly 2 critical points.

13 More generally for $[x]^{n+1} = x(x-h)(x-2h)...(x-nh)$ we have $D[x]^{n+1} = (n+1)D[x]^n$. Because $[x]^n$ has exactly n roots, the function $[x]^{n+1}$ has exactly n critical points. Keep the formula

$$\boxed{D[x]^n = n[x]^{n-1}}$$

in mind!

14 The function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D \exp_h(x) = \exp_h(x)$. Because this function has no roots and the derivative is the function itself, the function has no critical points.

Critical points lead to extrema as we will see later in the course. In our discrete setting we can say:

Fermat's maximum theorem If f is continuous and has a critical point a for h, then f has either a local maximum or local minimum inside the open interval $(a, a+h)$.

Look at the range of the function f restricted to $[a, a+h]$. It is a bounded interval $[c, d]$ by the intermediate value theorem. There exists especially a point u for which $f(u) = c$ and a point v for which $f(v) = d$. These points are different if f is not constant on $[a, a+h]$. There is therefore one point, where the value is different than $f(a)$. If it is larger, we have a local maximum. If it is smaller we have a local minimum.

15 **Problem.** Verify that a cubic polynomial has maximally 2 critical points. **Solution** $f(x) = ax^3 + bx^2 + cx + d$. Because the x^3 terms cancel in $f(x+h) - f(x)$, this is a quadratic polynomial. It has maximally 2 roots.

What we have called "critical point" here will in the limit $h \to 0$ be called "critical point" later in this course. While the h-critical point notion makes sense for any continuous function, we will need more regularity to take the limit $h \to 0$. This limit $h \to 0$ will be one of the major features.

Homework

1 Find the roots for $x^4 - 4x^3 - 7x^2 + 22x + 24$. You are told that all roots are integers.

2 Use the intermediate value theorem to verify that $f(x) = x^5 - 6x^4 + 8$ has at least two roots on $[-2, 2]$.

3 Miley' height is 165 cm. Gaga's height is 155 cm. Gaga was born March 28, 1986, Miley was born November 23, 1992. Assume Gaga owns now 500 millions and Miley owns 150 millions.
a) Can you argue that there was a moment when Miley's height was exactly half of Gaga's height?
b) Can you argue that there was a moment when Miley's fortune was exactly a tenth of Gagas fortune?
c) Many of you live in New York. Show that if you drive the 190 miles from here to New York in 4 hours then there are at least two moments of time when you drive with exactly 40 miles per hours. The trip is not part of a larger trip. Your start is in Boston and your Destination is New York.

4 Argue why there is a solution to
a) $5 + \sin(x) = x$.
b) $\exp(3x) = x$.
c) $\text{sinc}(x) = x^4$.
d) Why does the following argument not work:
The function $f(x) = 1/\cos(x)$ satisfies $f(0) = 1$ and $f(\pi) = -1$. There exists therefore a point x where $f(x) = 0$.
e) Does the function $f(x) = x + \log|\log|x||$ have a root somewhere?

5 a) Find a concrete function which has three local maxima and two local minima.
b) Let $h = 1$. Find a critical point for the function $f(x) = |x|$.
c) Verify that for any $h > 0$, the function $f(x) = x^5$ has no critical point in the sense given in the text: $[f(x+h) - f(x)]/h = 0$ is not possible.

Lecture 5: Worksheet

Last Sunday was Groundhog day. Punxsatawney Phil predicted the future, saw some shadow and predicts a couple of more winter weeks. We study here extrema and the intermediate value theorem.

The intermediate value theorem

1 Today the average temperature is 37° Fahrenheit. Argue that there had been a moment this fall/winter where the temperature had been exactly 40 degree Fahrenheit.

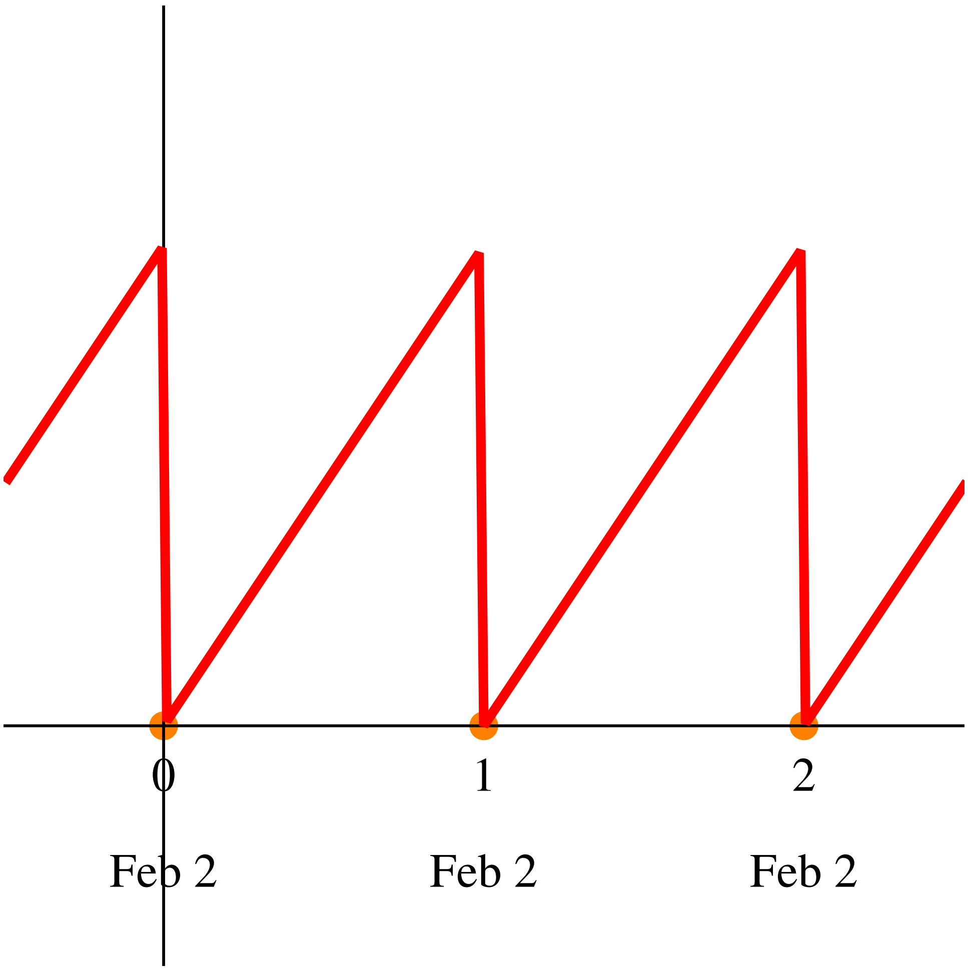
2 Is there a point x, where

$$\frac{1}{\sin(x)} = \frac{1}{2} \, .$$

Why does the intermediate value theorem not give such a point? We have $1/\sin(\pi/2) = 1$ and $1/\sin(3\pi/2) = -1$.

3 The earth's diameter is 12'756 km in average. Is there a point on earth where the distance to its antipiode is exactly 12'756 km?

4 The function $f(x) = x - \text{floor}(x)$ is called the **ground hog function**. If you know the movie with **Bill Murray**, you know why. Find an interval where the intermediate value theorem fails.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Lecture 6: Fundamental theorem

Calculus is the theory of **differentiation** and **integration**. We explore this concept in a simple setup and practice differentiation and integration without taking limits. We fix a positive constant h and take differences and sums. Without taking limits, we prove a fundamental theorem of calculus. We can so differentiate and integrate polynomials, exponentials and trigonometric functions. Later, we will do the same with actual derivatives and integrals. But now, we can work with arbitrary continuous functions. The constant h might confuse you a bit at first. Just think of it as something small. If you like, take $h = 1$ everywhere. One of the advantages of the calculus we are going to learn is that h will no more appear.

Given a function $f(x)$, define the **difference quotient**

$$\boxed{Df(x) = (f(x+h) - f(x))\,\frac{1}{h}}$$

If f is continuous then Df is a continuous. For shorthand, we call it simply the "derivative". It will in the limit $h \to 0$ become the derivative we are going to define later in the course but we keep h constant and positive here.

1 Lets take the constant function $f(x) = 5$. We get $Df(x) = (f(x+h)-f(x))/h = (5-5)/h = 0$ everywhere. You can see that in general, if f is a constant function, then $Df(x) = 0$.

2 $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$. You see in general that if f is a linear function $f(x) = ax + b$, then $Df(x) = a$ is constant.

3 If $f(x) = ax + b$, then $Df(x) = \boxed{a}$.

For constant functions, the derivative is zero. For linear functions, the derivative is the slope.

4 For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h$ which is $\boxed{2x+h}$.

Given a function f, define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \leq jh < x$. That is, if k is such that $(k-1)h$ is the largest below x, then

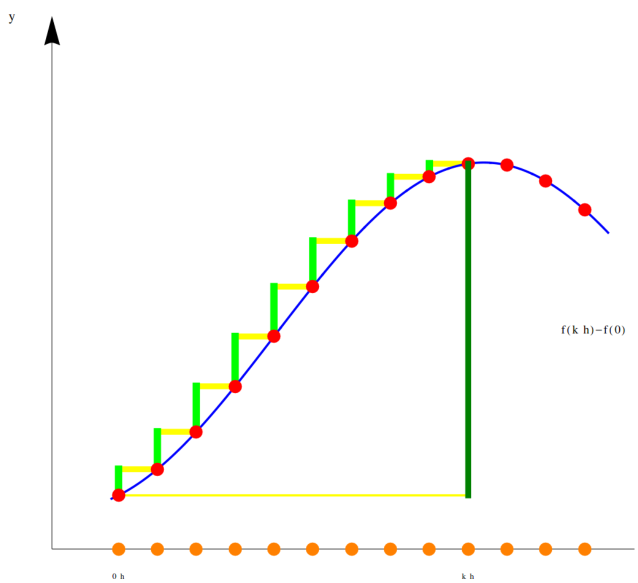
$$\boxed{Sf(x) = h[\ f(0)+f(h)+f(2h)+\cdots+f((k-1)h)\ ]}$$

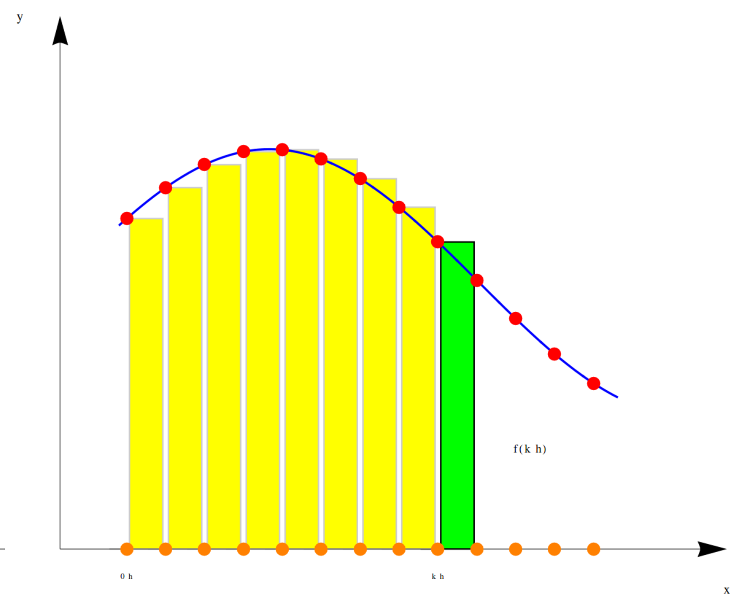
In short hand, we call Sf also the "integral" or "antiderivative" of f. It will become the integral in the limit $h \to 0$ later in the course.

5 Compute $Sf(x)$ for $f(x) = 1$. **Solution**. We have $Sf(x) = 0$ for $x \leq h$, and $Sf(x) = h$ for $h \leq x < 2h$ and $Sf(x) = 2h$ for $2h \leq x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but grows in quantized steps.

The difference	$Df(x)$	will become the	**derivative**	$f'(x)$.
The sum	$Sf(x)$	will become the	**integral**	$\int_0^x f(t)\, dt$.

Df	means rise over run	and is close to	the slope	of the	graph of f.
Sf	means areas of rectangles	and is close to	the area	under the	graph of f.





Theorem: Sum the differences and get
$$SDf(kh) = f(kh) - f(0)$$

Theorem: Difference the sum and get
$$DSf(kh) = f(kh)$$

6 For $f(x) = [x]_h^m = x(x-h)(x-2h)...(x-mh+h)$ we have

$f(x+h) - f(x) = (x(x-h)(x-2h)...(x-kh+2h))\,((x+h)-(x-mh+h)) = [x]^{m-1}hm$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. We have obtained the important formula $\boxed{D[x]^m = m[x]^{m-1}}$
We can establish from this differentiation formulas for **polynomials**. We will leave away the square brackets often and do the calculus we will do later on.

7 If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$.
The fundamental theorem allows us to integrate and get the right values at the points k/n:

8 Find Sf for the same function. The answer is $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero. As a rescaled exponential, it is continuous and monotone. Indeed, using rules of the logarithm we can see $\exp_h(x) = e^{x(\log(1+h)/h)}$. We see that it agrees with the exponential function after a rescaling of x.

9 You have already computed the derivative in a homework. Lets do it again. The function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. **Solution**: $\exp_h(x+h) = (1+h)\exp(x)$ shows that. $\boxed{D\exp_h(x) = \exp_h(x)}$

10 Define $\exp(a \cdot x) = (1 + ah)^{x/h}$. It satisfies $\boxed{D\exp_h(a \cdot x) = a\exp_h(a \cdot x)}$ We write a dot because $\exp_h(ax)$ is not equal to $\exp_h(a \cdot x)$. For now, only the differentiation rule for this function is important.

11 This is example which uses complex numbers $i = \sqrt{-1}$. If we allow a to become complex, we get $\exp(1 + ia)(1 + aih)^{x/h}$. We still have $D\exp_h^{ai}(x) = ai\exp_h^{ai}(x)$. Taking real and imaginary parts define new functions $\exp_h^{ai}(x) = \cos_h(a \cdot x) + i\sin_h(a \cdot x)$. These functions are real and morph into the familiar cos and sin functions for $h \to 0$. For any $h > 0$ and any a, we have now $\boxed{D\cos_h(a \cdot x) = -a\sin_h(a \cdot x) \text{ and } D\sin_h(a \cdot x) = a\cos_h(a \cdot x).}$ No worries, we will derive these identities for the usual trig functions.

Homework

We leave the h away in this homework. To have more fun, also define $\log_h$ as the inverse of $\exp_h$ and define $1/[x]_h = D\log_h(x)$ for $x > 0$. If we start integrating from 1 instead of 0 as usual we write $S_1 f$ and get $S_1 1/[x]_h = \log_h(x)$. We also write simply x^n for $[x]_h^n$ and write $\exp(a \cdot x) = e^{a \cdot x}$ instead of $\exp_h^a(x)$ and $\log(x)$ instead of $\log_h(x)$. Use the differentiation and integration rules on the right to find derivatives and integrals.

1 Find the derivatives $Df(x)$ of the following functions:

a) $f(x) = x^{11} - 3x^4 + 5x + 1$
a) $f(x) = -x^4 + 8\log(x)$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Df(0)$?

2 Find the integrals $Sf(x)$ of the following functions assuming $Sf(0) = 0$:

a) $f(x) = x^9$.
b) $f(x) = x^2 + 6x^7 - x$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Sf(1)$ in the case $h = 1$?

3 Find the derivatives $Df(x)$ of the following functions

a) $f(x) = \exp(9 \cdot x) + x^6$
b) $f(x) = 8\exp(-3 \cdot x) + 9x^6$
c) $f(x) = -\exp(5 \cdot x) + x^6$

4 Find the integrals $Sf(x)$ of the following functions

a) $f(x) = \exp(6 \cdot x) - 3x^6$
b) $f(x) = \exp(8 \cdot x) + x^6$
c) $f(x) = -\exp(5 \cdot x) + x^6$

5 Define $f(x) = \sin(4 \cdot x) - \exp(2 \cdot x) + x^5$.
a) Find $Df(x)$
b) Find $g(x) = Sf(x)$
c) Verify that $Dg(x) = f(x)$.

All calculus on 1/3 page

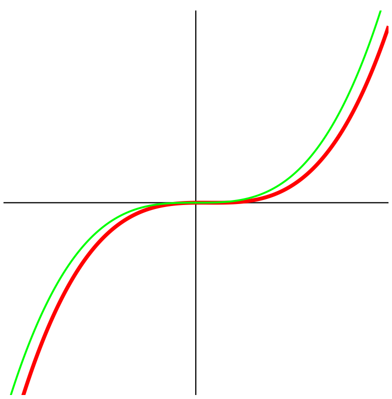
Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

Differentiation rules
$Dx^n = nx^{n-1}$
$De^{a \cdot x} = ae^{a \cdot x}$
$D\cos(a \cdot x) = -a\sin(a \cdot x)$
$D\sin(a \cdot x) = a\cos(a \cdot x)$
$D\log(x) = 1/x$

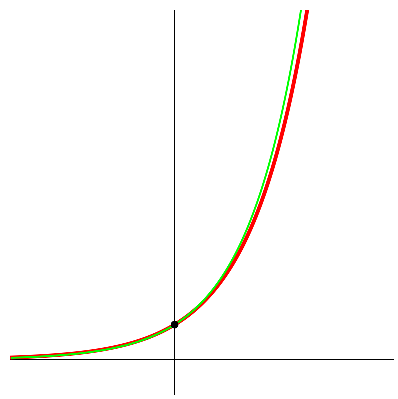
Integration rules (for $x = kh$)
$Sx^n = x^{n+1}/(n+1)$
$Se^{a \cdot x} = e^{a \cdot x}/a$
$S\cos(a \cdot x) = \sin(a \cdot x)/a$
$S\sin(a \cdot x) = -\cos(a \cdot x)/a$
$S\frac{1}{x} = \log(x)$

Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x+h)$.

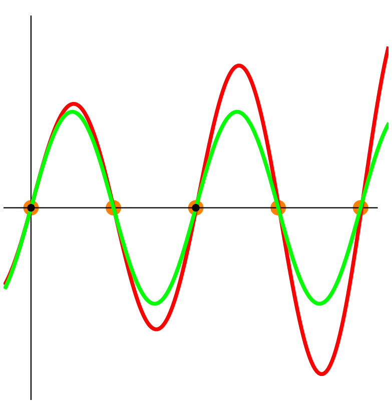
Pictures



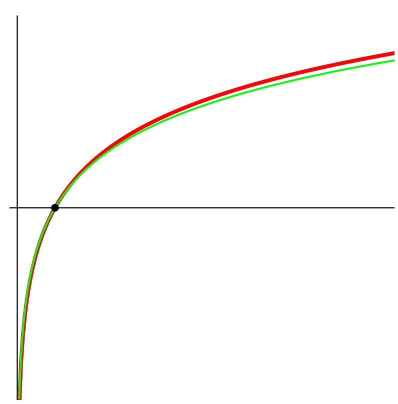
$[x]^3_h$ for $h = 0.1$



$\exp_h(x)$ for $h = 0.1$



$\sin_h(x)$ for $h = 0.1$



$\log_h(x)$ for $h = 0.1$

Lecture 6: Worksheet

The exponential function

We illuminate the fundamental theorem for the exponential function $\exp(x) = (1+h)^{x/h}$. We look again at the case $h = 1$ in which case $\exp(x) = 2^x$ maps positive integers to positive integers.

$$D\exp(x) = \exp(x) \; .$$

From the fundamental theorem, we get $SD\exp(x) = S\exp(x) = \exp(x) - \exp(0)$ we see

$$S\exp(x) = \exp(x) - 1 \; .$$

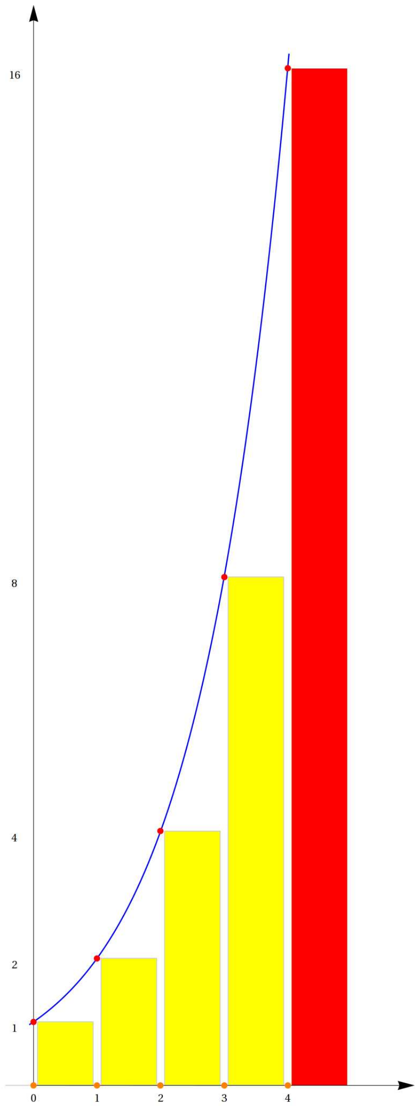
In other words, for the exponential function, we know both the derivative and the integral.

1 The formula $D\exp(x) = \exp(x)$ tells for $x = 3$ that $16 - 8 = 8$.

2 The formula $S\exp(x) = \exp(x) - 1$ tells for $x = 4$ that $1 + 2 + 4 + 8 = 16 - 1$.

3 The identity $DS\exp(x) = \exp(x)$ is illustrated in the left picture for $x = 4$.

4 The identity $SD\exp(x) = \exp(x) - 1$ is illustrated to the right for $x = 4$.


16
8
4
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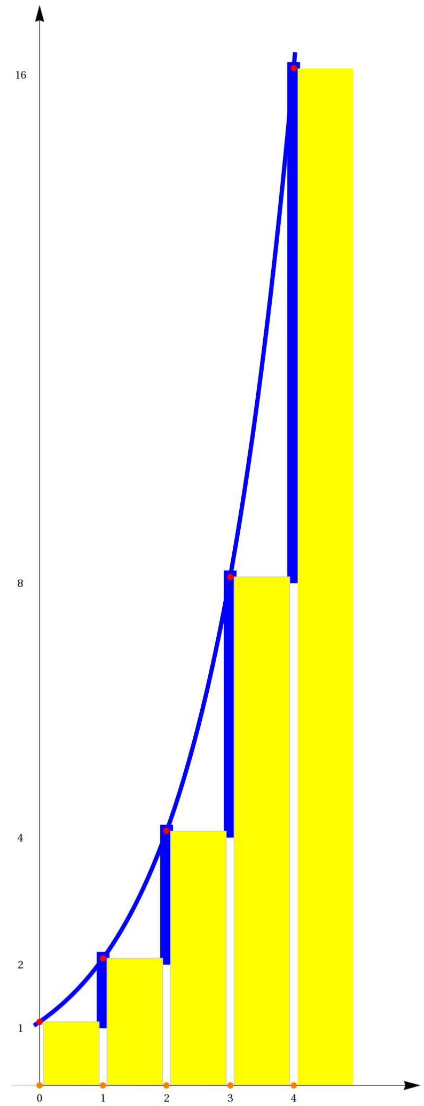
16
8
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4

Here are some worked out examples, similar to what we expect you to do for the homework of lecture 6.

The homework should be straightforward. When finding the value of $Sf(x)$, we want to add a constant such that $Sf(0) = 0$. In general, you will not need to evaluate functions and can leave terms like $\sin(5 \cdot x)$ as they are. If you have seen calculus already, then you could do this exercise by writing

$$\frac{d}{dx} f(x)$$

instead of $Df(x)$ and by writing

$$\int_0^x f(x)\, dx$$

instead of $Sf(x)$. Since we did not introduce the derivative df/dx nor the integral $\int_0^x$ yet, for now, just use the differentiation and integrations rules in the box to the right to solve the problems.

1 **Problem:** Find the derivative $Df(x)$ of the function $f(x) = \sin(5 \cdot x) + x^7 + 3$.
Answer: From the differentiation rules, we know $\boxed{Df(x) = 5\cos(5 \cdot x) + 7x^6}$.

2 **Problem:** Find the derivative $Df(0)$ of the same function $f(x) = \sin(5 \cdot x) + 5x^7 + 3$.
Answer: We know $Df(x) = 5\cos(5 \cdot x) + 35x^6$. Plugging in $x = 0$ gives $\boxed{5}$.

3 **Problem:** Find the integral $Sf(x)$ of the function $f(x) = \sin(5 \cdot x) + 5x^7 + 3$.
Answer: From the integration rules, we know $\boxed{Sf(x) = -\cos(5 \cdot x)/5 + 5x^8/8 + 3x}$.

4 **Problem:** Find the integral $Sf(1)$ of the function $f(x) = \exp(4 \cdot x)$.
Answer: From the integration rules, we know $Sf(x) = \exp(4 \cdot x)/4 - 1/4$. We have added a constant such that $Sf(0) = 0$. Plugging in $x = 1$ gives $\boxed{\exp(4)/4 - 1/4}$.

Lecture 7: Rate of change

Given a function f and $h > 0$, we can look at the new function

$$Df(x) = \frac{f(x+h) - f(x)}{h} .$$

It is the **rate of change** of the function with **step size** h. When changing x to $x + h$ and then $f(x)$ changes to $f(x+h)$. The quotient $Df(x)$ is "rise over run". In this lecture, we take the limit $h \to 0$ and derive the important formulas $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}\exp(x) = \exp(x)$, $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$ which we have seen already before in a discrete setting.

1 You walk up a snow hill of height $f(x) = 30 - x^2$ meters. Assume you walk with a step size of $h = 0.5$ meters. You are at position $x = 3$. How much do you climb or descend when making an other step? We have $f(3) = 21$ and $f(3.5) = 17.75$. We have walked down 3.25 meters. How steep was the snow hill at this point? We have to divide the height difference by the walking distance: $-3.25/0.5 = -7.5$. This is the slope with that step size.

Today, we take the limit $h \to 0$:

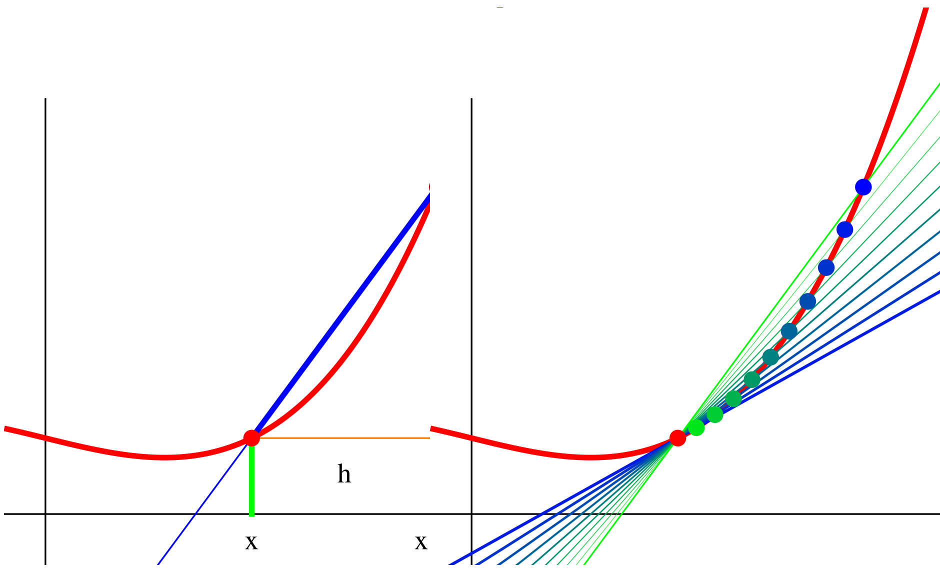
If the limit $\frac{d}{dx}f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist, we say f is **differentiable** at the point x. The value is called the **derivative** or **instantaneous rate of change** of the function f at x. We denote the limit also with $f'(x)$.

2 In the previous problem, $f(x) = 30 - x^2$ we have

$$f(x+h) - f(x) = [30 - (x+h)^2] - [30 - x^2] = -2xh - h^2$$

Dividing this by h gives $-2x - h$. The limit $h \to 0$ gives $-2x$. We have just seen that for $f(x) = x^2$, we get $f'(x) = -2x$. For $x = 3$, this is -6. The actual slope of the snow hill is a bit smaller than the estimate done by walking. The reason is that the hill gets steeper.

The derivative $f'(x)$ has a geometric meaning. It is the slope of the tangent at x. This is an important geometric interpretation. It is useful to think about x as "time" and the derivative as the rate of change of the quantity $f(x)$ in time.

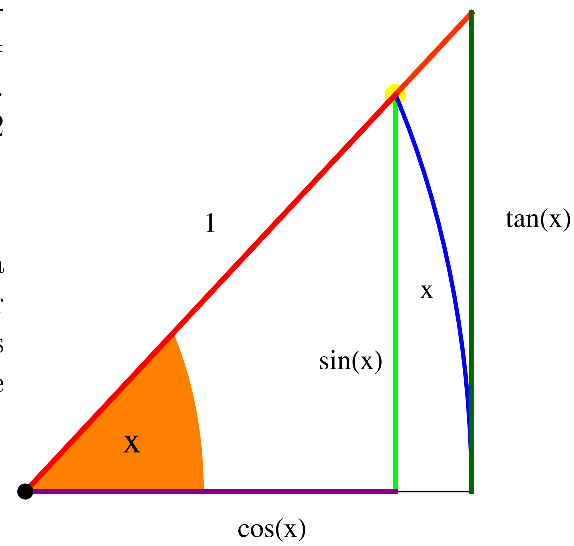


For $f(x) = x^n$, we have $f'(x) = nx^{n-1}$.

Proof: $f(x+h) - f(x) = (x+h)^n = (x^n + nx^{n-1}h + a_2h^2 + ... + h^n) - x^n = nx^{n-1}h + a_2h^2 + ... + h^n)$. If we divide by h, we get $nx^{n-1} + h(a_2 + \ldots + h^{n-2})$ for which the limit $h \to 0$ exists: it is nx^{n-1}. This is an important result because most functions can be approximated very well with polynomials.

For $f(x) = \sin(x)$ we have $f'(0) = 1$ because the differential quotient is $[f(0+h) - f(0)]/h = \sin(h)/h = \mathrm{sinc}(h)$. We have already seen that the limit is 1 before. Lets look at it again geometrically. For all $0 < x < \pi/2$ we have

$$\sin(x) \leq x \leq \tan(x) \ .$$



3 [dividing by 2 squeezes the area of the sector by the area of triangles.] Because $\tan(x)/\sin(x) = 1/\cos(x) \to 1$ for $x \to 0$, the value of $\mathrm{sinc}(x) = \sin(x)/x$ must go to 1 as $x \to 0$. Renaming the variable x with the variable h, we see the **fundamental theorem of trigonometry**

$$\boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1}$$

4 For $f(x) = \cos(x)$ we have $f'(x) = 0$. To see this, look at $f(0+h) - f(0) = \cos(h) - 1$. Geometrically, we can use Pythagoras $\sin^2(h) + (1-\cos(h))^2 \leq h^2$ to see that $2 - 2\cos(h) \leq h^2$ or $(1 - \cos(h) \leq h^2/2$ so that $(1 - \cos(h))/h \leq h/2$. This goes to 0 for $h \to 0$. We have just nailed down an other important identity

$$\boxed{\lim_{h\to 0} \frac{(1-\cos(h)}{h} = 0}.$$

The interpretation is that the tangent is **horizontal** for the cos function at $x = 0$. We will call this a critical point later on.

5 From the previous two examples, we get

$$\cos(x+h) - \cos(x) = \cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x) = \cos(x)(\cos(h) - 1) - \sin(x)\sin(h)$$

because $(\cos(h) - 1)/h \to 0$ and $\sin(h)/h \to 1$, we see that $[\cos(x+h) - \cos(x)]/h \to -\sin(x)$.

For $f(x) = \cos(ax)$ we have $f'(x) = -a\sin(ax)$.

6 Similarly,

$$\sin(x+h) - \sin(x) = \cos(x)\sin(h) + \sin(x)\cos(h) - \sin(x) = \sin(x)(\cos(h) - 1) + \cos(x)\sin(h)$$

because $(\cos(h) - 1)/h \to 0$ and $\sin(h)/h \to 1$, we see that $[\sin(x+h) - \sin(x)]/h \to \cos(x)$.

for $f(x) = \sin(ax)$, we have $f'(x) = a\cos(ax)$.

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

Like π, the Euler number e is irrational. Here are the first digits: 2.7182818284590452354. If you want to find an approximation, just pick a large n, like $n = 100$ and compute $(1 + 1/n)^n$. For $n = 100$ for example, we see $101^{100}/100^{100}$. We only need 101^{100} and then put a comma after the first digit to get an approximation. Interested why the limit exists: verify hat the fractions $A_n = (1 + 1/n)^n$ increase and $B_n = (1 + 1/n)^{(n+1)}$ decrease. Since $B_n/A_n = (1 + 1/n)$ which goes to 1 for $n \to \infty$, the limit exists. The same argument shows that $(1+1/n)^{xn} = \exp_{1/n}(x)$ increases and $\exp_{1/n}(x)(1+1/n)$ decreases. The limiting function $\exp(x) = e^x$ is called the **exponential function**. Remember that if we write $h = 1/n$, then $(1 + 1/n)^{nx} = \exp_h(x)$ considered earlier in the course. We can sandwich the exponential function between $\exp_h(x)$ and $(1 + h)\exp_h(x)$:

$$\exp_h(x) \leq \exp(x) \leq \exp_h(x)(1 + h), \quad x \geq 0 .$$

For $x < 0$, the inequalities are reversed.

7 Lets compute the derivative of $f(x) = e^x$ at $x = 0$. **Answer**. We have for $x \leq 1$

$$1 \leq (e^x - 1)/x \leq 1 + x .$$

Therefore $f'(0) = 1$. The exponential function has a graph which has slope 1 at $x = 0$.

8 Now, we can get the general case. It follows from $e^{x+h} - e^x = e^x(e^h - 1)$ that the derivative of $\exp(x)$ is $\exp(x)$.

For $f(x) = \exp(ax)$, we have $f'(x) = a\exp(ax)$.

It follows from the properties of taking limits that $(f(x) + g(x))' = f'(x) + g'(x)$. We also have $(af(x))' = af'(x)$. From this, we can now compute many derivatives

9 Find the slope of the tangent of $f(x) = \sin(3x) + 5\cos(10x) + e^{5x}$ at the point $x = 0$. **Solution**: $f'(x) = 3\cos(3x) - 50\sin(10x) + 5e^{5x}$. Now evaluate it at $x = 0$ which is $3 + 0 + 5 = 8$.

Finally, lets mention an example of a function which is not everywhere differentiable.

10 The function $f(x) = |x|$ has the properties that $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x =< 1$. The derivative does not exist at $x = 0$ evenso the function is continuous there. You see that the slope of the graph jumps discontinuously at the point $x = 0$.

For a function which is discontinuous at some point, we don't even attempt to differentiate it there. For example, we would not even try to differentiate $\sin(4/x)$ at $x = 0$ nor $f(x) = 1/x^3$ at $x = 0$ nor $\sin(x)/|x|$ at $x = 0$. Remember these bad guys?

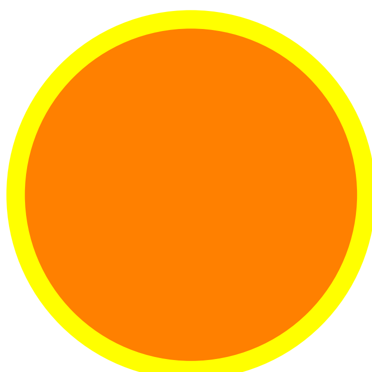
To the end, you might have noticed that in the boxes, more general results have appeared, where x is replaced by ax. We will look at this again but in general, the relation $f'(ax) = af(ax)$ holds ("if you drive twice as fast, you climb twice as fast").

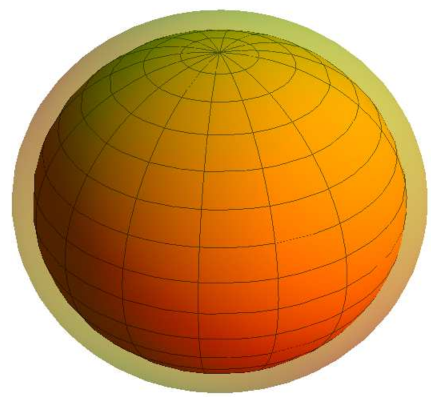
Homework

1 For which of the following functions does the derivative $f'(x)$ exist for every x?

a) $|\sin(x)|$ b) $|\exp(x)|$ c) $x + \exp(7x) + 3\cos(45x)$
d) $\sin^4(x)$ e) $\sin(1/x)$ f) $\exp(-x) + |1 + \cos(15x)|$

2 a) A circle of radius x encloses a disc of area $f(r) = \pi r^2$. Find $\frac{d}{dr}f(r)$. Evaluate the rate of change at $r = 1/10$.
b) The ball of radius r has the volume $f(r) = 4\pi r^3/3$. Find $\frac{d}{dr}f(r))$ and compare it with the surface area of the sphere bounding the ball. Evaluate $\frac{d}{dr}f(r)$ at $r = 1/10$.
c) A **hypersphere** of radius r has the **hyper volume** $f(r) = \pi^2 r^4/2$. Find $\frac{d}{dr}f(r)$ and evaluate it at $r = 1/10$.





3 Find the derivatives of the following functions at the point $x = 0$.
a) $f(x) = 7\exp(4x) + \sin(11x) + x + x^4 + x^6 + x^{10}$.
b) $f(x) = (x^5 - 1)/(x - 1) + \cos(2x)$. First heal this function.
c) $f(x) = \frac{1+4x+6x^2+4x^3+x^4}{x^2+2x+1}$. Also here, first heal!

4 In this problem we compute the derivative of $\sqrt{x}$ for $x > 0$. To do so, we have to find the limit

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} .$$

Hint: multiply the top and the bottom with $(\sqrt{x+h} + \sqrt{x})$ and simplify.

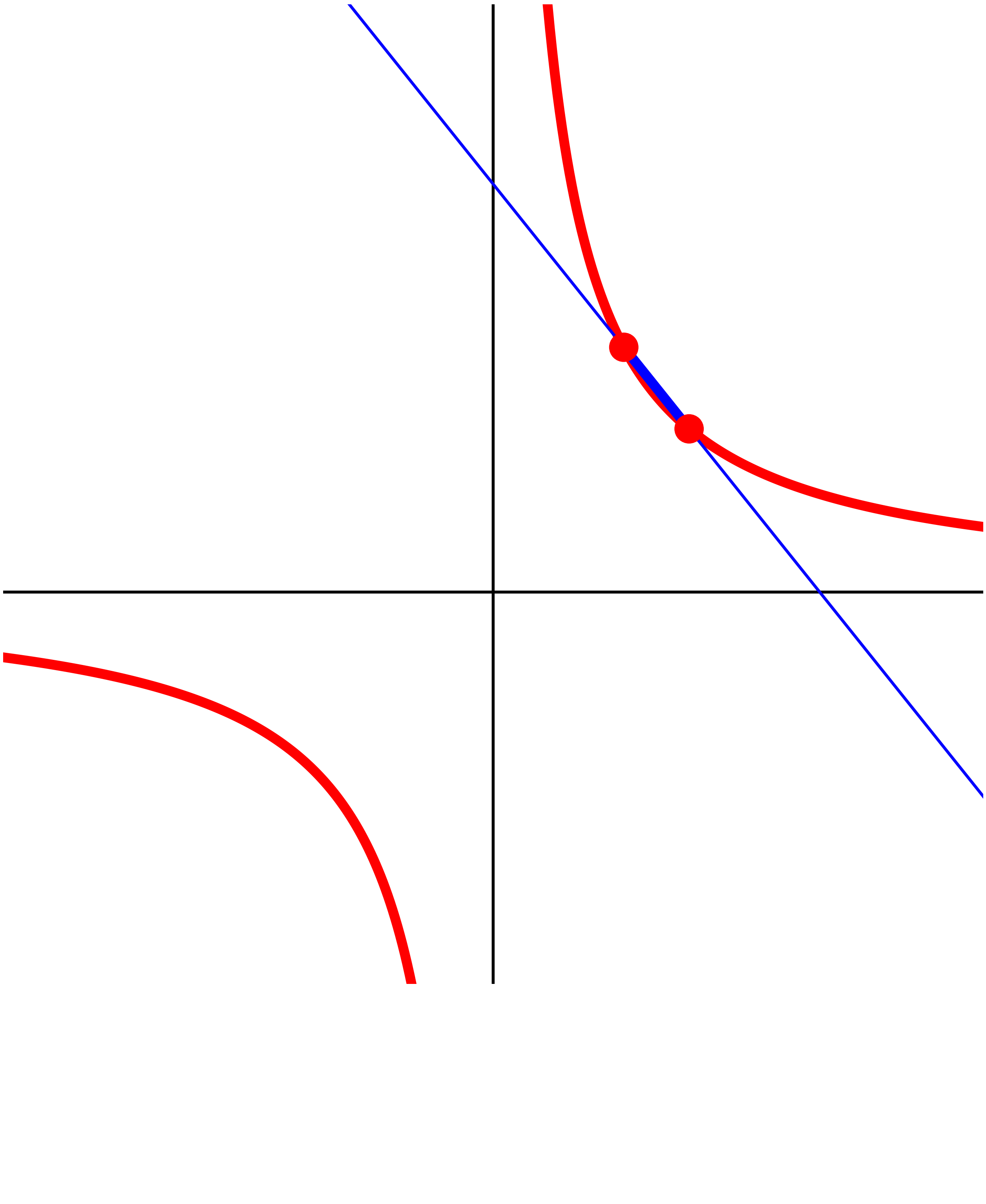
5 On March 21, 2014, asteroid 2003 QQ47 will zoom by the earth about 20 million kilometers from earth. It is a rock of 1.24 km diameter which would make quite a firework if hitting the earth. By Pythagoras, its distance $y(t)$ from the earth is $\sqrt{t^2 + 20^2}$.
Let $f(t) = y(t)^2 = t^2 + 400$. What is the rate of change of $f(t)$ at $t = 0$? Explain. Does this mean the asteroid stops, when closest to the earth? See http://neo.jpl.nasa.gov/news/news138.html

Lecture 7: Worksheet

Rate of change

We compute the derivative of $f(x) = 1/x$ by taking limits.

a) Simplify $\frac{1}{x+h} - \frac{1}{x}$.
b) Now take the limit $\frac{1}{h}[\frac{1}{x+h} - \frac{1}{x}]$ when $h \to 0$.
c) Is there any point where $f'(x) > 0$?



Derivatives

Differentiation rules

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$e^{ax} = ae^{ax}$$
$$\frac{d}{dx}\cos(ax) = -a\sin(ax)$$
$$\frac{d}{dx}\sin(ax) = a\cos(ax)$$

1 Find the derivatives of the function $f(x) = \sin(3x) + x^5$

2 Find the derivative of $f(x) = \cos(7x) - 8x^4$.

3 Find the derivative of $f(x) = e^{5x} + \cos(2x)$.

Two trigonometric identities

During lecture, we need the identities

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

and

$$\sin(x+y) = \cos(x)\sin(y) + \sin(x)\sin(y) \ .$$

You might know these identities from pre-calculus.

We do not work with complex numbers in this course but the verification of these identities is so elegant with the **Euler** formula

$$e^{ix} = \cos(x) + i\sin(x)$$

that it is a crime to prove the identities differently. Just compare the real and imaginary components of

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

with the real and imaginary parts of

$$\begin{aligned} e^{ix}e^{iy} &= (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) \\ &= \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)) \ . \end{aligned}$$

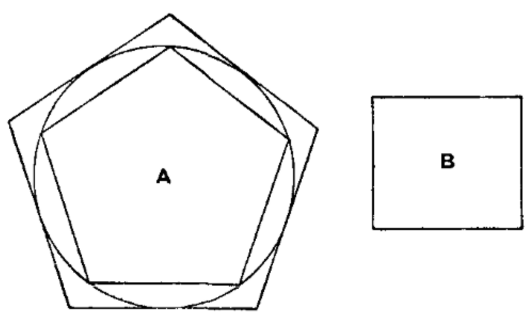
A page from Archimedes

Proposition 6.

"Similarly we can show that, *given two unequal magnitudes and a sector, it is possible to circumscribe a polygon about the sector and inscribe in it another similar one so that the circumscribed may have to the inscribed a ratio less than the greater magnitude has to the less.*

And it is likewise clear that, *if a circle or a sector, as well as a certain area, be given, it is possible, by inscribing regular polygons in the circle or sector, and by continually inscribing such in the remaining segments, to leave segments of the circle or sector which are* [*together*] *less than the given area.* For this is proved in the *Elements* [Eucl. XII. 2].

But it is yet to be proved that, *given a circle or sector and an area, it is possible to describe a polygon about the circle or sector, such that the area remaining between the circumference and the circumscribed figure is less than the given area.*"



The proof for the circle (which, as Archimedes says, can be equally applied to a sector) is as follows.

Let A be the given circle and B the given area.

Now, there being two unequal magnitudes $A+B$ and A, let a polygon (C) be circumscribed about the circle and a polygon (I) inscribed in it [as in Prop. 5], so that

$$C : I < A + B : A \text{ ....(1).}$$

The circumscribed polygon (C) shall be that required.

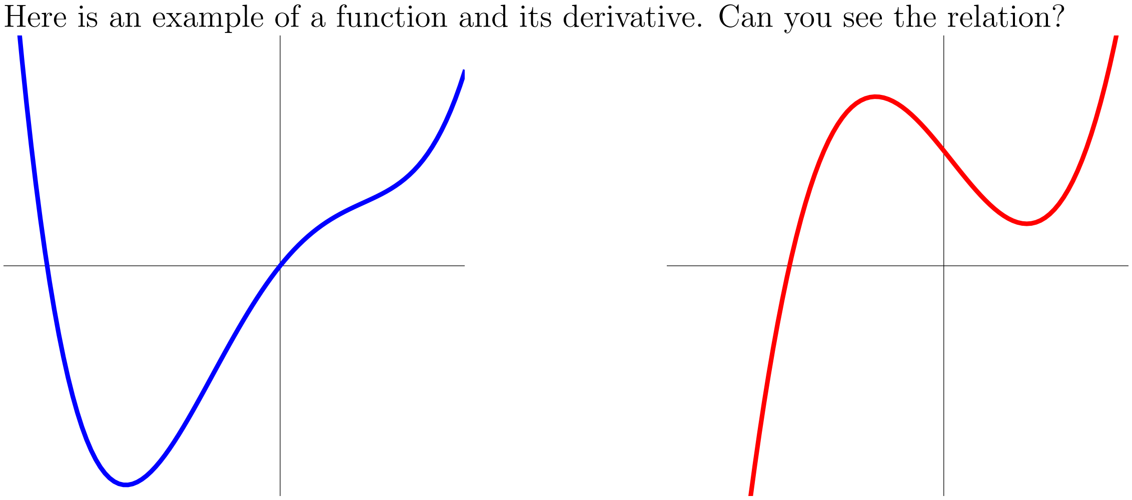
Lecture 8: The derivative function

We have defined the derivative $f'(x) = \frac{d}{dx} f(x)$ as a limit of $(f(x+h) - f(x))/h$ for $h \to 0$. We have seen that $\frac{d}{dx} x^n = n x^{n-1}$ holds for integer n. We also know already that $\sin'(ax) = a\cos(x), \cos'(ax) = -a\sin(x)$ and $\exp'(ax) = a\exp(x)$. We can already differentiate a lot of functions and evaluate the derivative $f'(x)$ at some point x. This is the slope of the curve at x.

1 Find the derivative $f'(x)$ of $f(x) = \sin(4x) + \cos(5x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$. **Solution**: $f'(x) = 4\cos(4x) - 5\sin(5x) - 1/(2\sqrt{x} - 1/x^2 + 4x^3$. Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

The function which takes the derivative at a given point is called the **derivative function**. For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$. In this lecture, we want to understand the new function and its relation with f. What does it mean if $f'(x) > 0$. What does it mean that $f'(x) < 0$. Do the roots of f tell something about f' or do the roots of f' tell something about f?

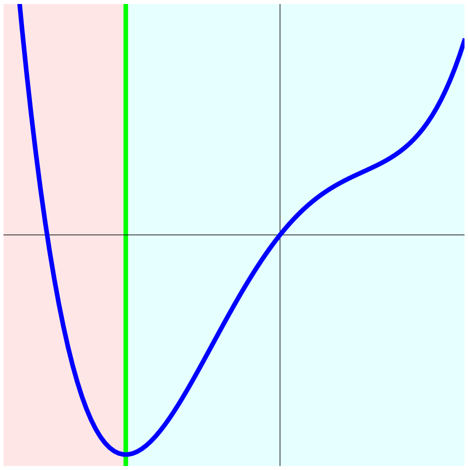
Here is an example of a function and its derivative. Can you see the relation?

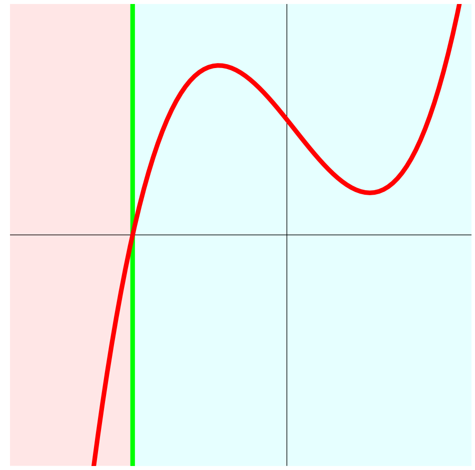


To understand the relation, it is good to distinguish intervals, where $f(x)$ is increasing or decreasing. This are the intervals where $f'(x)$ is positive or negative.

A function is called **monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Monotonoically increasing functions "go up" when you "increase x". Lets color that:





2 Can you find a function f on the interval $[0,1]$ which is bounded $|f(x)| \leq 1$ but such that $f'(x)$ is unbounded? Hint: square-"..."-beer.

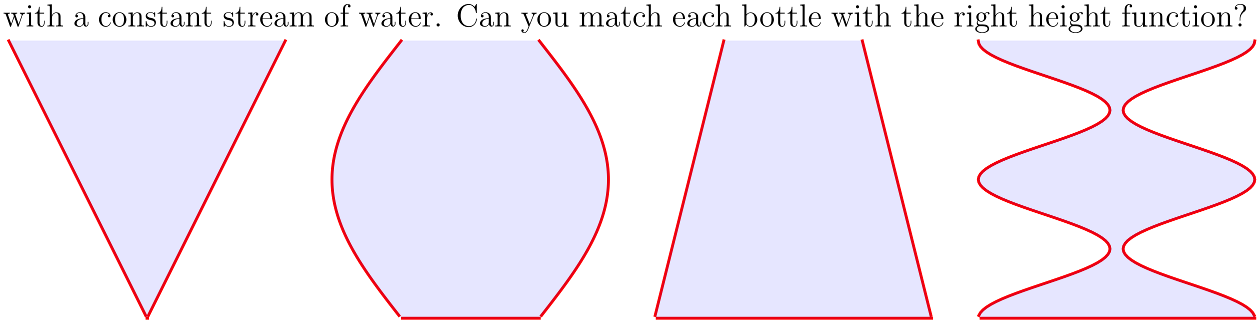
Given the function $f(x)$, we can define $g(x) = f'(x)$ and then take the derivative g' of g. This second derivative $f''(x)$ is called the **acceleration**. It measures the rate of change of the tangent slope. For $f(x) = x^4$, for example we have $f''(x) = 12x^2$. If $f''(x) > 0$ on some interval the function is called **concave up**, if $f''(x) < 0$, it is **concave down**.

3 Find a function f which has the property that its acceleration is constant equal to 6. **Solution.** We have to get a function such that its derivative is $6x$. That works for $3x^2$.

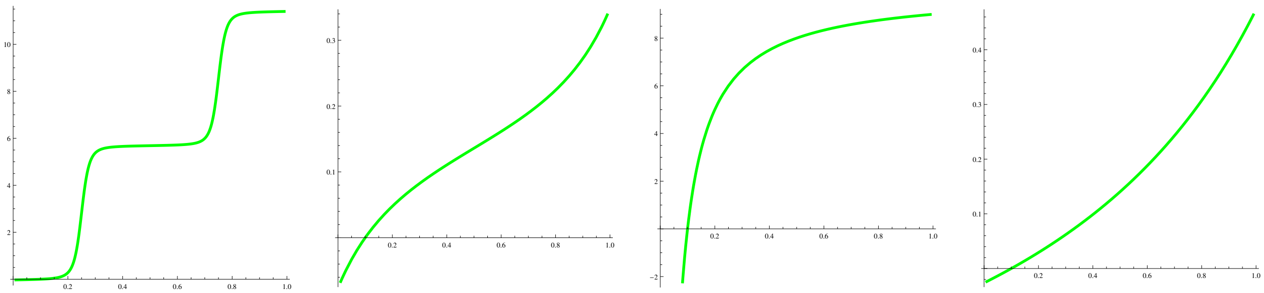
4 Find a function f which has the property that its acceleration is equal to its derivative. To do so, try basic functions you know and compute $f'(x), f''(x)$ in each case.

After matching some functions, we look at related inverse problem called **bottle calibration problem**. We fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time t. Assume the radius of the bottle is $f(z)$ at height z. Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that f is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of g and f. Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions g qualitatively.

5 In each of the bottles, we call g the height of the water level at time t, when filling the bottle with a constant stream of water. Can you match each bottle with the right height function?



a) b) d) c)



1) 2) 3) 4)

The key is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h)-g(t)]$ times the area πf^2 is a constant times the time difference $h = dt$, we have **calibration formula**

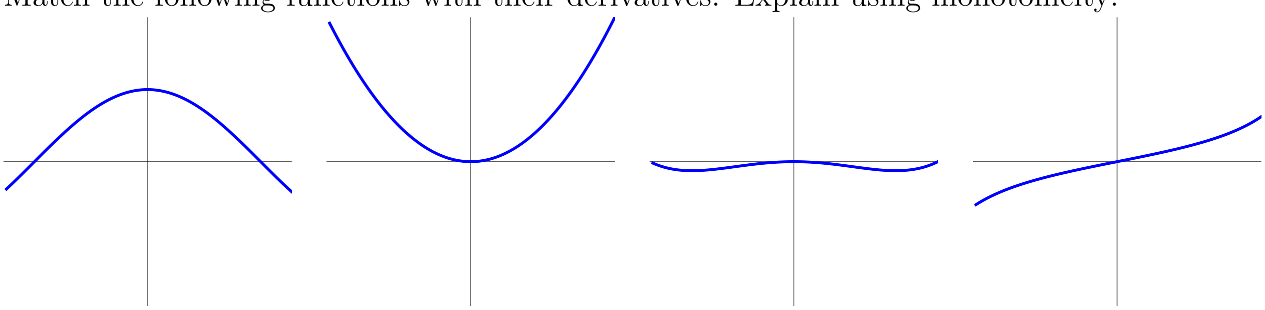
$$g' = \frac{1}{\pi f^2} .$$

It relates the derivative function of g with the thickness $f(t)$ of the bottle at height g. No need to learn this. It just explains the story completely. It tells that that if the bottle radius f is large, then the water level increase g' is small and if the bottle radius f is small, then the liquid level change g' is large.

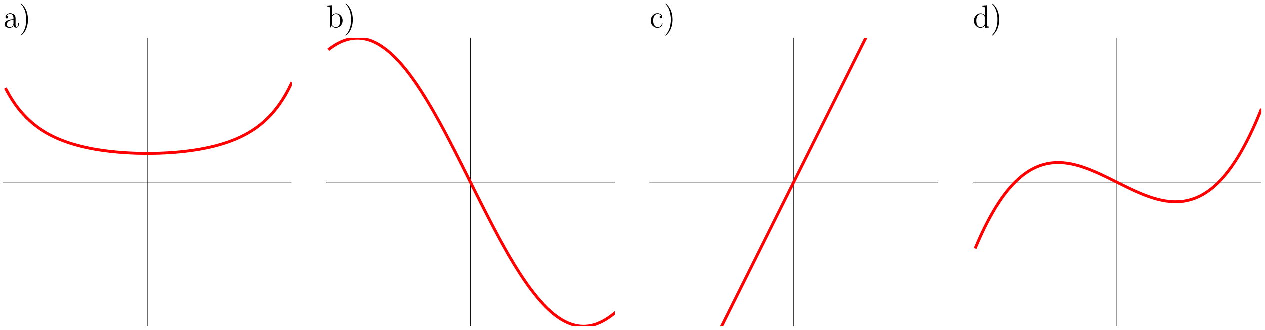
Homework

1 For the following functions, determine on which intervals the function is monotonically increasing or monotonically decreasing.
a) $f(x) = -x^3 + x$ on $[-2, 2]$
b) $f(x) = \sin(x)$ on $[-2\pi, 2\pi]$
c) $f(x) = -x^4 + 8x^2$ on $[-4, 4]$.

2 Match the following functions with their derivatives. Explain using monotonicity:

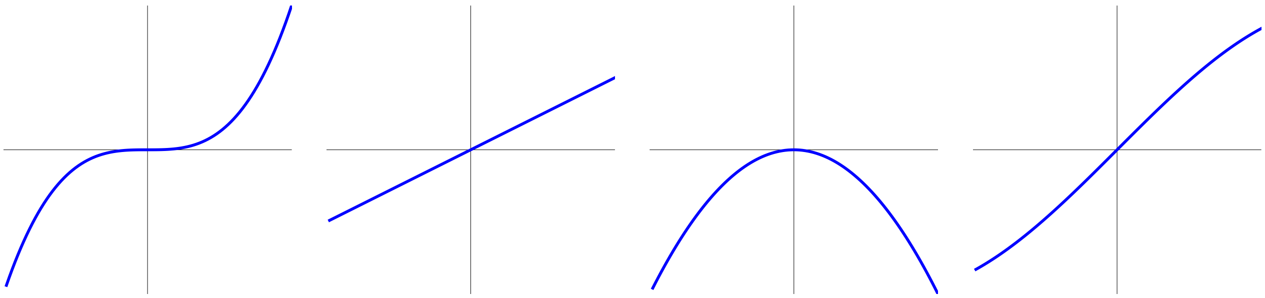


a) b) c) d)

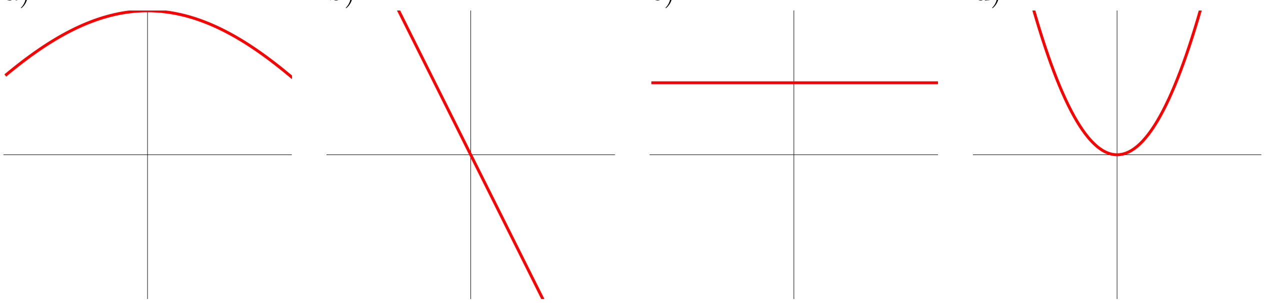


1) 2) 3) 4)

3 Match also the following functions with their derivatives. Give short explanations documenting your reasoning in each case.



a) b) c) d)



1) 2) 3) 4)

4 Draw for the following functions the graph of the function $f(x)$ as well as the graph of its derivative $f'(x)$. You do not have to compute the derivative analytically as a formula here since we do not have all tools yet to compute the derivatives. The derivative function you draw needs to have the right qualitative shape however.

a) The **"To whom the bell tolls"** function

$$f(x) = e^{-x^2}$$

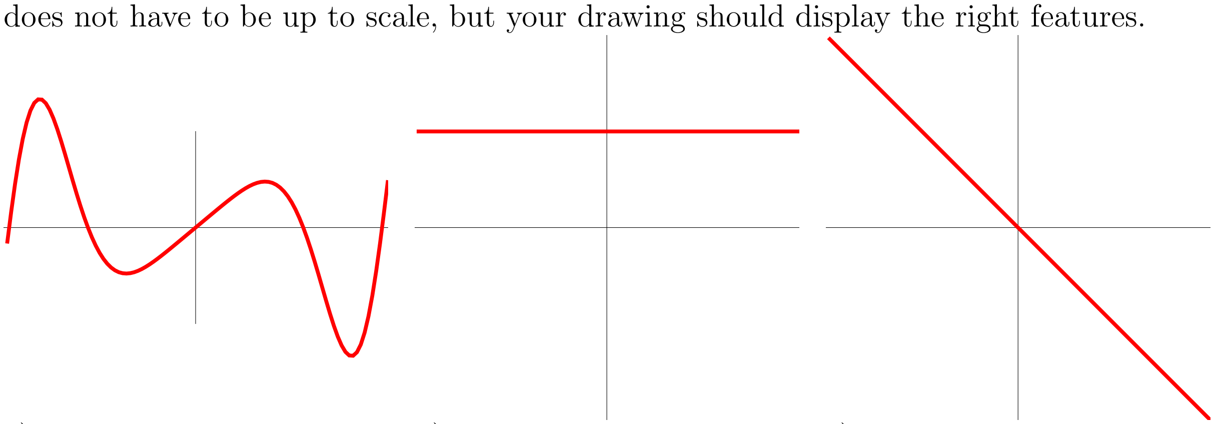
b) The **"witch of Maria Agnesi"** function:

$$f(x) = \frac{1}{1+x^2}$$

c) The **three gorges function**

$$f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} \; .$$

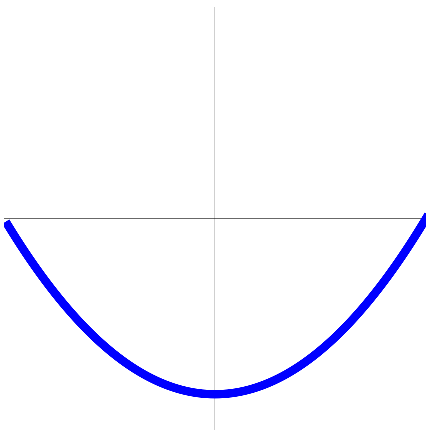
5 Below you the graphs of three derivative functions $f'(x)$. In each case you are told that $f(0) = 1$. Your task is to draw the function $f(x)$ in each of the cases a),b),c). Your picture does not have to be up to scale, but your drawing should display the right features.



a) b) c)

Lecture 8: Worksheet

Matching functions with their derivative



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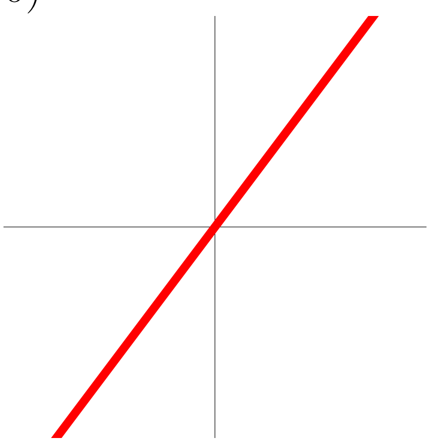
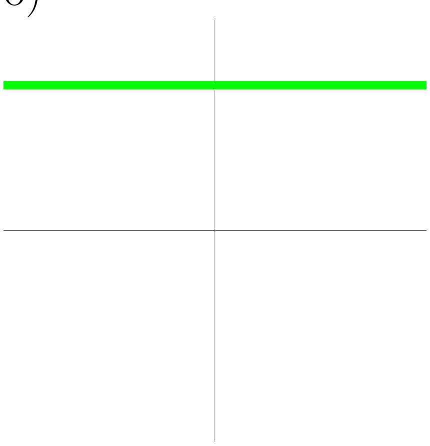


o)

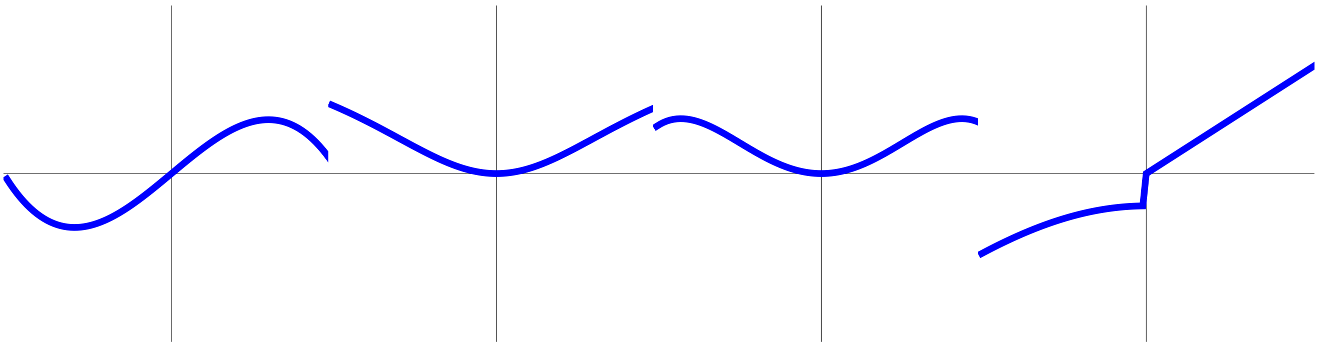


O)

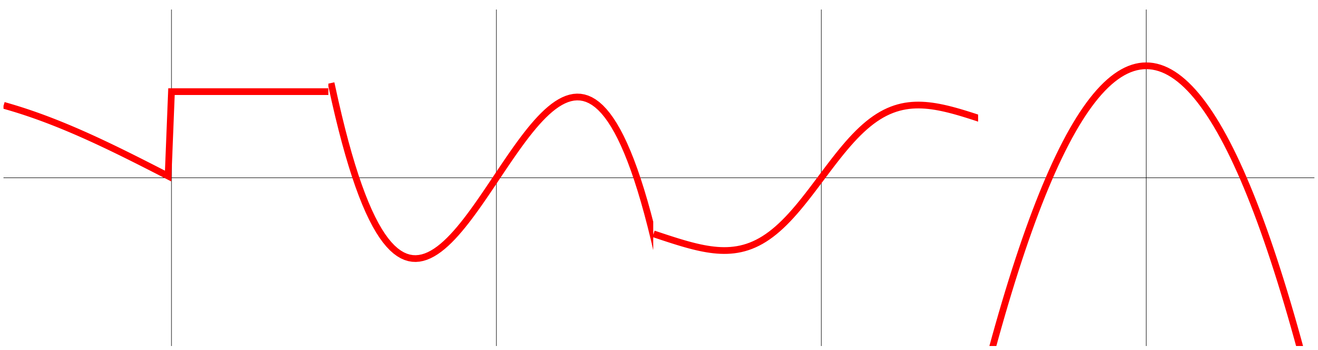
In this worksheet we want to match the graphs of functions with their derivatives and second derivatives. This is tougher than you might think. Here is an example:

The first graph shows the function, which is here the quadratic function. The slope on the right hand side is positive and increasing, on the left hand side the function is negative and decreasing. The middle graph shows the derivative function which is linear. The final graph shows the derivative function of the derivative function. It is constant in this case.

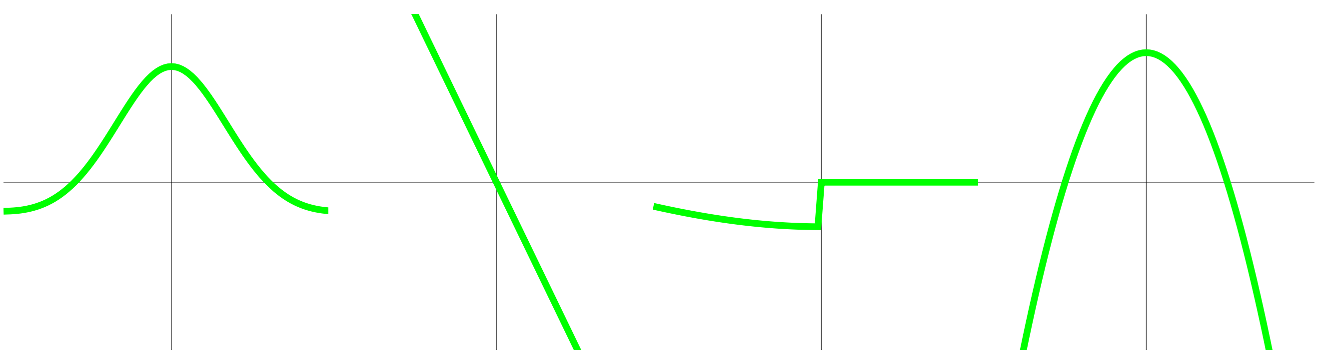
1 Match the following functions with their derivatives and then with the derivatives of the derivatives.



a) b) c) d)



1) 2) 3) 4)

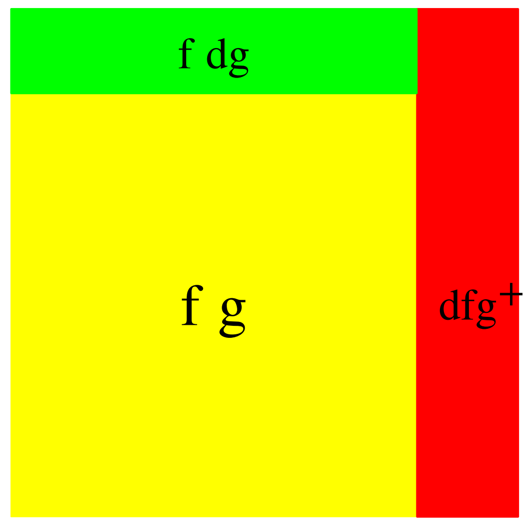


A) B) C) D)

Lecture 9: The product rule

In this lecture, we look at the derivative of a product of functions. The product rule is also called **Leibniz rule** named after **Gottfried Leibniz**, who found it in 1684. It is an important rule because it allows us to differentiate many more functions. We will be able to compute so the derivative of $f(x) = x\sin(x)$ for example without having to take the limit the limit $\lim(f(x+h) - f(x))/h$. We are too lazy for that. Lets start with the identity

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)] \ .$$



It can be written as $D(fg) = Dfg^+ + fDg$ with $g^+(x) = g(x+h)$. This **quantum Leibniz rule** can also be seen geometrically: the rectangle of area $(f+df)(g+dg)$ is the union of rectangles with area $f \cdot g$, $f \cdot dg$ and $df \cdot g^+$. Divide this relation by h to see

$$\begin{aligned} \frac{[f(x+h) - f(x)]}{h} \cdot g(x+h) &\to f'(x) \cdot g(x) \\ f(x) \cdot \frac{[g(x+h) - g(x)]}{h} &\to f(x) \cdot g'(x) \ . \end{aligned}$$

We get the extraordinarily important **product rule**:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \ .$$

Remark: the discrete Leibniz rule is therefore true in the **Babylonian calculus** developed in the first hour.

1 Find the derivative function $f'(x)$ for $f(x) = x^3\sin(x)$. **Solution:** We know how to differentiate x^3 and $\sin(x)$ so that $f'(x) = 3x^2\sin(x) + x^3\cos(x)$.

2 While we know

$$\frac{d}{dx}x^5 = 5x^4 \ ,$$

lets compute this with the Leibniz rule and write $x^5 = x^3 \cdot x^2$. We have

$$\frac{d}{dx}x^3 = 3x^2, \frac{d}{dx}x^2 = 2x \ .$$

The Leibniz rule gives us $d/dx^5 = 3x^4 + 2x^4 = 5x^4$.

3 Lets look at a few derivatives related to functions where we know the answer already but where we can check things using the product formula:

- $\frac{d}{dx}(x^3 \cdot x^5)$
- $\frac{d}{dx}e^{3x}e^{5x}$
- $\frac{d}{dx}\sqrt{x}/\sqrt{x}$
- $\frac{d}{dx}\sin(x)\cos(x)$

Before we look at the quotient rule which allows to differentiate $f(x)/g(x)$ we can also write the later as $f(x) \cdot 1/g(x)$ and use a rule telling us how to differentiate $1/g(x)$. This is the **reciprocal rule**:

If $g(x) \neq 0$, then
$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2} .$$

In order to see this $h = 1/g$ and differentiate the equation $1 = g(x)h(x)$ on both sides. The product rule gives $0 = g'(x)h(x) + g(x)h'(x)$ so that $h'(x) = -h(x)g'(x)/g(x) = -g'(x)/g^2(x)$.

4 Find the derivative of $f(x) = 1/x^4$. **Solution**: $f'(x) = -4x^3/x^8 = -4/x^5$. The same computation shows that $\frac{d}{dx}x^n = nx^{n-1}$ holds for all integers n.

The formula $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$ holds for all integers n.

The **quotient rule** is obtained by applying the product rule to $f(x) \cdot (1/g(x))$ and using the reciprocal rule:

If $g(x) \neq 0$, then
$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[f'(x)g(x) - f(x)g'(x)]}{g^2(x)} .$$

5 Find the derivative of $f(x) = \tan(x)$. **Solution**: because $\tan(x) = \sin(x)/\cos(x)$ we have
$$\tan'(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} .$$

6 Find the derivative of $f(x) = \frac{2-x}{x^2+x^4+1}$. **Solution**. We apply the quotient rule and get $[(-1)x^2 + x^4 + 1 + (2-x)(2x + 4x^3)]/(x^2 + x^4 + 1)$.

Here are some more problems with solutions:

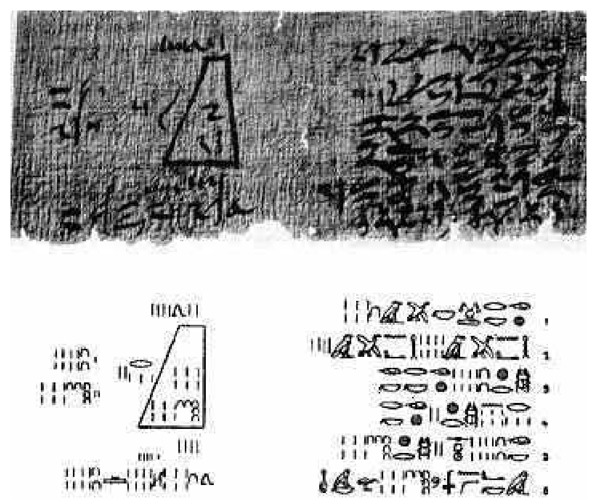
7 Find the second derivative of $\tan(x)$. **Solution**. We have already computed $\tan'(x) = 1/\cos^2(x)$. Differentiate this again with the quotient rule gives

$$\frac{-\frac{d}{dx}\cos^2(x)}{\cos^4(x)} \; .$$

We still have to find the derivative of $\cos^2(x)$. The product rule gives $\cos(x)\sin(x) + \sin(x)\cos(x) = 2\cos(x)\sin(x)$. Our final result is

$$2\sin(x)/\cos^3(x) \; .$$

8 A cylinder has volume $V = \pi r^2 h$, where r is the radius and h is the height. Assume the radius grows like $r(t) = 1 + t$ and the height shrinks like $1 - \sin(t)$. Does the volume grow or decrease at $t = 0$?
Solution: The volume $V(t) = \pi(1+t)^2(1-\sin(t))$ is a product of two functions $f(t) = \pi(1+t)^2$ and $g(t) = (1-\sin(t)$. We have $f(0) = 1, g'(0) = 2, f'(0) = 2, g(0) = 1$. The product rule gives gives $V'(0) = \pi 1 \cdot (-1) + \pi 2 \cdot 1 = \pi$. The volume increases in volume at first.



On the **Moscow papyrus** dating back to 1850 BC, the general formula $V = h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h appeared. Assume $h(t) = 1 + \sin(t), a(t) = 1 + t, b(t) = 1 - 2t$. Does the volume of the truncated pyramid grow or decrease at first? **Solution**. We could fill in
9 $a(t), b(t), h(t)$ into the formula for V and compute the derivative using the product rule. A bit faster is to write $f(t) = a^2 + ab + b^2 = (1+t)^2 + (1-3t)^2 + (1+t)(1-3t)$ and note $f(0) = 3, f'(0) = -6$ then get from $h(t) = (1+\sin(t))$ the data $h(0) = 1, h'(0) = 1$. So that $V'(0) = (h'(0)f(0) - h(0)f'(0))/3 = (1 \cdot 3 - 1(-6))/3 = -1$. The pyramid shrinks in volume at first.

10 We pump up a balloon and let it fly. Assume that the thrust increases like t and the resistance decreases like $1/\sqrt{1-t}$ since the balloon gets smaller. The distance traveled is $f(t) = t/\sqrt{1-t}$. Find the velocity $f'(t)$ at time $t = 0$.

Homework

1 Find the derivatives of the following functions:
a) $f(x) = \sin(11x)\cos(22x)$.
b) $f(x) = \cos^2(x)/x^3$.
c) $f(x) = x^4 \sin(x)\cos(x)$.
d) $f(x) = 3/\sqrt{x}$.
e) $f(x) = 6\cot(x) + 8\tan(x)$.

2 a) Verify that for $f(x) = g(x)h(x)k(x)l(x)$ the formula $f' = g'hkl + gh'kl + ghk'l + ghkl'$ holds.
b) Verify the following formula for derivative of $f(x) = g(x)^4$ we have $f'(x) = 4g^3(x)g'(x)$. We will derive this later using the chain rule. Don't use the chain rule yet, even if you know it.

3 If $f(x) = \text{sinc}(x) = \sin(x)/x$, find its derivative $g(x) = f'(x)$ and then the derivative of $g(x)$. Then evaluate this at $x = 0$.

4 Find the derivative of

$$\frac{\sin(x)}{1 + \cos(x) + \frac{x^4}{\sin(x)}}$$

at $x = 0$.

5 a) We have already computed the derivative of $g(x) = \sqrt{x}$ in the last homework. Introduce $f(x) = x^{1/4}$ and apply the product rule to $g(x) = f(x)f(x)$ to get the derivative of f.
b) Use problem 2b) applied to the identity $x = f(x)^4$ to get the derivative of f.

Remark: Also this last problem 5) is a preparation for the chain rule, we see next Monday. Avoid using the chain rule already here.

Lecture 9: Worksheet

The product rule

We practice the product, reciprocal and quotient rule

1 What is the slope of the graph of the function $f(x) = xe^{-x}$ at $x = 0$?

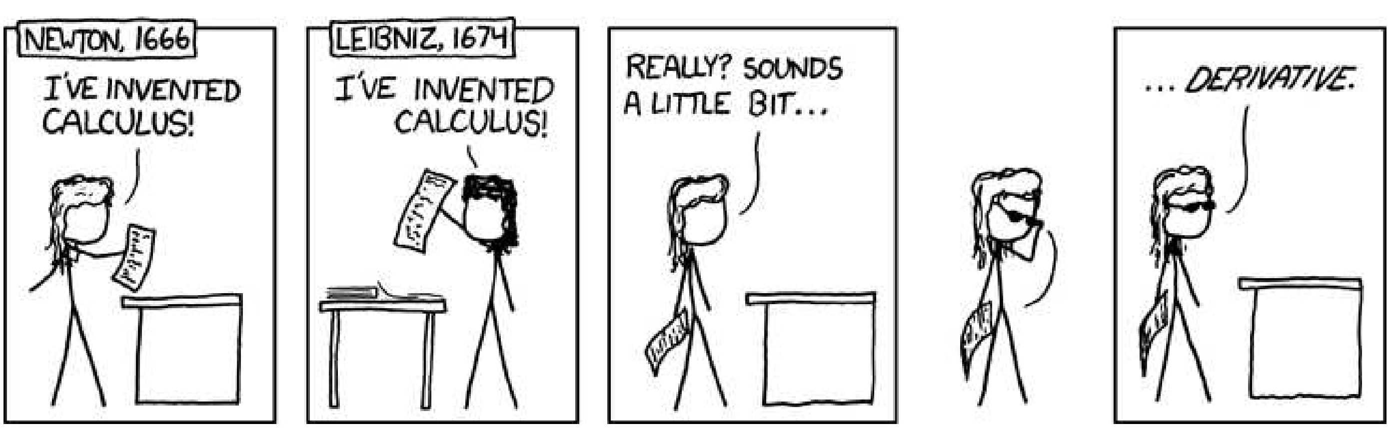
2 Find the derivative of the sinc-function $\sin(x)/x$ at the point $x = 0$.

3 Find the derivative of $\sqrt{x}/x$ at $x = 1$.

4 Find the derivative of $1/e^x$ at $x = 1$.

5 Assume we remember the formula $\sin(2x) = 2\sin(x)\cos(x)$. Differentiate both sides to get a formula for $\cos(2x)$.

6 Find the derivative of $x - 1/(x^2 + 1)$ at $x = 0$.



Source: XKCD

I.

NOVA METHODUS PRO MAXIMIS ET MINIMIS, ITEMQUE TANGENTIBUS, QUAE NEC FRACTAS NEC IRRATIONALES QUANTITATES MORATUR, ET SINGULARE PRO ILLIS CALCULI GENUS*).

Sit (fig. 111) axis AX, et curvae plures, ut VV, WW, YY, ZZ, quarum ordinatae ad axem normales, VX, WX, YX, ZX, quae vocentur respective v, w, y, x, et ipsa AX, abscissa ab axe, vocetur x. Tangentes sint VB, WC, YD, ZE, axi occurrentes respective in punctis B, C, D, E. Jam recta aliqua pro arbitrio assumta vocetur dx, et recta, quae sit ad dx, ut v (vel w, vel y, vel z) est ad XB (vel XC, vel XD, vel XE) vocetur dv (vel dw, vel dy, vel dz) sive differentia ipsarum v (vel ipsarum w, vel y, vel z). His positis, calculi regulae erunt tales.

Sit a quantitas data constans, erit da aequalis 0, et $d\overline{ax}$ erit aequalis adx. Si sit y aequ. v (seu ordinata quaevis curvae YY aequalis cuivis ordinatae respondenti curvae VV) erit dy aequ. dv. Jam *Additio et Subtractio*: si sit $z - y + w + x$ aequ. v, erit $d\overline{z - y + w + x}$ seu dv aequ. $dz - dy + dw + dx$. *Multiplicatio*: $d\overline{xv}$ aequ. $xdv + vdx$, seu posito y aequ. xv, fiet dy aequ. $xdv + vdx$. In arbitrio enim est vel formulam, ut xv, vel compendio pro ea literam, ut y, adhibere. Notandum, et x et dx eodem modo in hoc calculo tractari, ut y et dy, vel aliam literam indeterminatam cum sua differentiali. Notandum etiam, non dari semper regressum a differentiali Aequatione, nisi cum quadam cautione, de quo alibi. Porro *Divisio*: $d\frac{v}{y}$ vel (posito z aequ. $\frac{v}{y}$) dz aequ. $\frac{\pm vdy \mp ydv}{yy}$.

Quoad *Signa* hoc probe notandum, cum in calculo pro litera substituitur simpliciter ejus differentialis, servari quidem eadem signa, et pro $+ z$ scribi $+ dz$, pro $- z$ scribi $- dz$, ut ex addi-

*) Act. Erud. Lips. an. 1684.

Leibniz 1684 paper. The product and quotient rule is introduced.

Lecture 10: The chain rule

How do we take the derivative of a composition of functions like $f(x) = \sin(x^7)$? The product rule does not work here. The functions are "chained", we evaluate first x^7 then apply sin to it. In order to differentiate, we the derivative of the first function we evaluate x^7 then multiply this with the derivative of the function sin at x^7. The answer is $7x^6 \cos(x^7)$.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \; .$$

The chain rule follows from the identity

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h} \; .$$

Write $H(x) =$ g(x+h)-g(x) in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h} \; .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(g(x))$ and the second factor has $g'(x)$ as a limit.

1 Find the derivative of $f(x) = (4x-1)^{17}$. **Solution** The inner function is $g(x) = 4x - 1$. It has the derivative 4. We get therefore $f'(x) = 17(4x-1)^6 \cdot 4 = 28(4x-1)^6$. Remark. We could have expanded out the power $(4x-1)^{17}$ first and avoided the chain rule. Avoiding the **chain rule** is called the **pain rule**.

2 Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution**: applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

3 For linear functions $f(x) = ax + b, g(x) = cx + d$, the chain rule can readily be checked: we have $f(g(x)) = a(cx + d) + b = acx + ad + b$ which has the derivative ac. This agrees with the definition of f times the derivative of g. You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

4 Find the derivative of the natural logarithm function $\log(x)$ [1] **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x)) \log'(x) = x \log'(x) = 1$. Therefore $\log'(x) = 1/x$.

[1] We always write $\log(x)$ for the natural log. The ln notation is old fashioned and only used in obscure places like calculus books.

$\frac{d}{dx} \log(x) = 1/x$.

Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0, \pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$.

5 Find the derivative of $\arcsin(x)$. **Solution**. We write $x = \sin(\arcsin(x))$ and differentiate.

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} .$$

6 Find the derivative of $\arccos(x)$. **Solution**. We write $x = \cos(\arccos(x))$ and differentiate.

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}} .$$

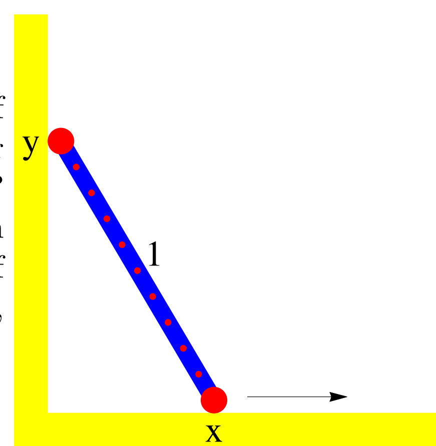
7 $f(x) = \sin(x^2 + 3)$. Then $f'(x) = \cos(x^2 + 3)2x$.

8 $f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x))) \cos(\sin(x)) \cos(x)$.

Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

9 Lets compute the derivative of $\sin(\sqrt{x^5 - 1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5 - 1}$ gives $\cos(\sqrt{x^5 - 1}) \frac{d}{dx} \sqrt{x^5 - 1}$. Apply now the chain rule again for the derivative on the right hand side.

10 Here is a famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0, y)$ on the wall with $(x, 0)$ on the floor. We want to express y as a function of x. We have $y = f(x) = \sqrt{1 - x^2}$. Taking the derivative, assuming $x' = 1$ gives $f'(x) = -2x/\sqrt{1 - x^2}$.



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

11 For the brave: find the derivative of $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))))$.

Practice

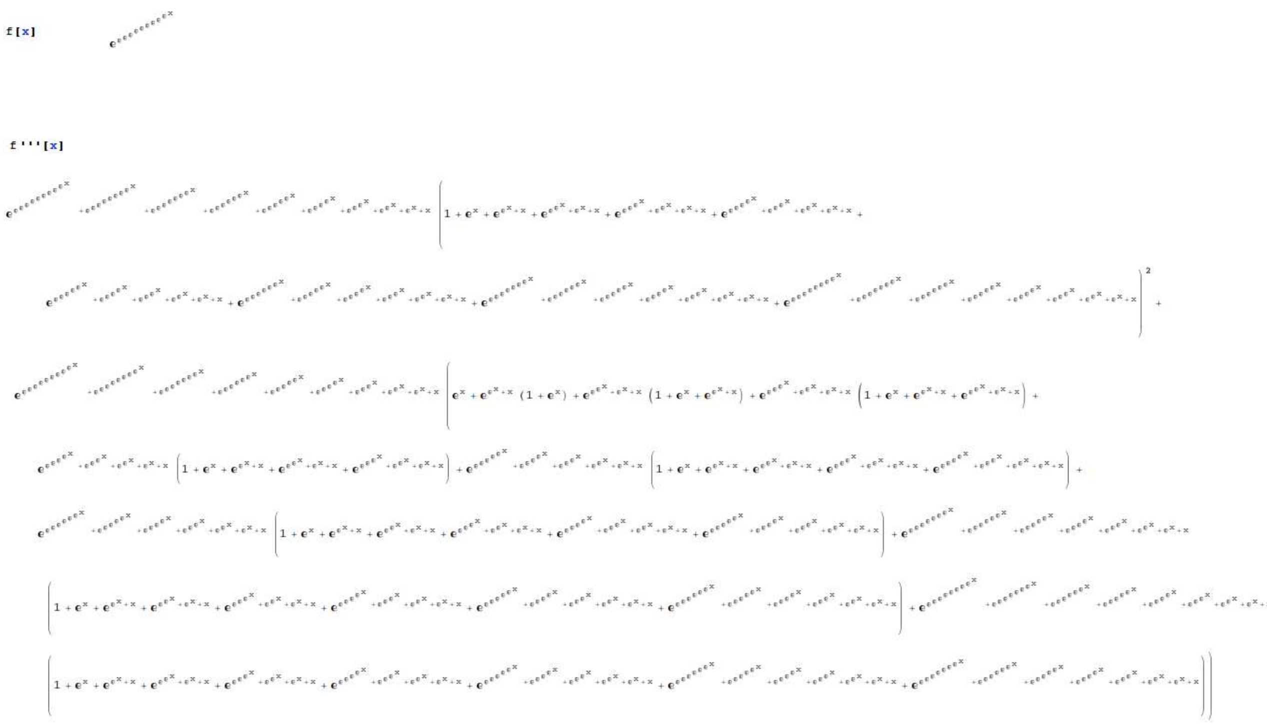
12 Take the derivative of $f_3(x) = e^{e^{e^x}}$.
Solution We can also write this as $\exp(\exp(\exp(x))$. The derivative is

$$\exp(\exp(\exp(x)))\exp(\exp(x))\exp(x) .$$

13 Lets push that to the extreme and differentiate

$$f(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x))))))))))$$

two times. Here is a picture of the formula obtained when running this in Mathematica:



14 Find the derivative of $1/\sin(x)$ using the quotient rule.
Solution $-\cos(x) \cdot 1/\sin^2(x)$.

15 Find the derivative of $f(x) = 1/\sin(x)$ using the chain rule.
Solution. The outer function is $f(x) = 1/x$. Therefore $f'(x) = -\cos(x)/\sin^2(x)$.

Homework

1 Find the derivatives of the following functions:
a) $f(x) = \sin(\log(x))$
b) $f(x) = \tan(x^9)$
c) $f(x) = \exp(1/(1+x^2))$
d) $(2+\sin(x))^{-5}$

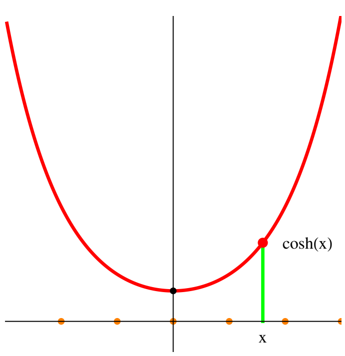
2 Find the derivatives of the following functions at $x = 1$.
a) $f(x) = x^8 \log(x)$. (where log is natural log)
b) $\sqrt{x^5+1}$
c) $(1+x^2+x^4)^{100}$
d) $\frac{5x^4}{2\sqrt{x^5+1}}$

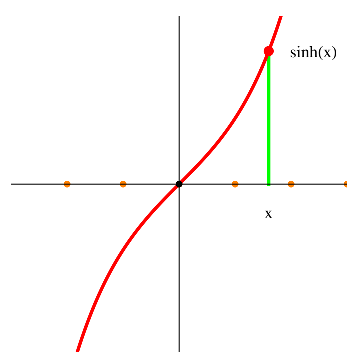
3 a) Find the derivative of $f(x) = 1/x$ by differentiating the identity $xf(x) = 1$ and using the product rule.
b) Find the derivative of $f(x) = \text{arccot}(x)$ by differentiating $\cot(\text{arccot}(x)) = x$ and using the chain rule.

4 a) Find the derivative of $\sqrt{x}$ by differentiating the identity $f(x)^2 = x$ leaving f as it is and solving for $f'(x)$.
b) Find the derivative of $x^{m/n}$ by differentiating the identity $f(x)^n = x^m$ leaving f as it is and solving for $f'(x)$.

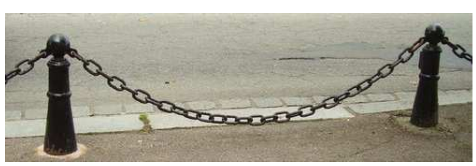
The function $f(x) = [\exp(x) + \exp(-x)]/2$ is called $\cosh(x)$.
The function $f(x) = [\exp(x) - \exp(-x)]/2$ is called $\sinh(x)$.
They are called **hyperbolic cosine** and **hyperbolic sine**. The first is even, the second is odd. You can see directly using $\exp'(x) = \exp(x)$ and $\exp'(-x) = -\exp(-x)$ that $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$. Furthermore $\exp = \cosh + \sinh$ writes exp as a sum of an even and odd function. You can easily check that $\cosh^2(x) - \sinh^2(x) = 1$.

5 a) Find the derivative of the inverse $\text{arccosh}(x)$ of $\cosh(x)$ by using the chain rule.
b) Find the derivative of the inverse $\text{arcsinh}(x)$ of $\sinh(x)$ by using the chain rule.





The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine. You can use the identity $\cosh^2(x) - \sinh^2(x) = 1$.



Lecture 10: Worksheet

The chain rule

The rule$(f(g(x))' = f'(g(x))g'(x)$ is called the **chain rule**.
For example, the derivative of $\sin(\log(x))$ is $\cos(\log(x))/x$.
We have also seen that we can compute the derivative of inverse functions using the chain rule.

1 Find the derivative of $\sqrt{1+x^2}$ using the chain rule

2 Find the derivative of $\sin^3(x)$ using the product rule.

3 Find the derivative of $\sin^3(x)$ using the chain rule.

4 Find the derivative of $\tan(\sin(x))$.

5 Find the derivative of $\sin(\cos(\exp(x)))$.

6 Find the derivative of $\arcsin(x)$ using the chain rule.

A lovely application of the chain rule

The **Valentine equation** (from last Friday). $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$ relates x with y, but we can not write the curve as a graph of a function $y = g(x)$. Extracting y or x is difficult. The set of points satisfying the equation looks like a heart. You can check that $(1, 1)$ satisfies the Valentine equation. Near it, the curve looks like the graph of a function $g(x)$. Lets fill that in and look at the function

$$f(x) = (x^2 + g(x)^2 - 1)^3 - x^2 g(x)^3$$

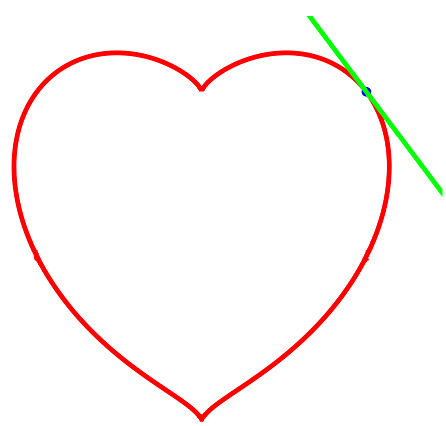
The key is that $f(x)$ is actually zero and if we take the derivative, then we get zero too. Using the chain rule, we can take the derivative

$$f'(x) = 3(x^2+g(x)^2-1)(2x+2g(x)g'(x))-2xg(x)^3-x^2 3g(x)^2 g'(x) = 0$$

We can now solve solve for g'

$$g'(x) = -\frac{3(x^2 + g(x)^2 - 1)2x - 2xg(x)^3}{3(x^2 + g(x)^2 - 1)2g(x) - 3x^2 g(x)^2} .$$

Filling in $x = 1, g(x) = 1$, we see this is $-4/3$. We have computed the slope of g without knowing g. Magic!



We will come back to this application of the chain rule later in the course.

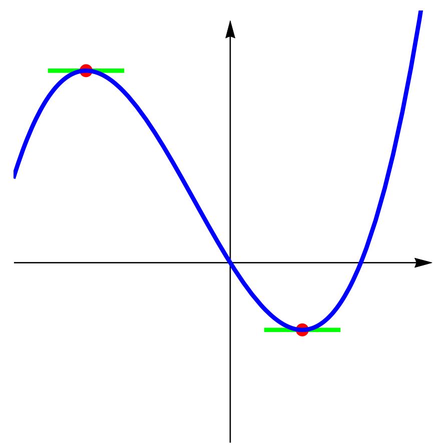
Lecture 11: Local extrema

Today, we look at the problem to find extrema. One of the major goals in calculus is to maximize nice quantities and minimize unpleasant ones. Extremizing quantities is also the most important principle nature follows. Laws in physics like Newtons law, Maxwell equations describing light, or equations describing matter can be based on the principle of extremization. The most important intuitive insight is that at maxima or minima, the tangent to the graph must be horizontal. This leads to the following notion for differentiable functions:

A point x_0 is a **critical point** of f if $f'(x_0) = 0$.

In some textbooks, critical points include points where f' is not defined. Others also include boundary points. [1] We do **not** include such points in the list of critical points. They are points outside the domain of definition of f' and we will deal with them separately.

1 Find the critical points of the function $f(x) = x^3 + 3x^2 - 24x$. **Solution:** we compute the derivative as $f'(x) = 3x^2 + 6x - 24$. The roots of f' are $2, -4$.



A point is called a **local maximum** of f, if there exists an interval $U = (p-a, p+a)$ around p, such that $f(p) \geq f(x)$ for all $x \in U$. A **local minimum** is a local maximum of $-f$. Local maxima and minima together are called **local extrema**.

2 The point $x = 0$ is a local maximum for $f(x) = \cos(x)$. The reason is that $f(0) = 1$ and $f(x) < 1$ nearby.

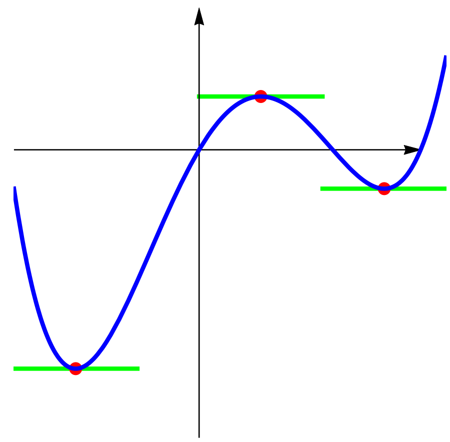
3 The point $x = 1$ is a local minimum for $f(x) = (x-1)^2$. The function is zero at $x = 1$ and positive everywhere else.

Fermat: If f is differentiable and has a local extremum at x, then $f'(x) = 0$.

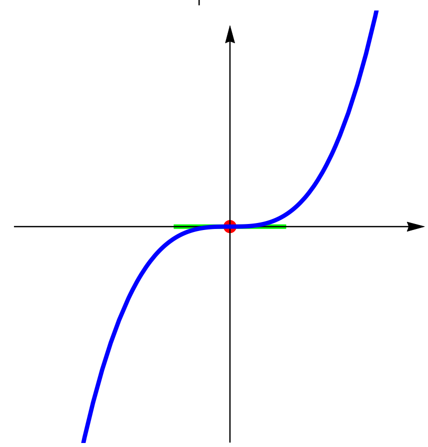
Why? Assume the derivative $f'(x) = c$ is not zero. We can assume $c > 0$ otherwise replace f with $-f$. By the definition of limits, for some large enough h, we have $f(x+h) - f(x)/h \geq c/2$. But this means $f(x+h) \geq f(x) + hc/2$ and x can not be a local maximum. Since also $(f(x) - f(x-h))/h \geq c/2$ for small enough h, we also have $f(x-h) \leq f(x) - hc/2$ and x can not be a local minimum.

[1] Important definitions have to be simple.

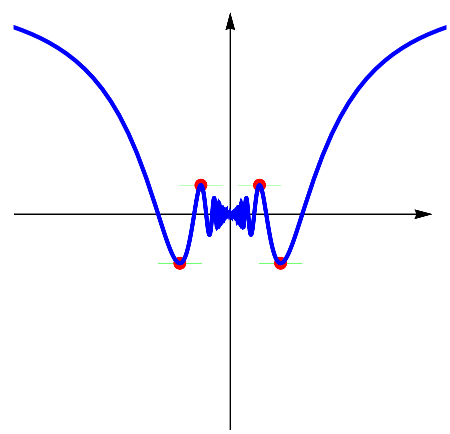
4 The derivative of $f(x) = 72x - 30x^2 - 8x^3 + 3x^4$ is $f'(x) = 72 - 60x - 24x^2 + 12x^3$. By plugging in integers (calculus teachers like integer roots because students like integer roots!) we can guess the roots $x = 1, x = 3, x = -2$ and see $f'(x) = 12(x-1)(x+2)(x-3)$. The critical points are $1, 3, -2$.



5 We have already seen that $f'(x) = 0$ does not assure that x is a local extremum. The function $f(x) = x^3$ is a counter example. It satisfies $f'(0) = 0$ but 0 is not a local extremum. It is an example of an **inflection point**, a point where f'' changes sign.



6 Lets look at one nasty example. The function $f(x) = x\sin(1/x)$ is continuous at 0 but there are infinitely many critical points near 0.



If $f''(x) > 0$, then the graph of the function is concave up. If $f''(x) < 0$ then the graph of the function is concave down.

Second derivative test. If x is a critical point of f and $f''(x) > 0$, then f is a local minimum. If $f''(x) < 0$, then f is a local maximum.

If $f''(x_0) > 0$ then $f'(x)$ is negative for $x < x_0$ and positive for $f'(x) > x_0$. This means that the function decreases left from the critical point and increases right from the critical point. Similarly, if $f''(x_0) < 0$ then $f'(x)$ is positive for $x < x_0$ and $f'(x)$ is positive for $x > x_0$. This means that the function increases left from the critical point and increases right from the critical point.

7 The function $f(x) = x^2$ has one critical point at $x = 0$. Its second derivative is 2 there.

8 Find the local maxima and minima of the function $f(x) = x^3 - 3x$ using the second derivative test. **Solution:** $f'(x) = 3x^2 - 3$ has the roots $1, -1$. The second derivative $f''(x) = 6x$ is negative at $x = -1$ and positive at $x = 1$. The point $x = -1$ is therefore a local maximum and the point $x = 1$ is a local minimum.

9 Find the local maxima and minima of the function $f(x) = \cos(\pi x)$ using the second derivative test.

10 For the function $f(x) = x^5 - x^3$, the second derivative test is inconclusive at $x = 0$. Can you nevertheless see the critical points?

11 Also for the function $f(x) = x^4$, the second derivative test is inconclusive at $x = 0$. The second derivative is zero. Can you nevertheless see whether the critical point 0 is local maximum of local minimum?

Finally, lets look at an example, where we can practice more the chain rule.

12 Find the critical points of $f(x) = 4\arctan(x) + x^2$. **Solution**. The derivative is

$$f'(x) = \frac{4}{1+x^2} + 2x = \frac{2x + 2x^3 + 4}{1+x^2} .$$

We see that $x = -1$ is a critical point. There are no other roots of $2x + 2x^3 + 4 = 0$. How did we get the derivative of arctan again? Differentiate

$$\tan(\arctan(x)) = x$$

and write $u = \arctan(x)$:

$$\frac{1}{\cos^2(\mathrm{u})} \arctan'(x) = 1 .$$

Use the identity $1 + \tan^2(\mathrm{u}) = 1/\cos^2(\mathrm{u})$ to write this as

$$(1 + \tan^2(\mathrm{u})) \arctan'(x) = 1 .$$

But $\tan(u) = \tan(\arctan(x)) = x$ so that $\tan^2(u) = x^2$. And we have

$$(1 + x^2) \arctan'(x) = 1 .$$

Now solve for $\arctan'(x)$:

$$\arctan'(x) = \frac{1}{1+x^2} .$$

Homework

1 Find all critical points for the following functions. If there are infinitely many, indicate their structure. For $f(x) = \cos(x)$ for example, the critical points can be written as $\pi/2 + k\pi$, where k is an integer.
a) $f(x) = x^4 - 2x^2$.
b) $f(x) = \sin(\pi x) + 1$
c) $f(x) = \exp(-x^2)x^2$.
d) $f(x) = \cos(\sin(x))$

2 Find all the critical points and use the second derivative test to determine whether they are maxima or minima.
a) $f(x) = x\log(x)$, where $x > 0$.
b) $f(x) = 1/(1 + x^2)$
c) $f(x) = x^2 - 2x + 1$.
d) $f(x) = 2x\tan(x)$, where $-\pi/2 < x < \pi/2$

3 Verify that a cubic equation $f(x) = x^3 + ax^2 + bx + c$ always has an inflection point, a point where $f''(x)$ changes sign.
Hint. Use the wobbling table!

4 Depending on c, the function $f(x) = x^4 - cx^2$ has either one or three critical points. Use the second derivative test to decide: a) For $c = 1$, find and determine the nature of the critical points.
b) For $c = -1$, find and determine the nature of the critical points.

5 a) Find a concrete function which has exactly one local maximum and local minimum.
b) Engineer a concrete function which has exactly 2 local maximum and 1 local minimum.
c) Try to find a differentiable function on the real line with 2 local maxima and no local minimum. Why is this not possible?

Lecture 11: Worksheet

Critical points and extrema

In this worksheet we want to find out which rectangle of fixed area $xy = 1$ has minimal circumference $2x + 2y$.

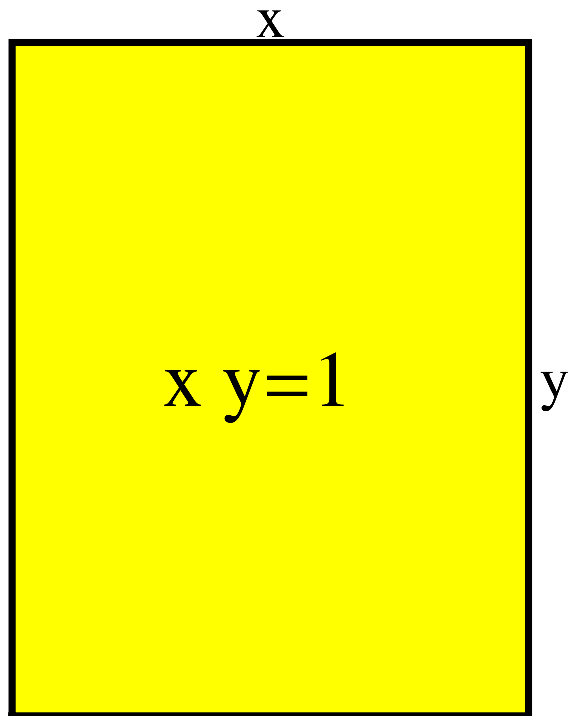
To solve this problem we have to extremize the function

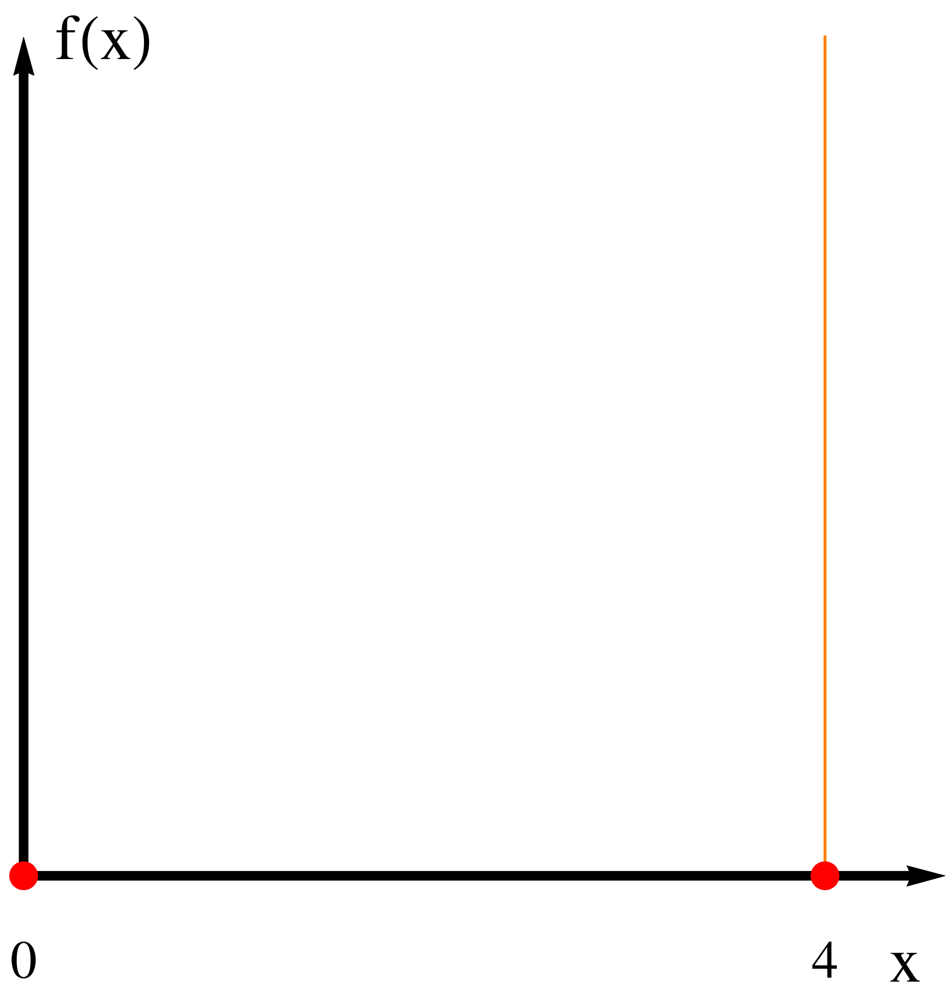
$$f(x) = 2x + \frac{2}{x} \ .$$

1 Differentiate the function f. For which x is it continuous?

2 Find the critical points of f, the places where $f'(x) = 0$.

3 Sketch the graph of f on the interval $(0, 4]$.


x
x y=1
y


f(x)
0
4
x

Lecture 12: Global extrema

In this lecture we are interested in the points where a function is maximal overall. These **global extrema** can occur at critical points of f or at the boundary of the domain, where f is defined.

A point p is called a **global maximum** of f if $f(p) \geq f(x)$ for all x. A point p is called a **global minimum** of f if $f(p) \leq f(x)$ for all x.

How do we find global maxima? The answer is simple: make a list of all local extrema and boundary points, then pick the largest. Global maxima or minima do not need to exist however. The function $f(x) = x^2$ has a global minimum at $x = 0$ but no global maximum. The function $f(x) = x^3$ has no global extremum at all. We can however look at global maxima on finite intervals.

1 Lets look at the example from last week where we found the square of maximal area among all squares of side length $x, 1-x$. The function $f(x) = x(1-x)$ had a maximum at $x = 1/2$. We also have to look at the boundary points. Why? Because both x and $f(x)$ can not become negative. We see that $f(x)$ has to be looked at on the interval $[0, 1]$.

2 Find the global maximum of $f(x) = x^2$ on the interval $[-1, 2]$. **Solution**. We look for local extrema at critical points and at the boundary. Then we compare all these extrema to find the maximum or minimum. The critical points are $x = 0$. The boundary points are $-1, 2$. Comparing the values $f(-1) = 1, f(0) = 0$ and $f(4) = 4$ shows that f has a global maximum at 2 and a global minimum at 0.

Extreme value theorem of Bolzano A continuous function f on a finite interval $[a, b]$ attains a global maximum and a global minimum.

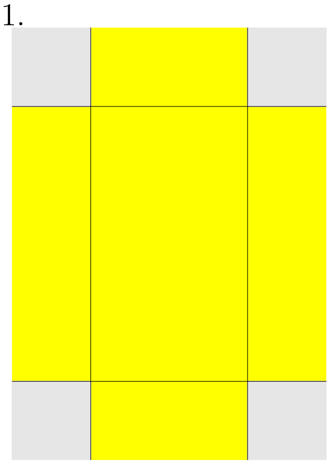
Proof: for every n, make a list of the points $x_k = (a + (k/n)(b - a))$ where $k = 1, \dots, n$. Pick the one where $f(x_k)$ is maximal one and call this x_n. Now we use the **Bolzano-Weierstrass theorem** which assures that any sequence of numbers on a closed interval $[a, b]$ has an accumulation point. Such an accumulation point is a maximum. Similarly, we can construct the minimum.
The **Bolzano-Weierstrass theorem** is verified constructively too: cut the interval in two equal parts and choose a part which contains infinitely many points x_n. We have reduced the problem to a smaller interval. Now take this interval and again divide it into two. Again chose the one in which x_n hits infinitely many times. Wash, rinse and repeat this again and again leads to smaller and smaller intervals of size $[b - a]/2^n$ in which there are infinitely many points. Note that these intervals are nested so that they lead to a limit (if the interval were $[0, 1]$ and we would cut each time into 10 pieces, then we would gain in every step one digit of the decimal expansion of the number we are looking for).

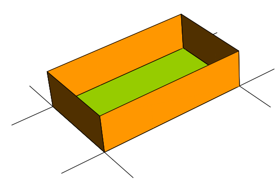
Note that the global maximum or minimum can also also on the boundary or points where the derivative dos not exist.

3 Find the global maximum and minimum of the function $f(x) = |x|$. The function has no absolute maximum as it goes to infinity for $x \to \infty$. The function has a global minimum at $x = 0$ but the function is not differentiable there. The point $x = 0$ is a point which does not belong to the domain of f'.

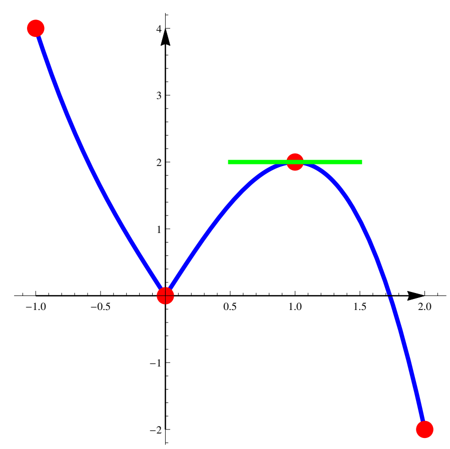
A **soda can** is a cylinder of volume $\pi r^2 h$. The surface area $2\pi r h + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is 2π,
4 we can solve the equation for $h = (1 - r^2)/r = 1/r - r$
Solution: The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.

5 Take a card of 2×2 inches. If we cut out 4 squares of equal side length x at the corners, we can fold up the paper to a tray with width $(2 - 2x)$ length $(2 - 2x)$ and height x. For which $x \in [0, 1]$ is the tray volume maximal?
Solution The volume is $f(x) = (2 - 2x)(2 - 2x)x$. To find the maximum, we need to compare the critical points which is at $x = 1/3$ and the boundary points $x = 0$ and $x = 1$.





Find the global maxima and minima of the function $f(x) = 3|x| - x^3$ on the interval $[-1, 2]$.
Solution. For $x > 0$ the function is $3x - x^3$ which can be differentiated. The derivative $3 - 3x^2$ is zero at $x = 1$. For $x < 0$ the function is $-3x - x^3$. The derivative is
6 $-3 - x^2$ and has no root. The only critical points are 1. There is also the point $x = 0$ which is not in the domain where we can differentiate the function. We have to deal with this point separately. We also have to look at the boundary points $x = -1$ and $x = 2$. Making a list of function values at $x = -1, x = 0, x = 1, x = 2$ gives the maximum.



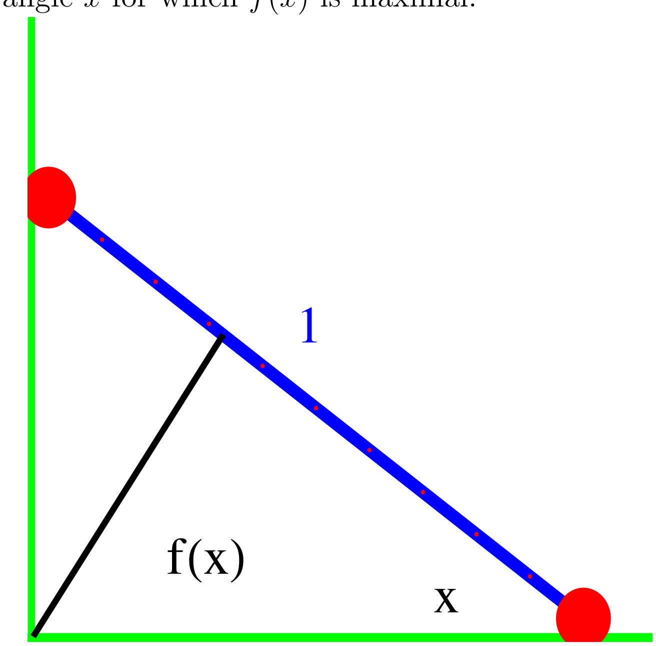
Homework

1 Find all the local maxima and minima as well as the global maximum and the global minimum of the function $f(x) = x^4 - 2x^2$ on the interval $[-2, 3]$.

2 Find the global maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 36x$ on the interval $[-4, 4]$

3 A candy manufacturer builds spherical candies. Its effectiveness is $A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r. Find the radius, where $f(r) = A(r) - V(r)$ has a global maximum for $r \geq 0$.

4 A ladder of length 1 is one side at a wall and on one side at the floor. a) Verify that the distance from the ladder to the corner is $f(x) = \sin(x)\cos(x)$.
b) Find the angle x for which $f(x)$ is maximal.



5 a) The function $S(x) = -x \log(x)$ is called **entropy**. Find the probability $0 < x \leq 1$ which maximizes entropy. One of the most important principles in all science is that nature tries to maximize entropy. In some sense we compute here the number of

maximal entropy.
b) We can write $1/x^x = e^{-x\log(x)}$. Find the positive number x, where x^{-x} has a local maximum. [1]

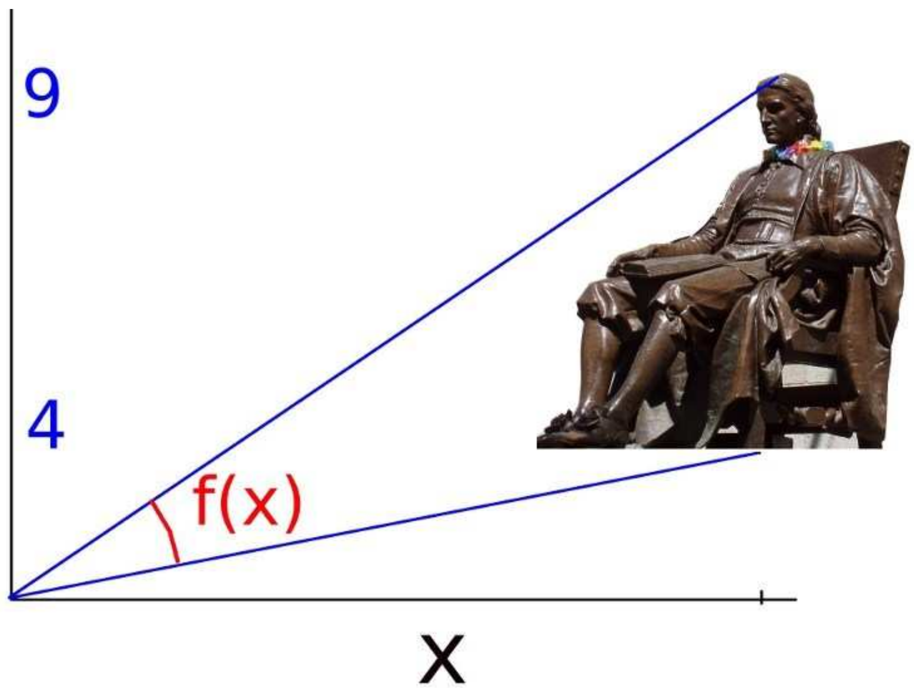
Entropy has been introduced by Boltzman. It is important in physics and chemistry.

[1]We have used the identity $a^b = e^{b\log(a)}$

Lecture 12: Worksheet

Extrema with boundaries

The following famous problem is usually asked with the Statue of liberty. At Harvard, we of course want to use the John Harvard Statue. It is a common situation. You want to look at a statue. If you are too close below it, the viewing angle becomes small. If you are far away, the viewing angle decreases again. There is an optimal distance where the viewing angle is maximal.



At which distance x do you see most of the John Harvard Statue? Assume the part you want to see 4 to 9 feet higher than your eyes.

1 Verify that the angle you see from the statue is

$$f(x) = \arctan(\frac{9}{x}) - \arctan(\frac{4}{x}) \ .$$

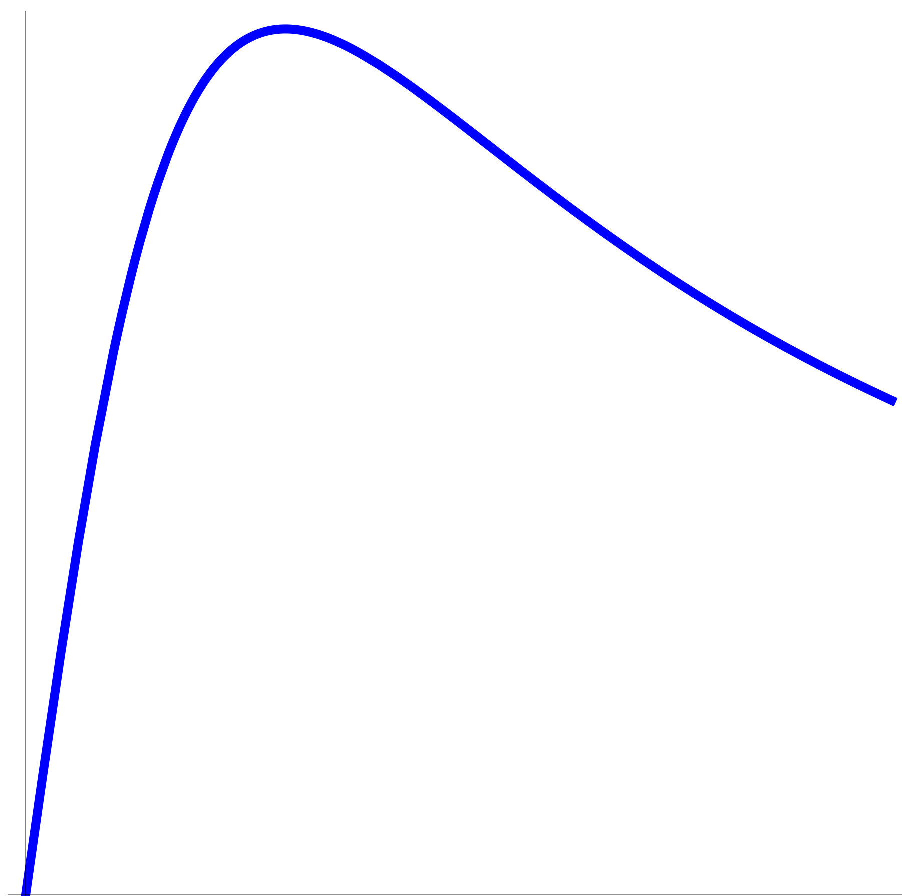
2 Differentiate $f(x)$ to find the minimum.

3 Are there any boundary points or points where f is not differentiable?

4 Find the global maximum of f.

5 Is there a global minimum of f?

Here is a graph of part of the function f.



Lecture 13: Hopitals rule

The rule

The Hopital's rule is a miracle procedure which solves all our worries about limits:

Hopital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} .$$

Lets see how it works:

1 Lets prove **the fundamental theorem of trigonometry** again:

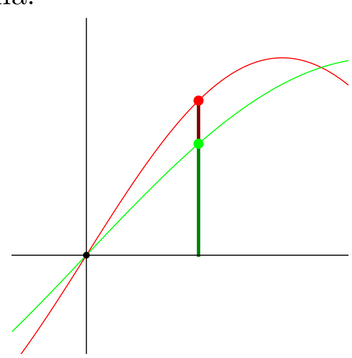
$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 .$$

Why did we work so hard for this? We used the fundamental theorem to derive the derivatives for cos and sin at all points. In order to apply l'Hopital, we had to know the derivative. Our work to establish the limit was not in vain.

The proof of the rule is comic in its simplicity. Especially after we will see how fantastically useful it is:
since $f(p) = g(p) = 0$ we have $Df(p) = f(p+h)/h$ and $Dg(p) = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$ the **quantum l'Hopital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} .$$

Now take the limit $h \to 0$. Voilà!



Sometimes, we have to administer a medicine twice. To use this, l'Hopital can be improved in that the condition $g'(0) = 0$ can be replaced by the requirement that the limit $\lim_{x \to p} f'(x)/g'(x)$ exists. Instead of having a rule which replaces a limit with an other limit and cure a disease with a new one, we formulate it how it is used. The second derivative case could easily be generalized for higher derivatives. There is no need to memorize this. Just remember that you can check in several times to a hospital.

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f''(x)}{g''(x)}$ if the limit to the right exists.

2 Find the limit $\lim_{x\to 0}(1-\cos(x))/x^2$. This limit had been pivotal to compute the derivatives of trigonometric functions. **Solution**: differentiation gives

$$\lim_{x\to 0} -\sin(x)/2x \ .$$

Now apply l'Hopital again.

$$\lim_{x\to 0} -\sin(x)/(2x) = \lim_{x\to 0} -\cos(x)/2 = -\frac{1}{2} \ .$$

3 **Problem.** Find the limit $f(x) = (\exp(x^2)-1)/\sin(x^2)$ for $x \to 0$.

4 **Problem:** What do you get if you apply l'Hopital to the limit $[f(x+h)-f(x)]/h$ as $h \to 0$?
Answer: Differentiate both sides with respect to h! And then feel awsome!

5 Find $\lim_{x\to\infty} x\sin(1/x)$. **Solution.** Write $y = 1/x$ then $\sin(y)/y$. Now we have a limit, where the denominator and nominator both go to zero.

The case when both sides converge to infinity can be reduced to the 0/0 case by looking at $A = f/g = (1/g(x))/(1/f(x))$ which has the limit $g'(x)/g^2(x)/f'(x)/f^2(x) = g'(x)/f'(x)((1/g)/(1/f))^2 = g'/f'(f^2/g^2) = (g'/f')A^2$, so that $A = f'(p)/g'(p)$. We see:

If $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = \infty$ for $x \to p$ and $g'(p) \neq 0$, then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)} \ .$$

2 What is the limit $\lim_{x\to 0} x^x$? This answers the question **What is 0^0?**
Solution: Because $x^x = e^{x\log(x)}$, it is enough to understand the limit $x\log(x)$ for $x \to 0$.

$$\lim_{x\to 0} \frac{\log(x)}{1/x} \ .$$

Now the limit can be seen as the limit $(1/x)/(-1/x^2) = -x$ which goes to 0. Therefore $\lim_{x\to 0} x^x = 1$. (We assume that $x > 0$ in order to have real values x^x)

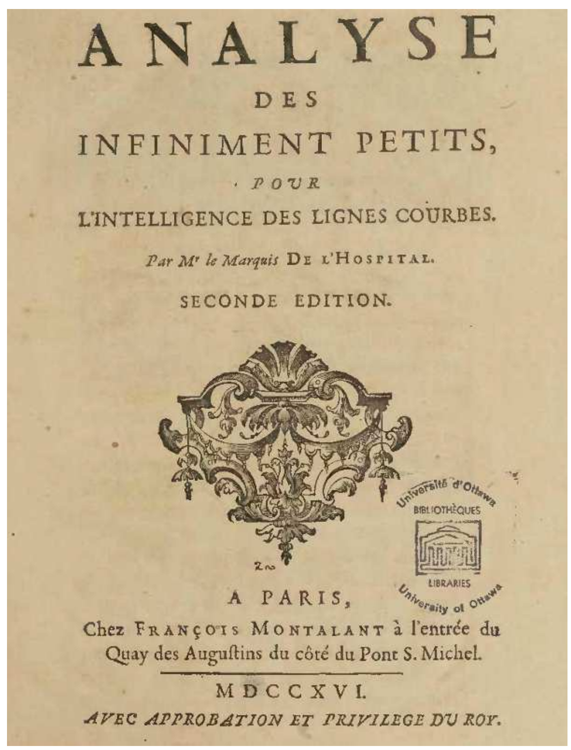
3 Find the limit $\lim_{x\to 2} \frac{x^2-4x+4}{\sin^2(x-2)}$.
Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.
Hopital's rule always works in calculus situations, where functions are differentiable. The rule can fail if differentiability of f or g fails. Here is a "rare" example:

4 **Deja Vue**: Find $\frac{\sqrt{x^2+1}}{x}$ for $x \to \infty$. L'Hopital gives $x/\sqrt{x^2+1}$ which in terms gives again $\frac{\sqrt{x^2+1}}{x}$. Apply l'Hopital again to get the original function. We got an infinite loop. If the limit is A, then the procedure tells that it is equal to $1/A$. The limit must therefore be 1. This case can be covered easily without l'Hopital: divide both sides by x to get $\sqrt{1+1/x^2}$. Now, we can see the limit 1.

5 **Trouble?** The limit $\lim_{x\to\infty}(2x+\sin(x))/3x$ is clearly 2/3 since we can take the sum apart. Hopital gives $\lim_{x\to\infty}(2+\cos(x))/3$ which has no limit. This is not trouble since Hopital applies only if the limit to right exists.

History

The "first calculus book", the world has known was "Analyse des Infiniment Petits pour l"intelligence des Lignes Courbes". It appeared in 1696 and was written by **Guillaume de l'Hopital**, a text if typeset in a modern font would probably fit onto 50-100 pages.[1] It is now clear that the mathematical content of Hopital's book is mostly due to **Johannes Bernoulli**: *Clifford Truesdell write in his article "The New Bernoulli Edition",* [2] *about this "most extraordinary agreement in the history of science": l'Hopital wrote: "I will be happy to give you a retainer of 300 pounds, beginning with the first of January of this year ... I promise shortly to increase this retainer, which I know is very modest, as soon as my affairs are somewhat straightened out ... I am not so unreasonable as to demand in return all of your time, but I will ask you to give me at intervals some hours of your time to work on what I request and also to communicate to me your discoveries, at the same time asking you not to disclose any of them to others. I ask you even not to send here to Mr. Varignon or to others any copies of the writings you have left with me; if they are published, I will not be at all pleased. Answer me regarding all this ..."* Bernoulli's response is lost, but a letter from l'Hopital indicates that it was quickly accepted. Clifford Truesdell also mentions that the book of l'Hopital has remained the standard for Calculus for a century.



[1]Stewart's book with 1200 pages probably contains about 4 million characters, about 12 times more than l'Hopital's book. It also contains more material of course. The OCR text of l'Hopital's book of 200 pages has 300'000 characters.

[2]Isis, Vol. 49, No. 1, 1958, pages 54-62

Homework

1 For the following functions, find the limits as $x \to 0$:
a) $8x/\sin(x)$
b) $(\exp(x)-1)/(\exp(3x)-1)$
c) $\sin^2(3x)/\sin^2(5x)$
d) $\frac{\sin(x^2)}{\sin^2(x)}$
e) $\sin(\sin(x))/x$.

2 For the following functions, find the limits as $x \to 1$:
a) $(x^2-x-1)/(\cos(x-1)-1)$
b) $(\exp(x)-e)/(\exp(3x)-e^3)$
c) $(x-4)/(4x+\sin(x)+8)$
d) Find the limit as $x \to \infty$:
$(x^2-x-1)/\sqrt{x^4+1}$.
(**Hint.** Find the limit of $(x^2-x-1)^2/(x^4+1)$ first, then take the square root of the limit).

3 Use l'Hopital to compute the following limits at $x = 0$:
a) $\log(5x)/\log|x|$.
b) $\lim_{x\to 0} 1/\log|x|$
c) $\mathrm{sinc}'(x) = (\cos(x)x - \sin(x))/x^2$
d) $\log|\log|1+x||/\log|\log|2+x||$.

4 We have seen how to compute limits with healing. Solve the following healing problems with l'Hopital at $x = 1$:
a) $\frac{x^{1000}-1}{x^{20}-1}$.
b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

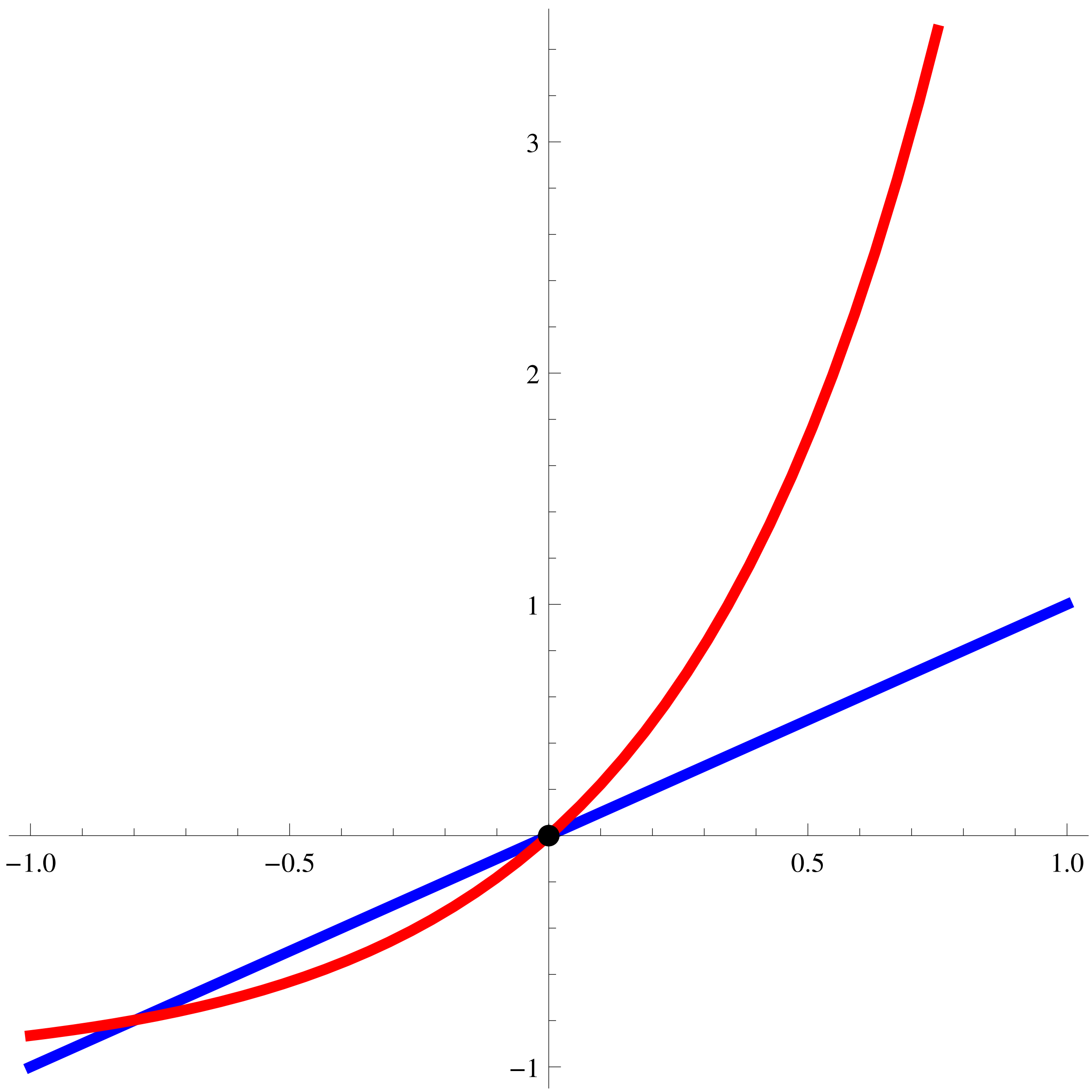
5 More practice.
a) Find the limit $\lim_{x\to 0} \frac{1-e^x}{x-x^3}$.
b) Find the limit $\lim_{x\to 0} \frac{\log(1+3x)}{x}$.
c) Find the limit $\lim_{x\to 1}(x^5-1)/(x^3-1)$.
d) Find the limit $\lim_{x\to 0} \frac{4x}{\tan(5x)}$.

Lecture 13: Worksheet

Hopital's rule

1 What does l'Hopital's rule say about

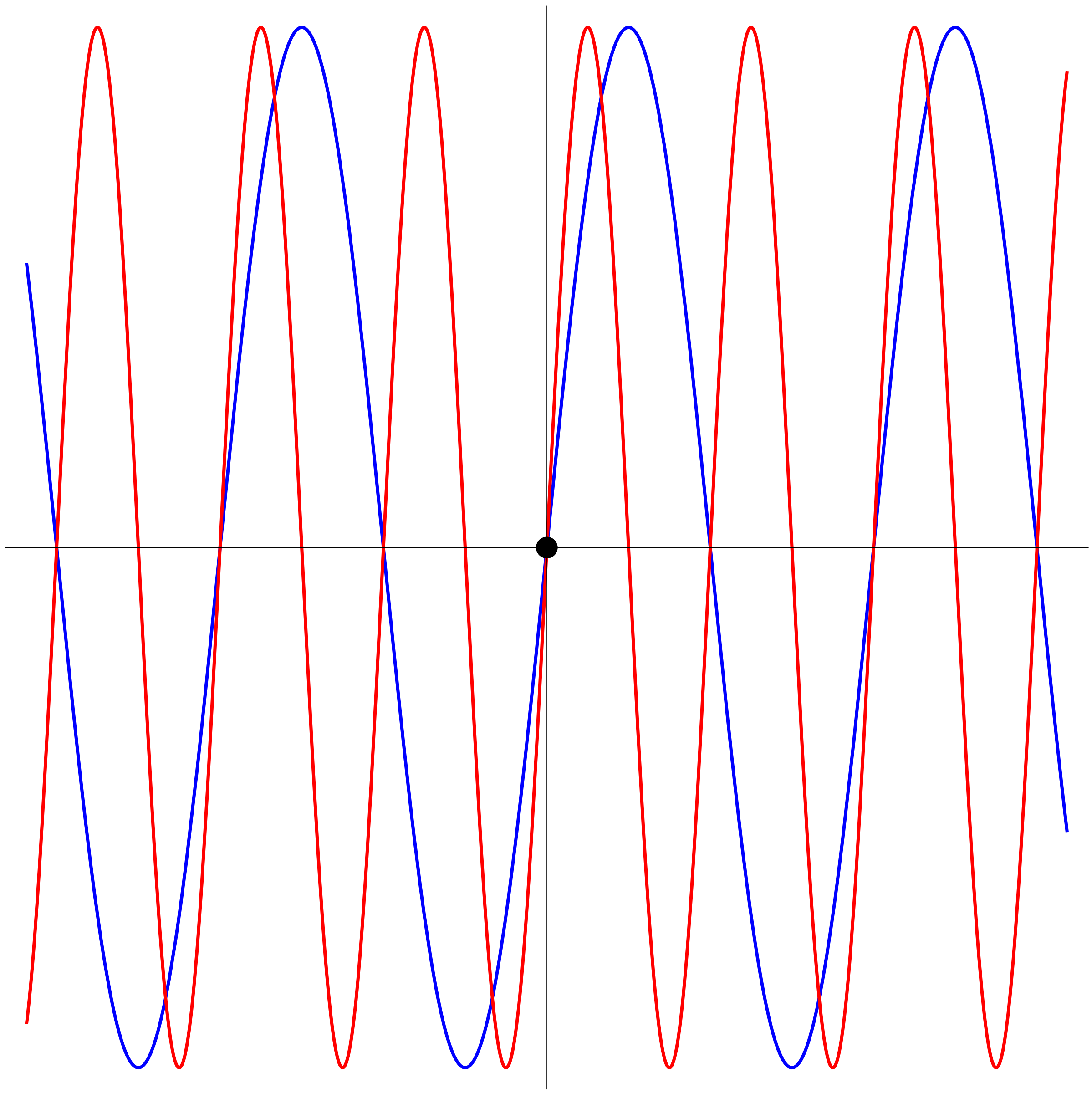
$$\lim_{x \to 0} \frac{\exp(2x) - 1}{x} \ ?$$



2 Apply l'Hopital's rule to get the limit of

$$f(x) = \frac{\sin(200x)}{\sin(300x)}$$

for $x \to 0$.



Lecture 14: Newton's method

Recall that a point a is called a **root** of a function f if $f(a) = 0$. We were able to find the roots of functions using a "divide and conquer" technique: start with an interval $[a, b]$ for which $f(a) < 0$ and $f(b) > 0$. If $f((a+b)/2)$ is positive, then use the interval $[a, (a+b)/2]$ otherwise $[(a+b)/2, b]$. After n steps, we are $(b-a)/2^n$ close to the root.
If the function f is differentiable, we can do much better. We can use the value of the derivative to get closer to a point $y = T(x)$. Lets find y. If we draw a tangent at $(x, f(x))$, then

$$f'(x) = \frac{f(x)}{x-y} \ .$$

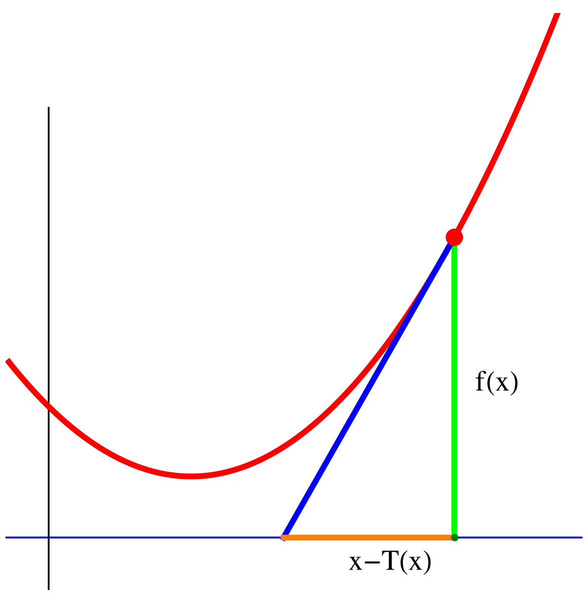
because $f'(x)$ is the slope of the tangent and the right hand side is "rise" over "run". If we solve for y we get

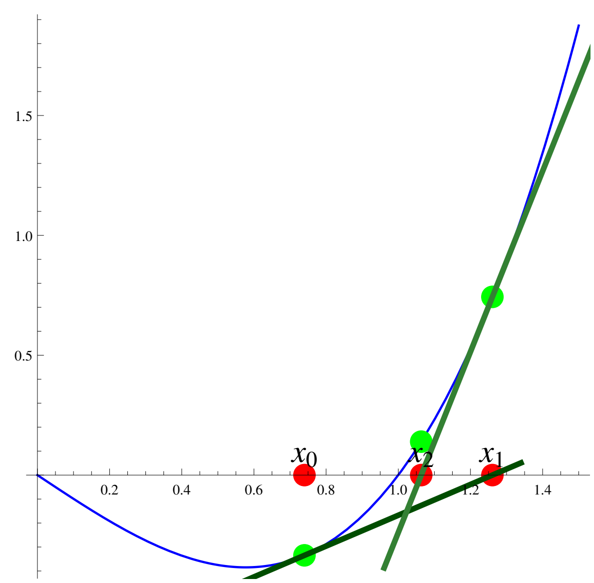
The **Newton map** is defined as

$$y = T(x) = x - \frac{f(x)}{f'(x)} \ .$$

Newton's method is the process of applying this map again and again until we are sufficiently close to the root. It is an extremely fast method to find the root of a function. Start with a point x, then compute a new point $x_1 = T(x)$, then $x_2 = T(x_1)$ etc.

If p is a root such that $f'(p) \neq 0$, and x_0 is close enough to p, then $x_1 = T(x), x_2 = T^2(x)$ converges to the root p.





1 If $f(x) = ax + b$, we reach the root in one step.

2 If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0 but not as fast as the method promises. Indeed, the root 0 is also a critical point of f. This slows us down.

3 The Newton map brings us to infinity if we start at a critical point.

Newton used this method to find the roots of polynomials. It is amazingly fast: Starting 0.1 close to the point, we have after one step 0.01 after 2 steps 0.0001 after 3 steps 0.00000001 and after 4 steps 0.0000000000000001.

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$ if we start sufficiently close to the root.

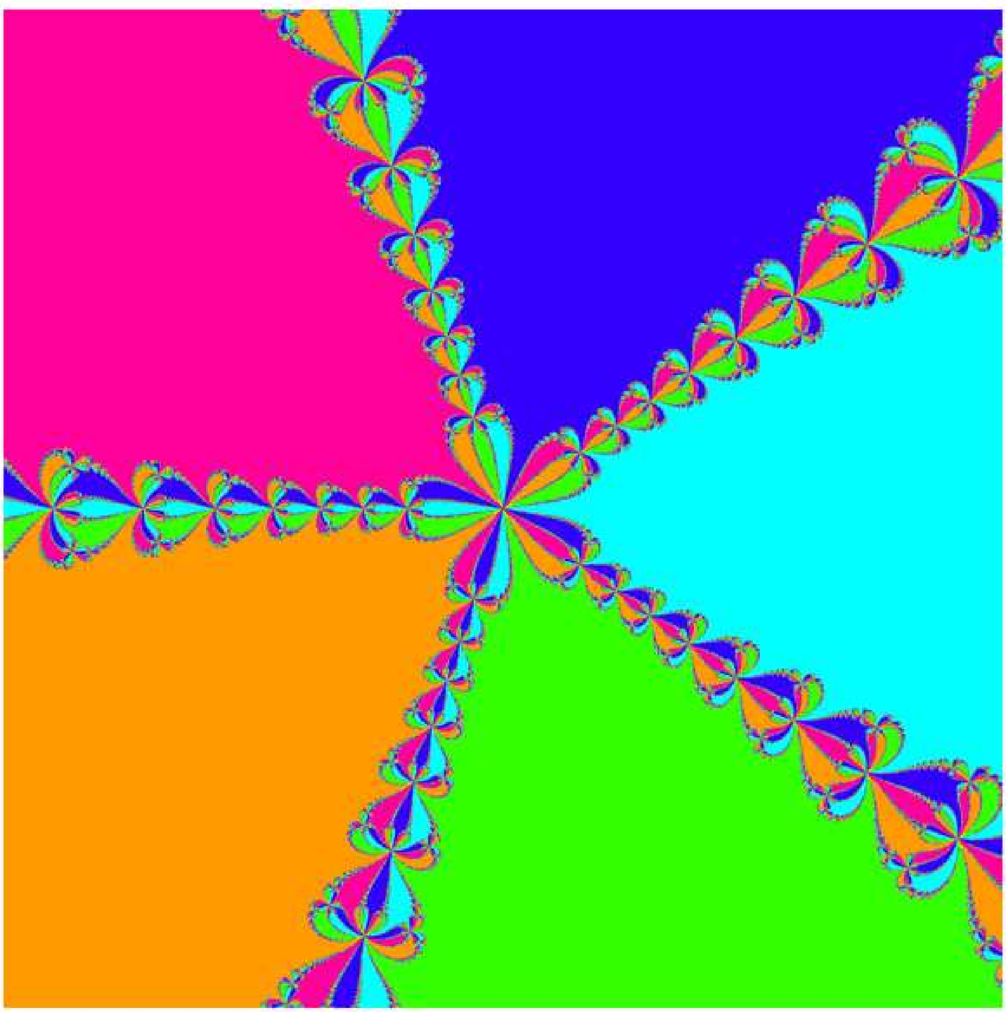
In 10 steps we can get a $2^{10} = 1024$ digits accuracy. Having a fast method to compute roots is useful. For example in computer graphics, where things can not be fast enough. Also in number theory, when working with integers having thousands of digits the Newton method can help. There is much theoretical use of the method. It goes so far as to explain stability of planetary motion or stability of plasma in fusion reactors.

4 Verify that the Newton map $T(x)$ in the case $f(x) = (x-1)^3$ has the property that we approach the root $x = 1$. **Solution**. You see that the approach is not that fast: we get $T(x) = x + (1-x)/3 = (1+2x)/3$. It converges exponentially fast, but not super exponential. The reason is that the derivative at $x-1$ is not zero. That slows us down.

If we have several roots, and we start at some point, to which root will the Newton method converge? Does it at all converge? This is an interesting question. It is also historically intriguing because it is one of the first cases, where "chaos" was observed at the end of the 19'th century.

5 Find the Newton map in the case $f(x) = x^5 - 1$. **Solution** $T(x) = x - (x^5-1)/(5x^4)$.

If we look for roots in the complex like for $f(x) = x^5 - 1$ which has 5 roots in the complex plane, the "basin of attraction" of each of the roots is a complicated set which we call the **Newton fractal**. Here is a picture:



6 Lets compute $\sqrt{2}$ to 12 digits accuracy. We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned} T(1) &= 1 - (1-2)/2 = 3/2 \\ T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\ T(17/12) &= 577/408 \\ T(577/408) &= 665857/470832 . \end{aligned}$$

This is already $1.6 \cdot 10^{-12}$ close to the real root! 12 digits, by hand! My grandfather Sigbert Bader (who was a work-safety officer at the SIG industries, and competed at the world championship in gymnastics and later owned and operated a bicycle shop) mastered this technique very well and continued to ask me while I studied mathematics: "How accurately can you compute square roots?" - I can now proudly answer: 12 digits, Grand pa! He would probably laugh and get 20 digits.

7 To find the cube root of 10 we have to find a root of $f(x) = x^3 - 10$. The Newton map is $T(x) = x - (x^3 - 10)/(3x^2)$. If we start with $x = 2$, we get the following steps $2, 13/6, 3277/1521, 105569067476/49000820427$. After three steps we have a result which is already $2.2 \cdot 10^{-9}$ close to the root.

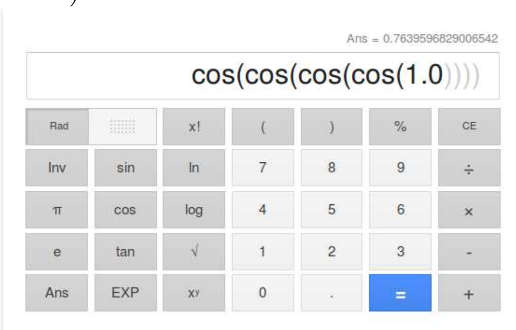
The Newton method is scrumtrulescent!

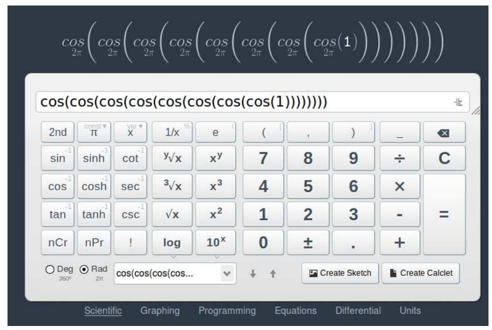
Will Ferrell's sketch: "Inside the Actors Studio" at Saturday Night live created the term "scrumtrulescent". See http://www.hulu.com/watch/3524

Homework

1 Find a formula for the Newton map $T(x) = x - f(x)/f'(x)$ in the following cases
a) $f(x) = (x-2)^2$
b) $f(x) = e^{5x}$
c) $f(x) = 2e^{-x^2}$
d) $f(x) = \cot(x)$.

2 The function $f(x) = \cos(x) - x$ has a root between 0 and 1. We get closer to the root by doing one Newton step starting with $x = 1$. (Compare with the actual root $x = 0.739085...$, which you can get on any calculator by entering 1, then punching the key "cos" a lot of times).

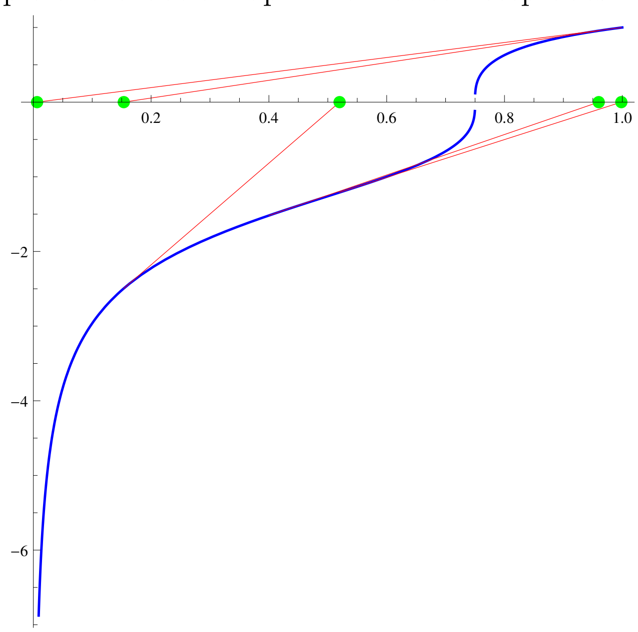




3 We want to find the square root of 102. We have to solve $\sqrt{102} = x$ or $f(x) = x^2 - 102 = 0$. Perform two Newton steps starting at $x = 10$.

4 Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$. What happens if you apply the Newton step again and again?

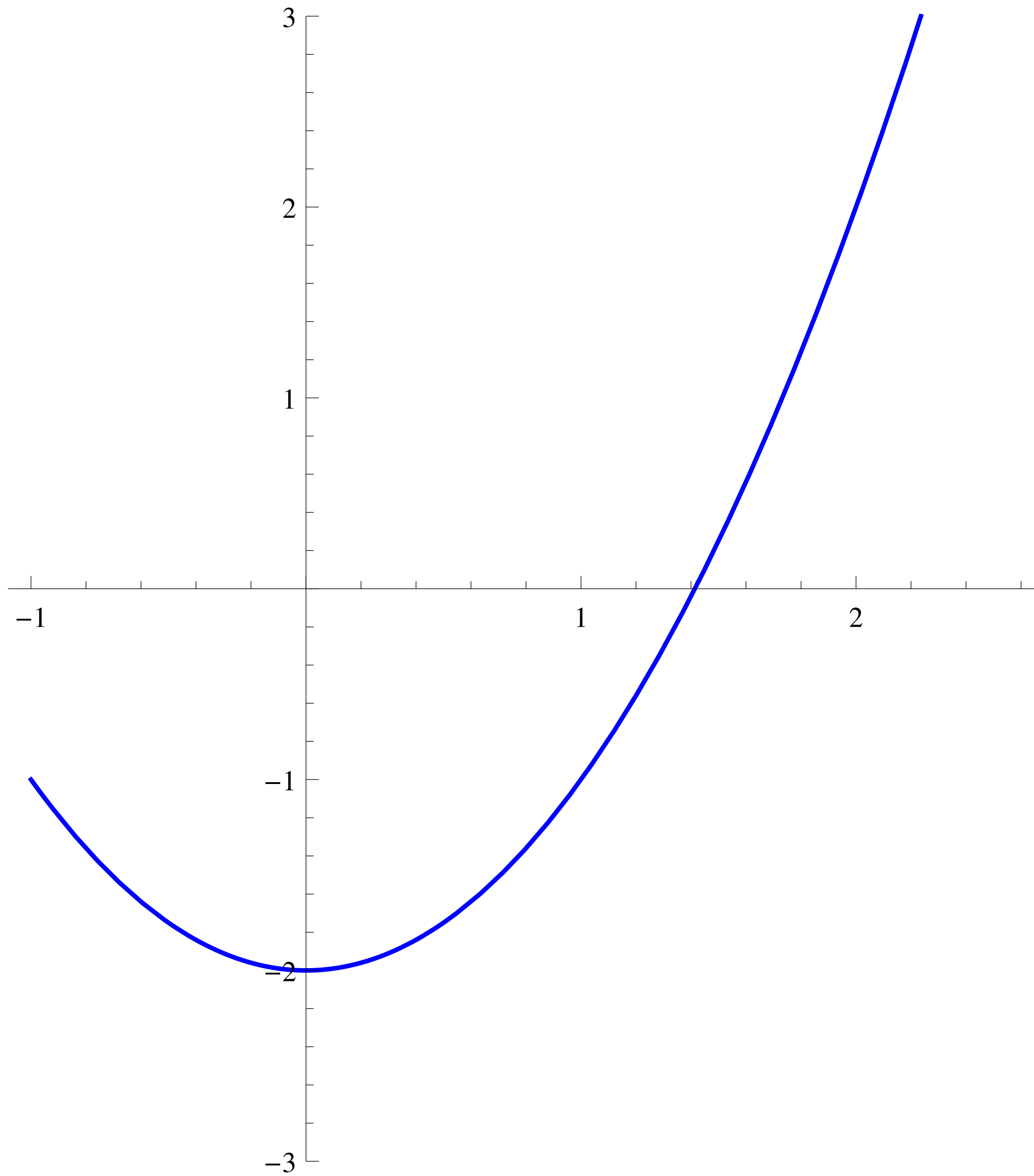
5 Verify that the Newton map in the case $f(x) = (4 - 3/x)^{1/3}$ is the quadratic map $T(x) = 4x(1-x)$.
This is an example of a chaotic map. The Newton step does not converge.



Lecture 14: Worksheet

Newton Method

1 In the following graph $y = x^2 - 2$ trace a Newton step starting at $x = 1.0$.



2 Now do a second step.

Lecture 15: Review for first midterm

Major points

A function is **continuous**, if whenever x, y are close, also $f(x), f(y)$ are close. Formally, for every a there exists $b = f(a)$ such that $\lim_{x \to a} f(x) = b$ for every a. The intermediate value theorem: $f(a) > 0, f(b) < 0$ implies f having a root in (a, b).

If $f'(x) = 0$ and $f''(x) > 0$ then x is a **local minimum**. If $f'(x) = 0$ and $f''(x) < 0$ then x is a **local maximum**. To find **global minima or maxima**, compare local extrema and boundary values.

If f changes sign we have a **root** $f = 0$, if f' changes sign, we have a **critical point** $f' = 0$ if f'' changes sign, we have an **inflection points**. A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$. Odd functions have 0 as roots, even functions have 0 as critical point.

If $f' > 0$ then f is increasing, if $f' < 0$ it is decreasing. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a horizontal tangent.

Hôptial's theorem applies for $0/0$ or ∞/∞ situations. In that case, $\lim_{x \to p} f(x)/g(x)$, where $f(p) = g(p) = 0$ or $f(p) = g(p) = \infty$ with $g'(p) \neq 0$ are given by $f'(p)/g'(p)$.

With $Df(x) = (f(x+h) - f(x))/h$ and $S(x) = h(f(0) + f(2h) + \cdots + f((k-1)h))$ we have a **preliminary fundamental theorem of calculus** $SDf(kh) = f(kh) - f(0)$ and $DS(f(kh)) = f(kh)$.

Roots of $f(x)$ with $f(a) < 0, f(b) > 0$ can be obtained by the dissection method by applying the **Newton map** $T(x) = x - f(x)/f'(x)$ again and again.

Algebra reminders

Healing:	$(a+b)(a-b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)$
Denominator:	$1/a + 1/b = (a+b)/(ab)$
Exponential:	$(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b \log(a)}$
Logarithm:	$\log(ab) = \log(a) + \log(b)$. $\log(a^b) = b \log(a)$
Trig functions:	$\cos^2(x) + \sin^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots:	$a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	$x^3 + 2x^2 + 3x + 1$	Exponential	$5e^{3x}$
Rational functions	$(x+1)/(x^3 + 2x + 1)$	Logarithm	$\log(3x)$
Trig functions	$2\cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$f(x) = x^n$	nx^{n-1}	$f(x) = \sin(ax)$	$a\cos(ax)$
$f(x) = e^{ax}$	ae^{ax}	$f(x) = \tan(x)$	$1/\cos^2(x)$
$f(x) = \cos(ax)$	$-a\sin(ax)$	$f(x) = \log(x)$	$1/x$

Differentation rules

Addition rule	$(f+g)' = f' + g'$.	Quotient rule	$(f/g)' = (f'g - fg')/g^2$.
Scaling rule	$(cf)' = cf'$.	Chain rule	$(f(g(x))' = f'(g(x))g'(x)$.
Product rule	$(fg)' = f'g + fg'$.	Easy rule	simplify before deriving

Extremal problems

1 Build a fence of length $x + 2y = 12$ and dimensions x and y with maximal area $A = xy$.

2 Find the largest area $A = 4xy$ of a rectangle with vertices $(x, y), (-x, y), (-x, -y), (x, -y)$ inscribed in the ellipse $x^2 + 2y^2 = 1$.

3 Which isosceles triangle of height h and base $2x$ and area $xh = 1$ has minimal circumference $2x + 2\sqrt{x^2 + h^2}$?

To extremize f on an interval $[a, b]$, find all critical points inside the interval, evaluate f on the boundary $f(a), f(b)$ and then compare the values to find the global maximum. Do not make the second derivative test at the boundary.

Limit examples

$\lim_{x\to 0} \sin(x)/x$	l'Hopital 0/0	$\lim_{x\to 1}(x^2 - 1)/(x - 1)$	heal
$\lim_{x\to 0}(1 - \cos(x))/x^2$	l'Hopital 0/0 twice	$\lim_{x\to\infty} \exp(x)/(1 + \exp(x))$	l'Hopital
$\lim_{x\to 0} x\log(x)$	l'Hopital ∞/∞	$\lim_{x\to 0}(x + 1)/(x + 5)$	no work necessary

Important things

Summation and taking differences is at the heart of calculus
The 3 major types of discontinuities are jump, oscillation, infinity
Dissection and Newton methods are algorithms to find roots.
The fundamental theorem of trigonometry is $\lim_{x\to 0} \sin(x)/x = 1$.
The derivative is the limit $Df(x) = [f(x + h) - f(x)]/h$ as $h \to 0$.
The rule $D(1 + h)^{x/h} = (1 + h)^{x/h}$ leads to $\exp'(x) = \exp(x)$.

More Examples

1 Is $1/\log|x|$ continuous at $x = 0$. Answer: yes with $f(0) = 0$

2 Is $\log(1/|x|)$ continuous at $x = 0$. Answer: no.

3 $\lim_{x\to 1}(x^{1/3} - 1)/(x^{1/4} - 1)$. Answer: 4/3.

Lecture 15: Worksheet

Checklist

Make a list of the most important definitions and a list of the most important results in this course.

Functions

Make a list of the most 10 most important functions and make sure you can plot them.

Function values

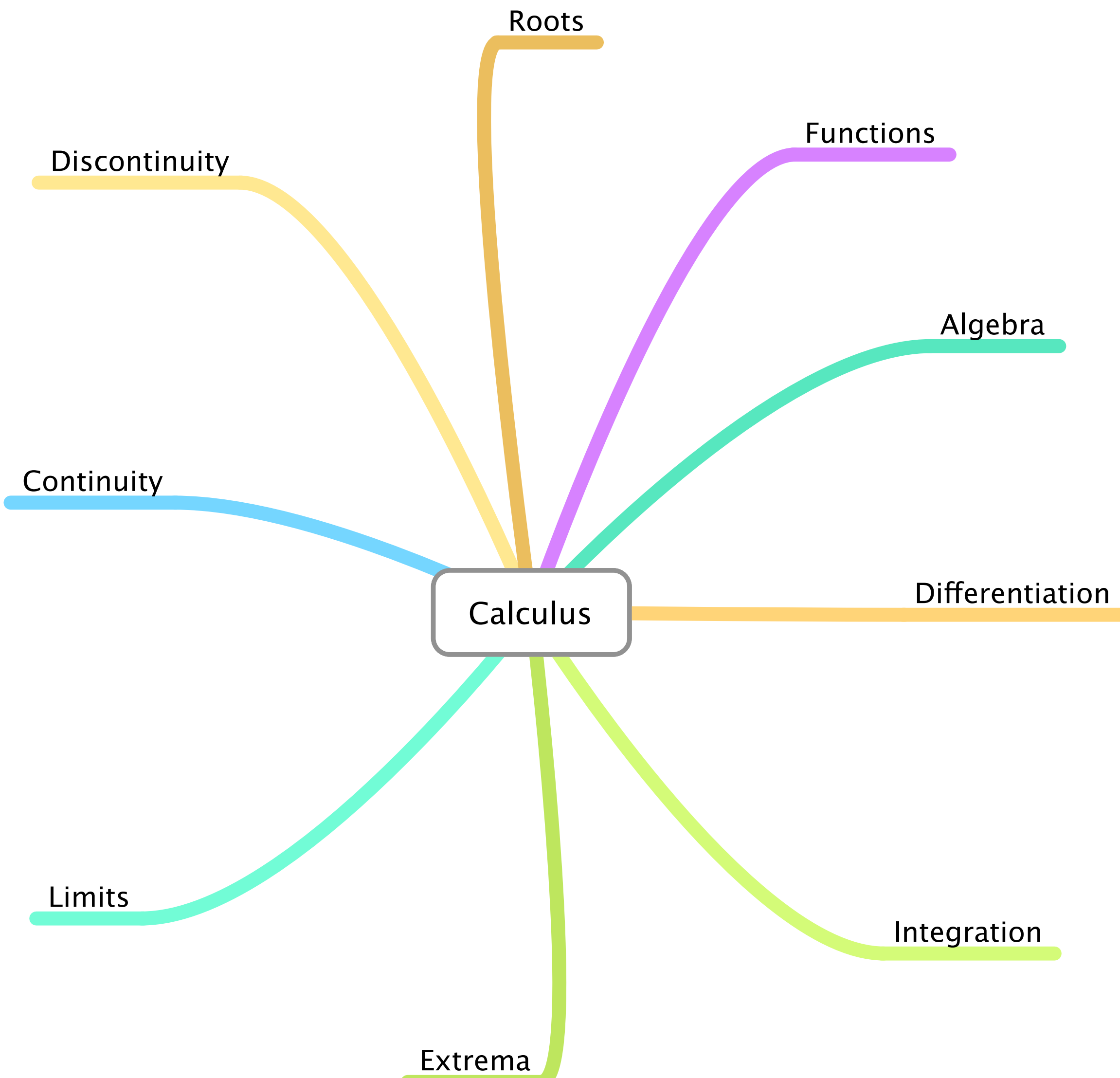
Make sure you know values like $\exp(0), \exp(1), \log(1), \sin(0) \cos(0), \sin(\pi/2),$

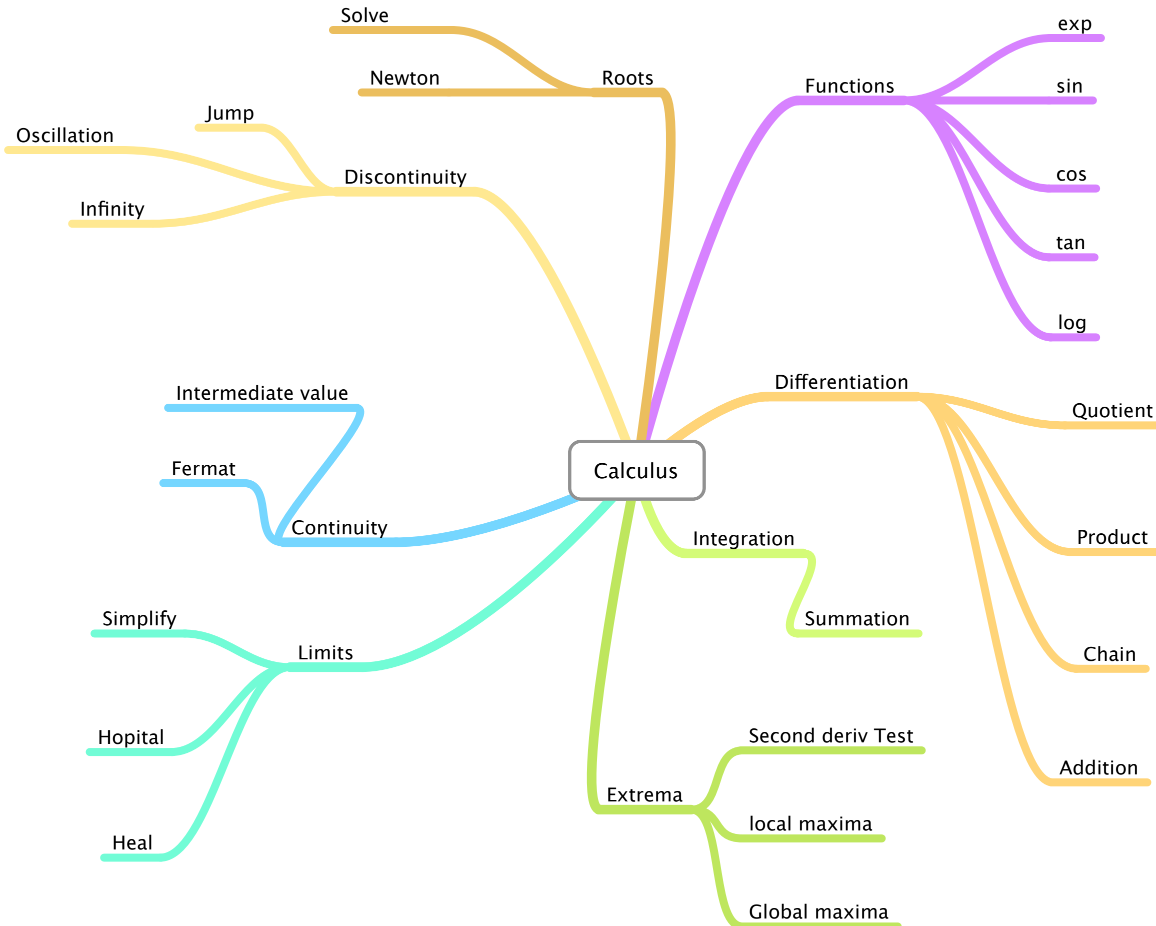
Derivatives

Make sure you know how to derive derivatives of inverse functions like $\arctan(x), \arcsin(x)$ or $\log(x)$.

Mind map

Produce your own mind map of the course. Here are some starting points.





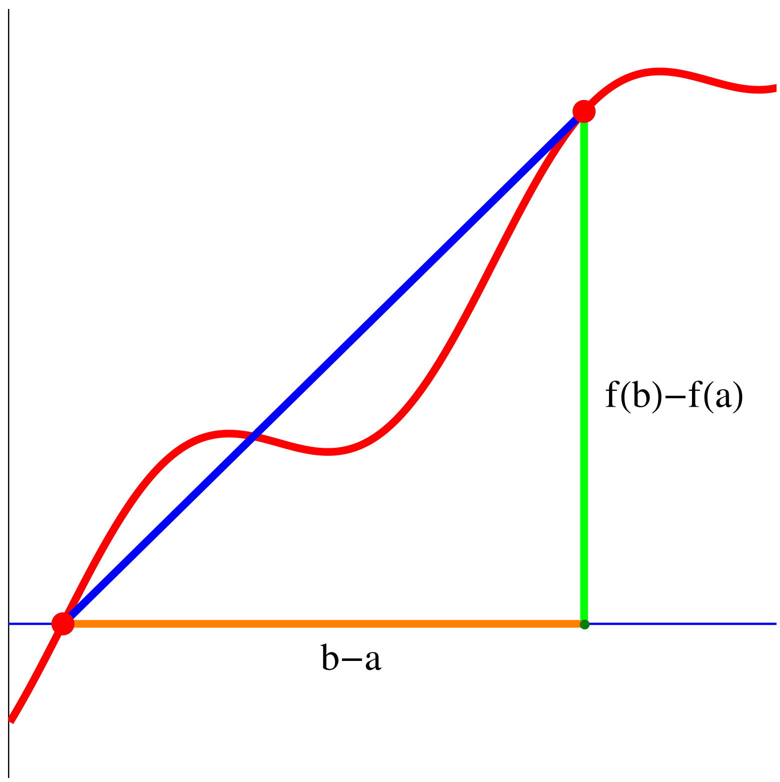
Calculus
Roots
Solve
Newton
Functions
exp
sin
cos
tan
log
Discontinuity
Jump
Oscillation
Infinity
Differentiation
Quotient
Product
Chain
Addition
Continuity
Intermediate value
Fermat
Integration
Summation
Limits
Simplify
Hopital
Heal
Extrema
Second deriv Test
local maxima
Global maxima

Lecture 16: The mean value theorem

In this lecture, we look at the **mean value theorem** and a special case called **Rolle's theorem**. Unlike the intermediate value theorem which applied for continuous functions, the mean value theorem involves derivatives. We assume therefore today that all functions are differentiable unless specified.

Mean value theorem: Any interval (a, b) contains a point x such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} .$$



Here are a few examples which illustrate the theorem:

1 Verify with the mean value theorem that the function $f(x) = x^2 + 4\sin(\pi x) + 5$ has a point where the derivative is 1.
Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1 - 0) = 5$.

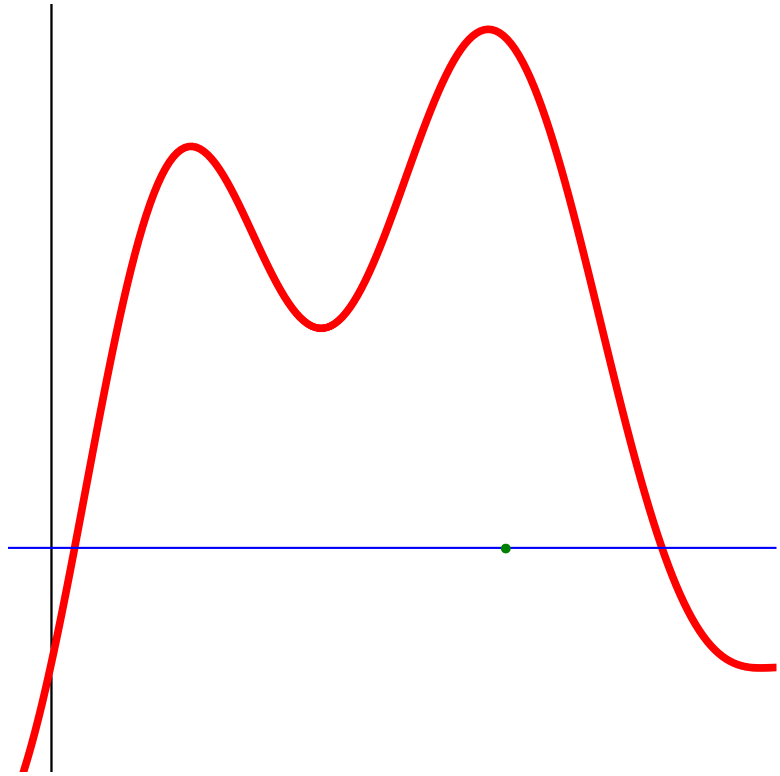
2 Verify that the function $f(x) = 4\arctan(x)/\pi - \cos(\pi x)$ has a point where the derivative is 3.
Solution. We have $f(0) = -1$ and $f(1) = 2$. Apply the mean value theorem.

3 A biker drives with velocity $f'(t)$ at position $f(b)$ at time b and at position a at time a. The value $f(b) - f(a)$ is the distance traveled. The fraction $[f(b) - f(a)]/(b - a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

4 The function $f(x) = \sqrt{1 - x^2}$ has a graph on $(-1, 1)$ on which every possible slope is taken.
Solution: We can see this with the intermediate value theorem because $f'(x) = x/\sqrt{1 - x^2}$ gets arbitrary large near $x = -1$ or $x = 1$. The mean value theorem shows this too because we can take intervals $[a, b] = [-1, -1 + c]$ for which $[f(b) - f(a)]/(b - a) = f(-1 + c)/ci \sim \sqrt{c}/c = 1/\sqrt{c}$ gets arbitrary large.

Proof of the theorem: the function $h(x) = f(a) + cx$, where $c = (f(b) - f(a)/(b-a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are done. By tilting the picture, we have reduced the statement to a special case which is important by itself:

Rolle's theorem: If $f(a) = f(b)$ then f has a critical point in (a, b).



Proof: If it were not true, then either $f'(x) > 0$ everywhere implying $f(b) > f(a)$ or $f'(x) < 0$ implying $f(b) < f(a)$.

Second proof: Fermat's theorem assures a local maximum or local minimum of f exists in (a, b). At this point $f'(x) = 0$.

Recall that if f is continuous and $f(a) = f(b)$ then there is a local maximum or local minimum in the interval (a, b). This applies to every continuous function like $f(x) = |x|$ on $[-1, 1]$ which has a minimum at 0 without that $f'(x)$ exists at 0.

5 There is a point in $[0, 1]$ where $f'(x) = 0$ with $f(x) = x(1 - x^2)(1 - \sin(\pi x))$. **Solution:** We have $f(0) = f(1) = 0$. Use Rolle's theorem.

6 Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$. **Solution:** The function is differentiable and nonnegative. It is zero at $0, \pi$. By Rolle's theorem, there is a critical point. Remark. We can not use Rolle's theorem to show that there is a local maximum even so the extremal value theorem assures us that this exist.

7 Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g. [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is 1/2 where $g(1/2) = 21/2 > 0$.]

It is not clear who discovered the **mean value theorem**? **Joseph Louis Lagrange** is a candidate. Also **Augustin Louis Cauchy** is credited for a modern formulation of the theorem.

Joseph Louis Lagrange, 1736-1813.

Augustin Louis Cauchy, 1789-1857.

What about **Michel Rolle**? He lived from 1652 to 1719, mostly in Paris. No picture of him seems available. Rolle also introduced the n'th root notation like when writing the cube root as

$$\sqrt[3]{x} \ .$$

It is still used today even so we prefer to write $x^{1/3}$ to make algebra easier. The identity $\sqrt[3]{x}\sqrt[3]{x^2} = x$ for example can be seen easier without the root symbol with $x^{1/3}x^{2/3} = x^1$.

Homework

1 Let $f(x) = \sin(5x) + \sin(x)$.
a) Use Rolle's theorem to see that f has a critical point on the interval $[0, 2\pi]$.
b) Use the mean value theorem to see that f has a point on $[\pi/2, 3\pi/2]$, where $f'(x) = -4/\pi = 1.2734....$
c) Use the mean value theorem to see that f has a point on $[5\pi/6, 7\pi/6]$, where $f'(x) = -6/\pi = 1.9098...$

2 a) The function $f(x) = 1 - |x|$ satisfies $f(-1) = f(1) = 0$ but there is no point in $(-1, 1)$ where $f'(x) = 0$. Why is this not a counter example to Rolle's theorem?
b) The function $f(x) = 1/x^2$ satisfies $f(-1) = f(1) = 1$ but there is no point in $(-1, 1)$ where $f'(x) = 0$. Why is this not a counter example to Rolle's theorem?

3 We look at the function $f(x) = x^{10} + x^4 - 20x$ on the positive real line. Use the **mean value theorem** on some interval (a, b) to assure the there exists x, where $f'(x) = 500$.

4 Write down the mean value theorem, the intermediate value theorem, the extreme value theorem and the Fermat theorem. Enter in the following table "yes" or "no", if the property is needed.

Property needed?	Mean value	Intermediate value	Extreme value	Fermat
f is continuous				
f is differentiable				
f is positive				

5 Use the mean value theorem to verify the following statement: if a function f has a minimum at a and a maximum at b, then there exists an inflection point between a and b.

Lecture 16: Worksheet

Theorems, Theorems, Theorems

1 Formulate the mean value theorem.

2 Formulate the intermediate value theorem.

3 Formulate the Rolle's theorem.

4 Formulate the extreme value theorem.

5 Formulate Fermat's principle.

6 Recite the second derivative test.

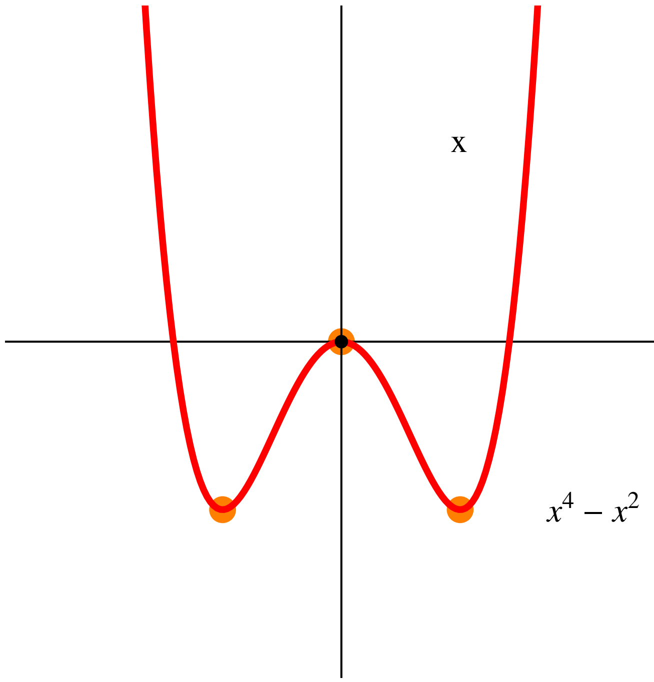
7 What is the fundamental theorem of trigonometry.

Lecture 17: Catastrophes

In this lecture, we look closer at extrema problems. More precisely, we are interested how extrema change when a parameter changes. Nature, economies, processes favor extrema. Extrema change smoothly with parameters. How come that the outcome is often not smooth? What is the reason that political change can go so fast once a tipping point is reached? One can explain this with mathematical models. We look at a simple example, which explains the general principle:

If a local minimum seizes to be a local minimum, a new stable position is favored. This new equilibrium can be far away from the original situation.

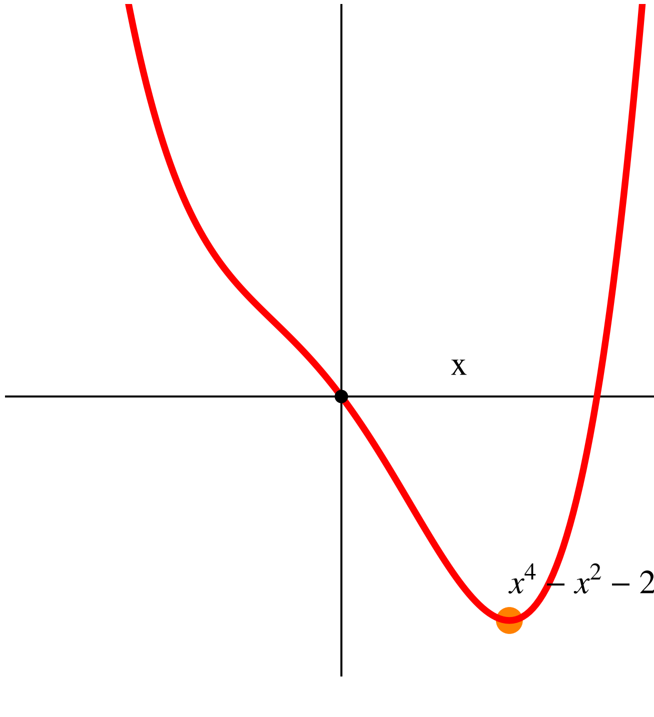
To get started, lets look at an extremal problem



1 Find all the extrema of the function

$$f(x) = x^4 - x^2$$

Solution: $f'(x) = 4x^3 - 2x$ is zero for $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$. The second derivative is $12x^2 - 2$. It is negative for $x = 0$ and positive at the other two points. We have two local minima and one local maximum.



2 Now find all the extrema of the function

$$f(x) = x^4 - x^2 - 2x$$

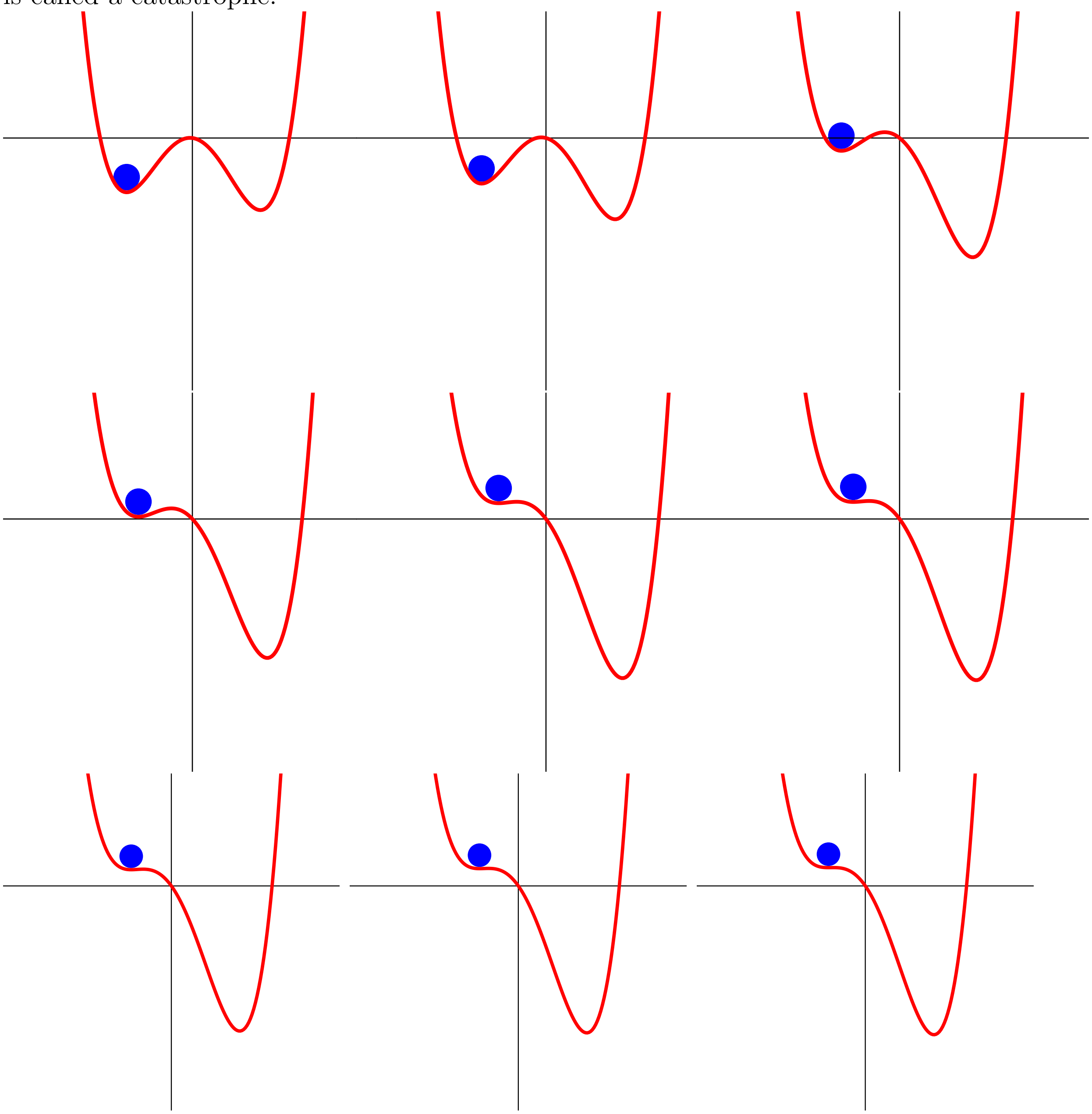
There is only one critical point. It is $x = 1$.

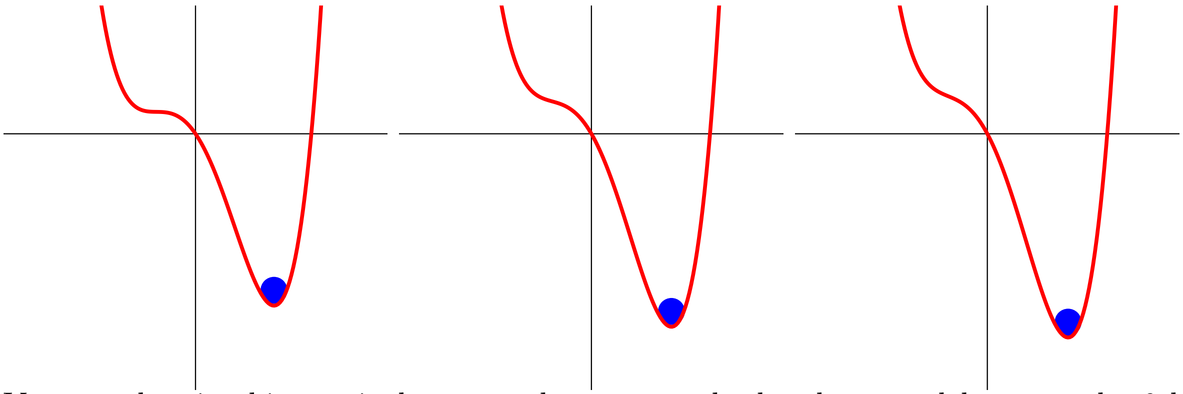
When the first graph was was morphed to the second example, the local minimum to the left has disappeared. Assume the function f measures the prosperity of some kind and c is a parameter. We look at the position of the first critical point of the function. Catastrophe theorists call it the **Delay assumption**:

A **stable equilibrium** is a local minimum of the function. Assume the system depends on a parameter, then the minimum depends on this parameter. It remains a stable equilibrium until it disappears. If that happens, the system settles in a neighboring stable equilibrium.

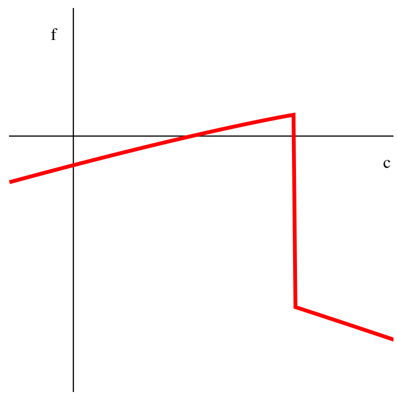
A parameter value c_0 at which a stable minimum disappears, is called a **catastrophe**. In other words, if for $c < c_0$ a different number of local minima exist than for $c > c_0$, then the parameter value c_0 is called a **catastrophe**.

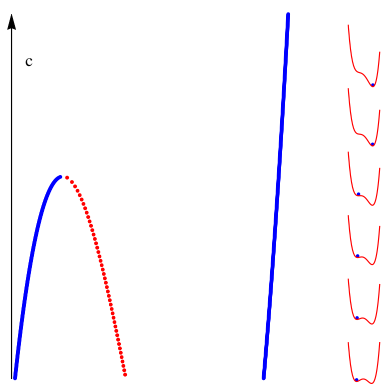
In order to visualize a catastrophe, we draw the graphs of the function $f_c(x)$ for various parameters c and look at the local minima. At a parameter value, where the number of local minima changes, is called a catastrophe.





You see that in this particular case, the catastrophe has happened between the 9th and 10th picture.
Here is the position of the equilibrium point in dependence of c.





A **Bifurcation diagram** displays the equilibrium points as they change in dependence of the parameter c. The vertical axes is the parameter c, the horizontal axes is x. At the bottom for $c = 0$, we have three equilibrium points, two local minima and one local maximum. At the top for $c = 1$ we have only one local minimum.

Principle: Catastrophes often lead to a decrease of the critical value. It is not possible to reverse the process in general.

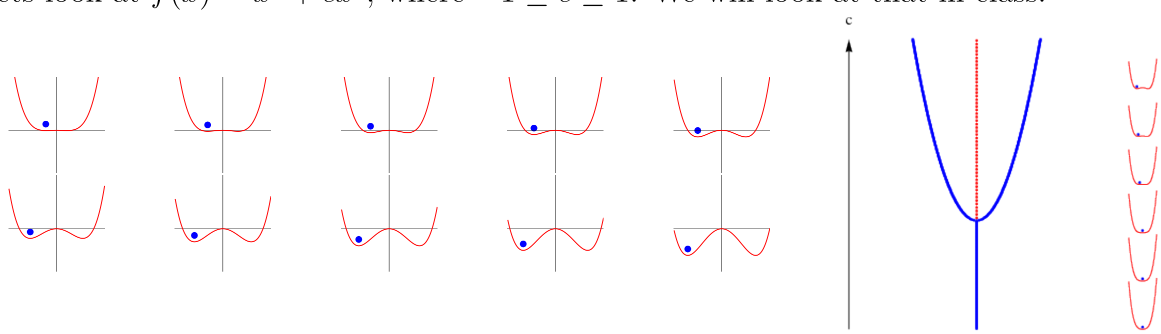
Look again at the above "movie" of graphs. But run it backwards and use the same principle, we do not end up at the position we started with. The new equilibrium remains the equilibrium nearby.

Catastrophes are in general irreversible.

We know this from experience: it is easy to screw up a relationship, get sick, have a ligament torn or lose trust. Building up a relationship, getting healthy or gaining trust usually happens continuously. Ruining the economy of a country or a company or losing a good reputation of a brand is easy but it takes time to regain it.

Local minima can change discontinuously, when a parameter is changed. This can happen with perfectly smooth functions and smooth parameter changes.

3 Lets look at $f(x) = x^4 + cx^2$, where $-1 \leq c \leq 1$. We will look at that in class.



Homework

We study a catastrophe for the function

$$f(x) = x^6 - x^4 + cx^2 \ ,$$

where c is a parameter between 0 and 1.

1 a) Find all the critical points in the case $c = 0$ and analyze their stability. b) Find all the critical points in the case $c = 1$ and analyze their stability.

2 Plot the graph of the function $f(x)$ for 10 values of c between 0 and 1. You can use software, a graphing calculator or Wolfram alpha. Mathematica code is below.

3 If you change from $c = -0.3$ to 0.6, pinpoint the value for the catastrophe and show a rough plot of $c \to f(x_c)$, the value at the first local minimum x_c in dependence of c. The text above provides this graph for an other function. It is the graph with a discontinuity.

4 If you change back from $c = 0.6$ to -0.3 pinpoint the value for the catastrophe. It will be different from the one in the previous question.

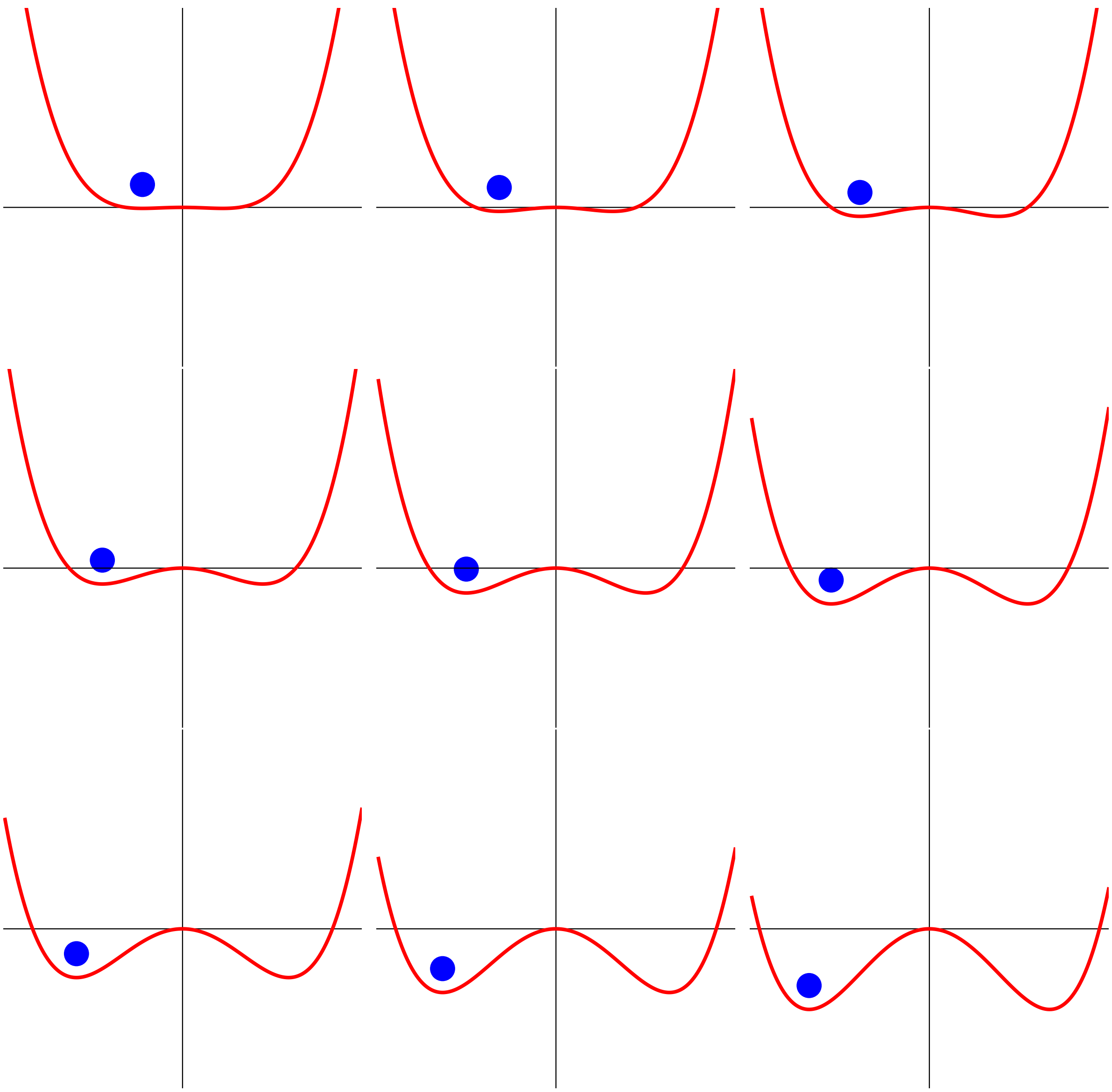
5 Sketch the bifurcation diagram. That is, if $x_k(c)$ is the k'th equilibrium point, then draw the union of all graphs of $x_k(c)$ as a function of c (the c-axes pointing upwards). As in the two example provided, draw the local maximum with dotted lines.

```
Manipulate[ Plot[x^6 - x^4 + c x^2, {x, -1, 1}], {c, 0, 1}]
```

Lecture 17: Worksheet

Catastrophes

We see here graphs of the function $f(x) = x^4 - cx^2$ for c between 0 and 1:



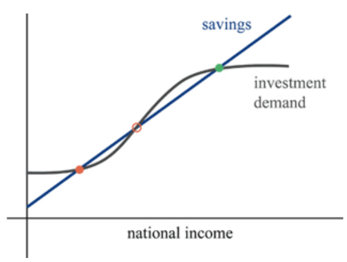
1 Draw the bifurcation diagram in this case. The vertical axes is the c axes.

Dangerous intersections

The following text is from Strogatz (http://opinionator.blogs.nytimes.com/2012/10/08/dangerous-intersection):

"In economics, a similar mechanism underlies the jumps predicted by a model of the business cycle. The idea goes back to **Nicholas Kaldor***, a disciple of Keynes, in the 1940s, and was recast in the framework of catastrophe theory in 1979 by Hal Varian, currently the chief economist at Google. The assumptions behind the model - and their limitations - are explained in detail here and in Varian's elegant paper, so let me focus on the model's tangential intersections and what they suggest about the economy.*

In these diagrams of the economy, the horizontal axis shows the level of economic activity, as reflected by the national income. Values on the far right mean a booming economy, while values on the left mean a sluggish economy like the one we've been in for the past few years. The model says that the economy will be in equilibrium when the supply of funds available for investment matches the demand for such funds. On the graph, equilibrium occurs where the "savings line" crosses the "investment curve."



Depending on how the line and curve are situated, one, two or three intersections can occur. The upper intersection (the green dot) represents a strong economy with high levels of national income. The lower equilibrium (solid red dot) depicts an economy stuck in the doldrums. The middle equilibrium (open red circle) turns out to be unstable and acts like a watershed; when economic conditions are near it, they drift away toward one of the other two equilibriums. Now comes the crucial idea. The amount of investment doesn't depend only on national income but also on how much investment has already been accumulated. At some point enough is enough. During the housing boom, for example, increases in income fueled the demand for housing investment. But as the stock of housing rose, the demand for investment dropped. In the model this drags the S-shaped investment curve down. It's like the straw being added to the camel's back.

As the next animation shows, that little straw - that gradual lowering of the demand for investment - can suddenly tip a strong economy into recession" See http://vimeo.com/49423671."

So much from Strogatz's blog. If you want to try this effect out with Mathematica.

```
Manipulate[Plot[{x,Cos[2x]+x/2+2.4-a},{x,0,6}],{a,0,2}]
```

Lecture 18: Riemann integral

In this lecture, we define the integral $\int_0^x f(t)\, dt$ if f is a differentiable function. We then compute it for some basic functions.

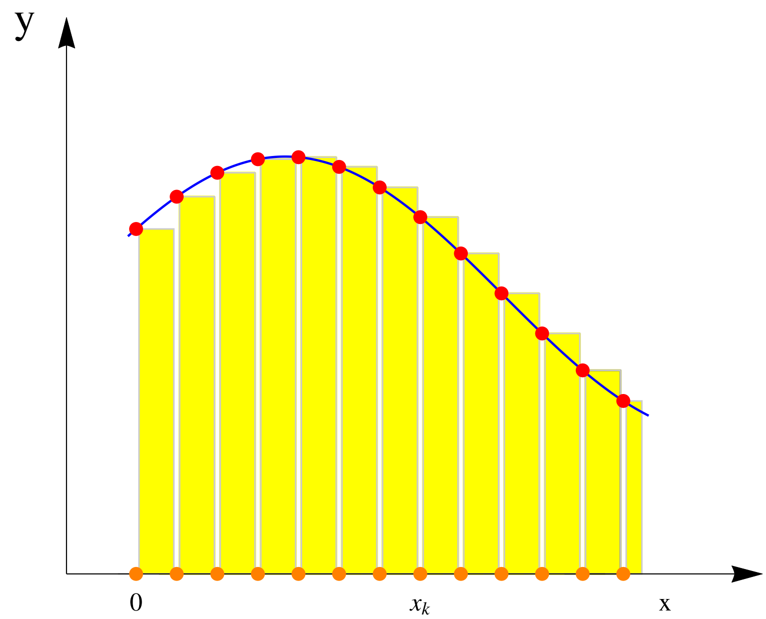
We have previously defined the **Riemann sums**

$$Sf(x) = h[\, f(0) + f(h) + f(2h) + \cdots + f(kh)]\ ,$$

where k is the largest integer such that $kh < x$. Lets write S_n if we want to stress that the parameter $h = 1/n$ was used in the sum. We define the **integral** as the limit of these sums $S_n f$, when the **mesh size** $h = 1/n$ goes to zero.

Define

$$\int_0^x f(t)\, dt = \lim_{n \to 0} S_n f(x)\ .$$



For any differentiable function, the limit exists

Proof: Lets first assume $f \geq 0$ on $[0, x]$. Let M be such that $f \leq M$ and $f' \leq M$ on $[0, x]$. The Riemann sum $S_n f(x)$ is the total area of k rectangles. Let $Sf(x)$ denote the area under the curve. If M is the maximal slope of f on $[0, x]$, then on each interval $[j/n, (j+1)/n]$, we have $|f(x) - f(j/n)| \leq M/n$ so that the area error is smaller than M/n^2. Additionally, we have a piece above the interval $[kh, x]$ with area $\leq M/n$. If we add all the $k \leq xn$ "roof area errors" and the "side area" up, we get

$$Sf(x) - S_n f(x) \leq \frac{kM}{n^2} + \frac{M}{n} \leq \frac{xnM}{n^2} + \frac{M}{n} = \frac{xM + M}{n}\ .$$

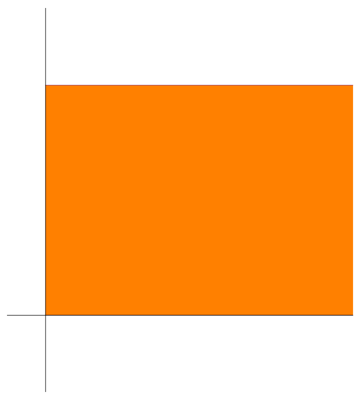
This converges to 0 for $n \to \infty$. The limit is therefore the area $Sf(x)$. For a general, not necessarily nonnegative function, we write $f = g - h$, where g, h are nonnegative (see homework) and have $\int_0^x f(x)\, dx = \int_0^x g(x)\, dx - \int_0^x h(x)\, dx$.

For nonnegative f, the value $\int_0^x f(x)\, dx$ is the **area between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

Remark: the Riemann integral is defined here as the limit $h\sum_{x_k=kh\in[0,x)} f(x_k)$. It converges to the area under the curve for all **continuous** functions but since we work with differentiable functions in calculus we restricted to that. Not **all** bounded functions can be integrated naturally like this. There are discontinuous functions like the **salt and pepper function** which is defined to be $f(x) = 1$ if x is rational and zero else. Now $Sf(x) = 1$ for rational h and $Sf(x) = 0$ if h is irrational. Therefore, an other integral, the **Lebesgue integral** is used too: it can be defined as the limit $\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ where x_k are **random points** in $[0, x]$. This **Monte-Carlo integral** definition of the Lebesgue integral gives the integral 0 for the salt and pepper function because rational numbers have zero probability.

Remark: Some calculus books define the Riemann integral with partitions $x_0 < x_1 < \cdots < x_n$ of points of the interval $[0, x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring x_j goes to zero. The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$, where y_k is arbitrarily chosen inside the interval (x_k, x_{k+1}). For continuous functions, the limiting result is the same the $Sf(x)$ sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated and keep a wide mesh, where the function does not change much. This leads to **numerical analysis** of integrals.

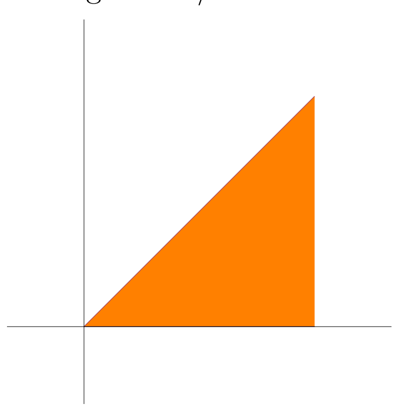
1 Let $f(x) = c$ be constant everywhere. Now $\int_0^x f(t)\ dt = cx$. We can see also that $cnx/n \leq S_n f(x) \leq c(n+1)x/n$.



2 Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Remark: we could also have added up the Riemann sum but thats more painful: for every $h = 1/n$, let k be the largest integer smaller than $xn = x/h$. Then (remember Gauss's punishment?)

$$S_n f(x) = \frac{1}{n}\sum_{j=1}^{k} \frac{cj}{n} = \frac{ck(k+1)/2}{n^2} .$$

Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\ dt = cx^2/2$.

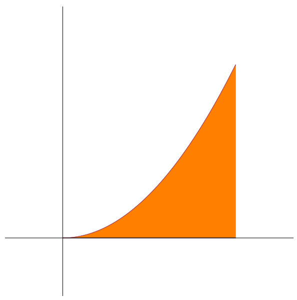


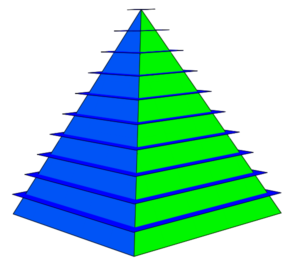
3 Let $f(x) = x^2$. In this case,we can not see the numerical value of the area geometrically. But since we have computed $S[x^2]$ in the first lecture of this course and seen that it is

$[x^3]/3$ and since we have defined $S_h f(x) \to \int_0^x f(t)\, dt$ for $h \to 0$ and $[x^k] \to x^k$ for $h \to 0$, we know that

$$\int_0^x t^2\, dt = \frac{x^3}{3} \; .$$

This example actually computes the **volume of a pyramid** which has at distance t from the top an area t^2 cross section. Think about $t^2 dt$ as a slice of the pyramid of area t^2 and height dt. Adding up the volums of all these slices gives the volume.

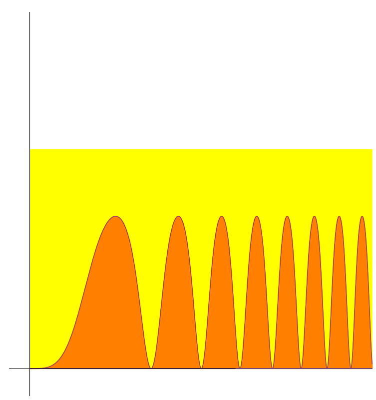




Linearity of the integral (see homework) $\int_0^x f(t)+g(t)\, dt = \int_0^t f(t)\, dt + \int_0^x g(t)\, dt$ and $\int_0^x \lambda f(t)\, dt = \lambda \int_0^x f(t)\, dt$.

Upper bound: If $0 \leq f(x) \leq M$ for all x, then $\int_0^x f(t)\, dt \leq Mx$.

4 $\int_0^x \sin^2(\sin(\sin(t)))/x\, dt \leq x$. **Solution**. The function $f(t)$ inside the interval is nonnegative and smaller or equal to 1 The graph of f is therefore contained in a rectangle of width x and height 1.



We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If f and g satisfy $|f(x) - g(x)| \leq c$, then

$$\int_0^x |f(x) - g(x)|\, dx \leq cx \; .$$

We know identities like $S_n [x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$ already. Since $[x]_h^k - [x]^k \to 0$ we have $S_n[x]_h^k - S_n[x]^k \to 0$ and from $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$.
The other equalities are the same since $\exp_h(x) = \exp(x) \to 0$. This gives us:

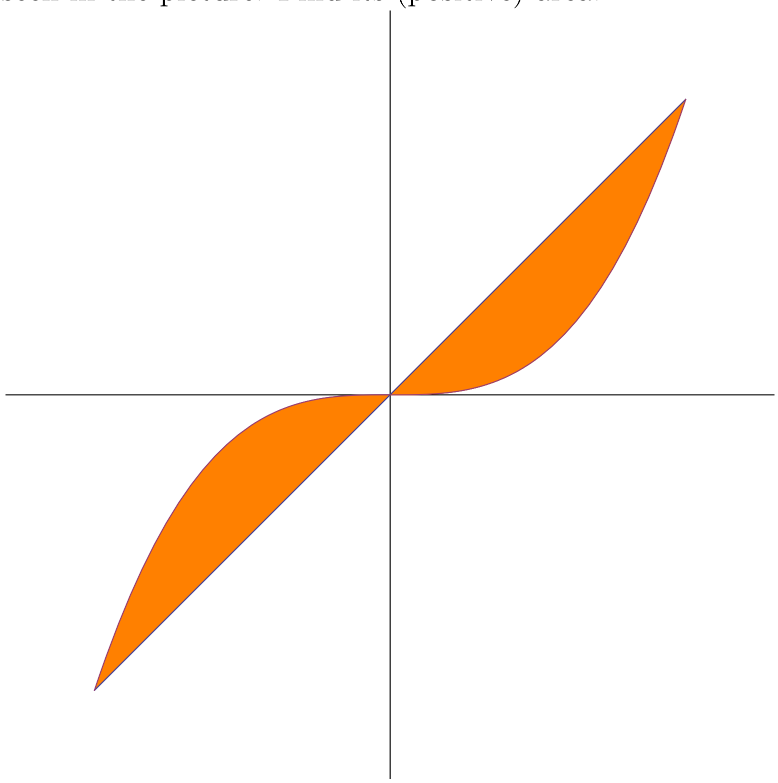
$\int_0^x t^n\, dt = \frac{x^{n+1}}{n+1}$	$\int_0^x \cos(t)\, dt = \sin(x)$
$\int_0^x e^t\, dt = e^x - 1$	$\int_0^x \sin(t)\, dt = 1 - \cos(x)$

Homework

In the following homework you can use that $\boxed{\int_a^b f(x)\ dx = F(b) - F(a)}$ if F is a function which satisfies $F'(x) = f(x)$. We have already verified it for sums.

1 a) What is the integral $\int_1^2 x^6\ dx$?
b) Find the integral $\int_0^1 8t^7 + e^t\ dt$.
c) Calculate $\int_{-1}^1 \frac{1}{1+x^2}\ dx$.
d) Find $\int_0^{\pi/2} \sin^2(t)\ dt$.
For d), use a double angle formula.

2 The region enclosed by the graph of x and the graph of x^3 has a propeller type shape as seen in the picture. Find its (positive) area.



3 Make a geometric picture for each of the following statements:

- $\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$.
- $\int_a^b f(x)\ dx - \int_a^b g(x)\ dx = \int_a^b f(x) - g(x)\ dx$.
- $\int_a^b \lambda f(x)\ dx = \lambda \int_a^b f(x)\ dx$.

4 Here are some more challenging integrals. Maybe you have to guess:
a) $\int_0^1 (3/2)\sqrt{1+x}\ dx$
b) $\int_0^{\sqrt{\log(2)}} 16xe^{-x^2}\ dx$
c) $\int_0^{\pi} \sin^4(x)\ dx$
d) $\int_1^e 5\log(x)/x\ dx$ For c), use double angle formulas, twice.

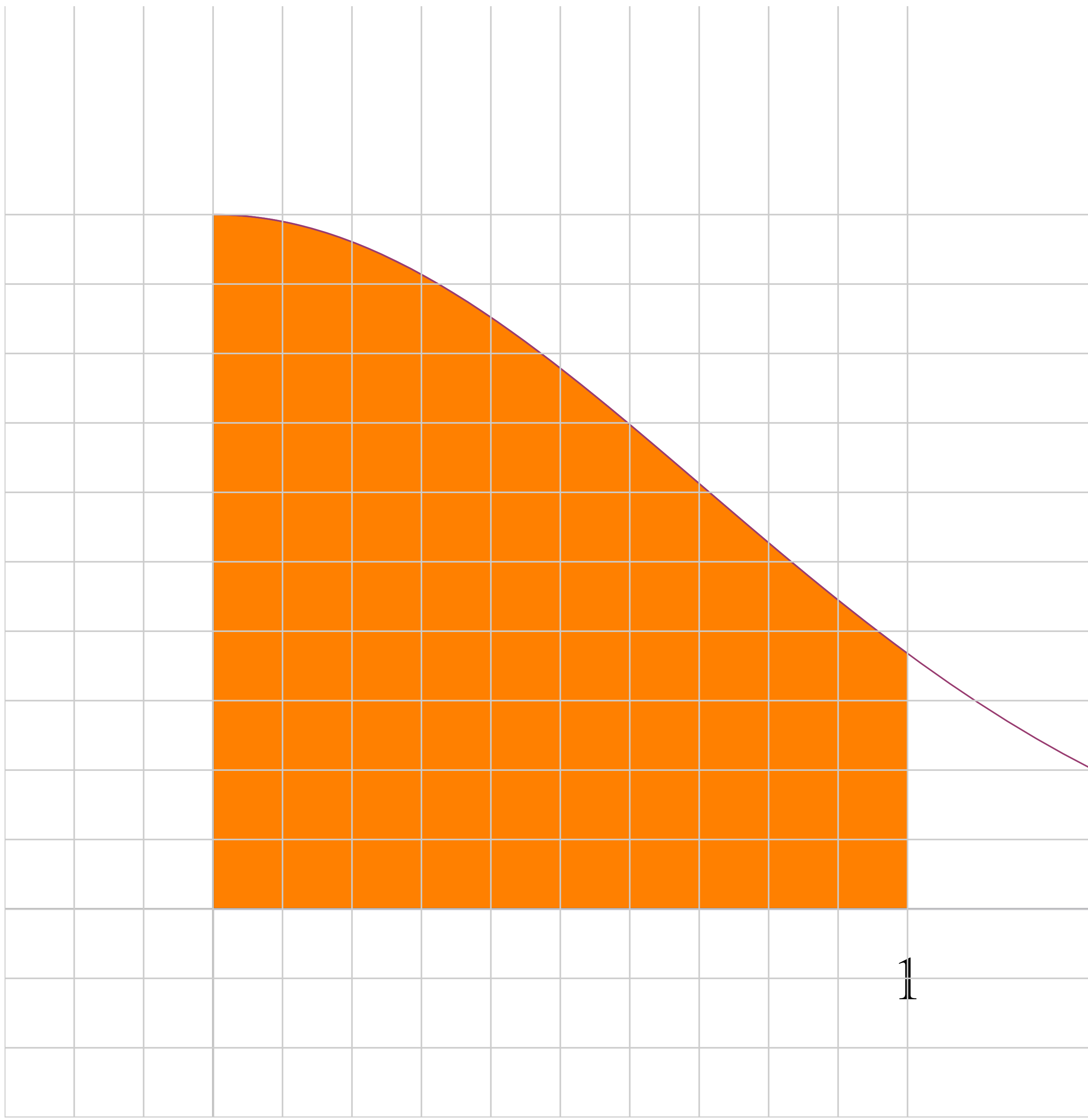
5 In this problem, it is crucial that you plot the function first. Split the integral up into parts.
a) Find $\int_0^3 |x-1|\ dx$. Distinguish cases.
b) Find $\int_0^3 f(x)\ dx$ for $f(x) = |x - |x-1||$. Also here, distinguish cases

Lecture 18: Worksheet

1 The following picture shows the graph of $\exp^{-x^2}$ from 0 to 1. You also see a grid of width $h = 1/10$. Estimate the area

$$\int_0^1 e^{-x^2}\, dx$$

by estimating the Riemann sum.



In the following, you can use that if $f = \frac{d}{dx}F(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)\ .$$

2 $\int_0^2 x^5\,dx$.

3 $\int_{-1}^1 x(1-x)\,dx$.

4 $\int_0^1 e^x\,dx$.

5 $\int_1^2 \frac{1}{x}\,dx$

6 $\int_0^1 \frac{1}{1+x^2}\,dx$

Lecture 19: Fundamental theorem

In this lecture we come back to the **fundamental theorem of calculus** for differentiable functions. This will allow us in general to compute integrals of functions which appear as derivatives. You have already made use of this theorem when doing the homework for today. We have seen earlier that with $Sf(x) = h(f(0) + \cdots + f(kh))$ and $Df(x) = (f(x+h) - f(x))/h$ we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$. This becomes now:

Fundamental theorem of calculus: Assume f is differentiable and f' is continuous. Then

$$\int_0^x f'(t)\, dt = f(x) - f(0) \text{ and } \frac{d}{dx}\int_0^x f(t)\, dt = f(x)$$

Proof. Using notation of Euler, we write $A \sim B$. We say "A and B are close" and mean $A-B \to 0$ for $h \to 0$. From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k+1)h$ because f is continuous. We also know $\int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$ because $Df(t) \sim f'(t)$ uniformly for all $0 \le t \le x$ by the definition of the derivative and the assumption that f' is continuous and using Bolzano on the bounded interval. We also know $SDf(x) = f(x) - f(0)$ for $x = kh$. By definition of the Riemann integral, $Sf(x) \sim \int_0^x f(t)\, dt$ and so $SDf(x) \sim \int_0^x Df(t)\, dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$$

as well as

$$f(x) \sim DSf(x) \sim D\int_0^x f(t)\, dt \sim \frac{d}{dx}\int_0^x f(t)\, dt\ .$$

Bolzano and Weierstrass would write $A \sim B$ using $\epsilon - \delta$ notation: $\forall \epsilon > 0 \exists \delta > 0$ so that $|h| < \delta$ implies $|A - B| < \epsilon$. The $\epsilon - \delta$ notation is still used today but not intuitive. We avoid it thus.

1 $\int_0^5 3x^7\, dx = \frac{x^8}{8}|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

2 $\int_0^{\pi/2} \cos(x)\, dx = \sin(x)|_0^{\pi/2} = 1$.

3 In the following example, the result is in terms of a variable x. To integrate we have to use an other variable: $\int_0^x \sqrt{1+t}\, dt = \int_0^x (1+t)^{1/2}\, dt = (1+t)^{3/2}(2/3)|_0^x = [(1+x)^{3/2} - 1](2/3)$. You have seen that $1+t$ in the interior of the function does not make a big difference. Keep such examples in mind.

4 Also in this example $\int_0^2 \cos(t+1)\, dt = \sin(x+1)|_0^2 = \sin(2) - \sin(1)$, the additional term $+1$ does not make a big dent.

5 $\int_{\pi/6}^{\pi/4} \cot(x)\, dx$. This is an example where the anti derivative is difficult to spot. It is easy if we know where to look for: the function $\log(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.

6 The example $\int_1^2 1/(t^2 - 9)\, dt$ is a bit challenging. We need a hint and write $-6/(x^2-9) = 1/(x+3) - 1/(x-3)$. The function $f(x) = \log|x+3| - \log|x-3|$ has therefore $-6/(x^2-9)$ as a derivative. We know therefore $\int_1^2 -6/(t^2-9)\, dt = \log|3+x| - \log|3-x||_1^2 = \log(5) - \log(1) - \log(4) + \log(2) = \log(5/2)$. The original task is now $(-1/6)\log(5/2)$.

7 $\int_0^x \cos(\sin(x))\cos(x)\, dx = \sin(\sin(x))$ because the derivative of $\sin(\sin(x))$ is $\cos(\sin(x))\cos(x)$. The function $\sin(\sin(x))$ is called the **antiderivative** of f. If we differentiate this function, we get $\cos(\sin(x))\cos(x)$.

8 Find $\int_0^\pi \sin(x)\, dx$. **Solution**: This has a very nice answer.

Here is an important notation, which we have used in the example and which might at first look silly. But it is a handy intermediate step when doing the computation.

$F|_a^b = F(b) - F(a)$.

We give reformulations of the fundamental theorem in ways in which it is mostly used:

If f is the derivative of a function F then

$$\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a) \ .$$

In some textbooks, this is called the "second fundamental theorem" or the "evaluation part" of the fundamental theorem of calculus. The statement $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$ is the "antiderivative part" of the fundamental theorem. They obviously belong together and are two different sides of the same coin.

Here is a version of the fundamental theorem, where the boundaries are functions of x.

Given functions g, h and if F is a function such that $F' = f$, then

$$\int_{h(x)}^{g(x)} f(t)\, dt = F(g(x)) - F(h(x)) \ .$$

9 $\int_{x^4}^{x^2} \cos(t)\, dt = \sin(x^2) - \sin(x^4)$.

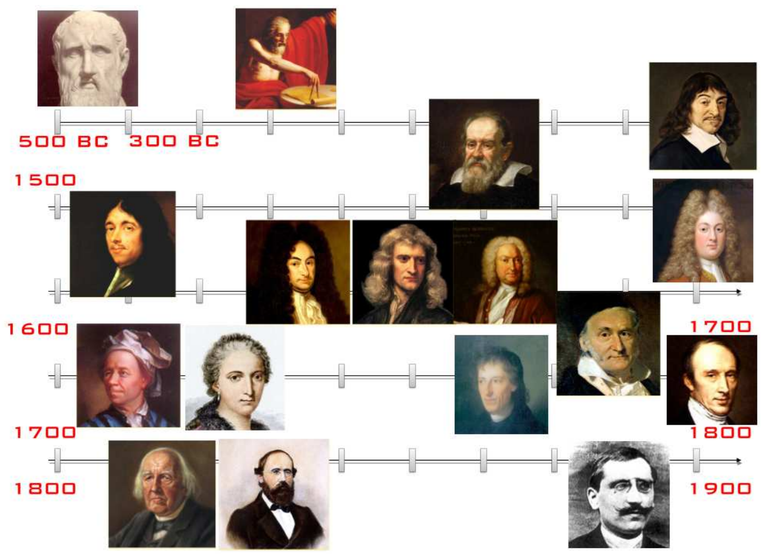
The function F is called an antiderivative. It is not unique but the above formula does always give the right result.

Lets look at a list of important antiderivatives. You should have as many antideriatives "hard wired" in your brain. It really helps. Here are the core functions you should know. They appear a lot.

function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$\frac{-\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x\log(x) - x$

Make your own table!

Meet **Isaac Newton** and **Gottfried Leibniz**. They have discovered the fundamental theorem of calculus. You can see from the expression of their faces, they are not pleased that Oliver has added other calculus pioneers. The sour faces might also have to do with the fact that they have to live forever the same handout with **Austin Powers** and **Doctor Evil**! But thats ok. Celebrities deserve to suffer.



Zeno of Elea 490-430 notion of limit
Democritus 460-370 cone and pyramid
Eudoxus 408-355 BC method of exhaustion
Archimedes 287-212 BC circle and sphere
Johannes Kepler 1571-1630, velocity
Rene Descartes 1596-1650, tangents
Bonaventura Cavalieri 1598-1647
Pierre de Fermat 1601-1665 maxima
John Wallis 1616-1703 infinite series
Christiaan Huygens 1629-1695 waves
Blaise Pascal 1623-1662, triangle
Isaac Barrow 1630-1677 tangents
James Gregory 1638-1675 fund. thm.
Robert Hooke 1635-1703 square law
Isaac Newton 1643-1727 Fluxions
Gottfried Leibniz 1646-1716 notation
Michel Rolle 1652-1719 Rolles theorem
Guillaume de L'Hopital 1661-1704 law
Johann Bernoulli 1667-1748 textbook
Brook Taylor 1685-1731 series, difference
Leonard Euler 1707-1783 Basel problem
Maria Agnesi 1718-1799 textbook
Bernard Bolzano 1781-1848 $\epsilon - \delta$, extrema
Augustin Cauchy 1789-1857, continuity
Karl Weierstrass 1815-1897 foundation
Bernhard Riemann 1826-1866 integral
Henri Lebesgue 1875-1941 integration

Homework

1 For the following integrals $\int f$, find a function F such that $F' = f$, then integrate
a) $\int_0^2 4x^3 + 10x \, dx$.
b) $\int_0^1 6(x+4)^3 \, dx$.
c) $\int_2^3 5/x + 7/(x-1) \, dx$.
d) $\int_0^{\sqrt{\pi}} \cos(x^2)x + \sin(x^2)x \, dx$

2 Find the following integrals by finding a function F satisfying $F' = f$. We will learn techniques to find the function. Here, we just use our knowledge about derivatives:
a) $\int_2^3 5x^4 + 4x^3 \, dx$.
b) $\int_{\pi/4}^{\pi/2} \sin(3x) + \cos(x) \, dx$.
c) $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2(x)} \, dx$.
d) $\int_2^3 \frac{1}{x-1} \, dx$.

3 Evaluate the following integrals:
a) $\int_1^2 2^x \, dx$.
b) $\int_{-1}^1 \cosh(x) \, dx$. (Remember $\cosh(x) = (e^x + e^{-x})/2$.)
c) $\int_0^1 \frac{1}{1+x^2} \, dx$.
d) $\int_{1/3}^{2/3} \frac{1}{\sqrt{1-x^2}} \, dx$.

4 a) Compute $F(x) = \int_0^{x^3} \sin(t) \, dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \, dt$ then find $G'(x)$

5 a) **A clever integral:** Evaluate the following integral:
$\int_0^{2\pi} \sin(\sin(\sin(\sin(\sin(x))))) \, dx$
Explain the answer you get.

b) **An evil integral**: Evaluate $\int_e^{e^e} \frac{1}{\log(x)x} \, dx$.
Hint: Can you figure out a function $f(x)$ which has $1/(\log(x)x)$ as the derivative?

Lecture 19: Worksheet

Fundamental theorem

Find the following integrals

1 $\int_1^3 x^3 \, dx$

2 $\int_{-2}^1 1 - x^5 \, dx$

3 $\int_0^1 \frac{1}{x^2+1} \, dx$

4 $\int_2^4 \frac{1}{x} \, dx$

Now a bit harder ones:

5 $\int_1^4 x^{1/3}\,dx$

6 $\int_0^1 \sqrt{1+x}\,dx$

7 $\int_1^2 \frac{1}{\sqrt{x}} + \frac{5}{x^2}\,dx$

8 $\int_0^e \frac{1}{1+x}\,dx$

Lecture 20: Antiderivatives

The definite integral $\int_a^b f(t)\ dt$ is **signed area under the curve**. The area of the region below the curve is counted in a negative way. There is something else to mention:

For every C, the function $F(x) = \int_0^x f(t)\ dt + C$ is called an **anti-derivative** of g. The constant C is arbitrary and not fixed.

The fundamental theorem of calculus assured us that

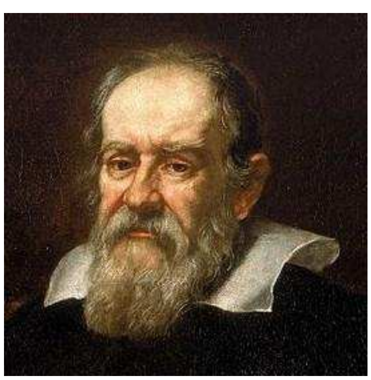
The anti derivative gives us from a function f a function F which has the property that $F' = f$. Two different anti derivatives F differ only by a constant.

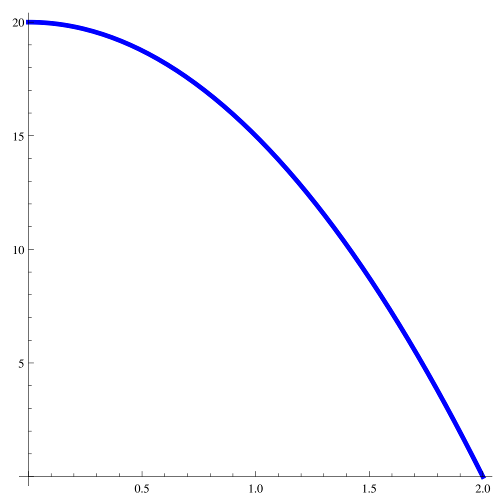
Finding the anti-derivative of a function is in general harder than finding the derivative. We will learn some techniques but it is in general not possible to give anti derivatives for even very simple functions.

1 Find the anti-derivative of $f(x) = \sin(4x) + 20x^3 + 1/x$. Solution: We can take the anti-derivative of each term separately. The anti derivative is $F(x) = -\cos(4x)/4 + 4x^4 + \log(x) + C$.

2 Find the anti derivative of $f(x) = 1/\cos^2(x) + 1/(1-x)$. **Solution:** we can find the anti derivatives of each term separately and add them up. The result is $F(x) = \tan(x) + \log|1-x| + C$.

3 **Galileo** measured **free fall**, a motion with constant acceleration. Assume $s(t)$ is the height of the ball at time t. Assume the ball has zero velocity initially and is located at height $s(0) = 20$. We know that the velocity is $v(t)$ is the derivative of $s(t)$ and the acceleration $a(t)$ is constant equal to -10. So, $v(t) = -10t + C$ is the antiderivative of a. By looking at v at time $t = 0$ we see that $C = v(0)$ is the initial velocity and so zero. We know now $v(t) = -10t$. We need now to compute the anti derivative of $v(t)$. This is $s(t) = -10t^2/2 + C$. Comparing $t = 0$ shows $C = 20$. Now $s(t) = 20 - 5t^2$. The graph of s is a parabola. If we give the ball an additional horizontal velocity, such that time t is equal to x then $s(x) = 20 - 5x^2$ is the visible trajectory. We see that jumping from 20 meters leads to a fall which lasts 2 seconds.





4 The **total cost** is the antiderivative of the **marginal cost** of a good. Both the marginal cost as well as the total cost are a function of the quantity produced. For instance, suppose

the total cost of making x shoes is 300 and the total cost of making $x+4$ shoes is 360 for all x. The marginal cost is $60/4 = 15$ dollars. In general the marginal cost changes with the number of goods. There is additional cost needed to produce one more shoe if 300 shoes are produced. **Problem:** Assume the marginal cost of a book is $f(x) = 5 - x/100$ and that producing the first 10 books costs 1000 dollars. What is the total cost of producing 100 books? **Answer:** The anti derivative $5 - x/100$ of f is $F(x) = 5x - x^2/100 + C$ where C is a constant. By comparing $F(10) = 1000$ we get $50 - 100/100 + C = 1000$ and so $C = 951$. the result is $951 + 5 * 100 - 10'000/100 = 1351$. The average book prize has gone down from 100 to 13.51 dollars.

5 The **total revenue** $F(x)$ is the antiderivative of the **marginal revenue** $f(x)$. Also these functions depend on the quantity x produced. We have $F(x) = P(x)x$, where $P(x)$ is the prize. Then $f(x) = F'(x) = P'(x)x + P$. For a **perfect competitive market**, $P'(x) = 0$ so that the prize is equal to the marginal revenue.

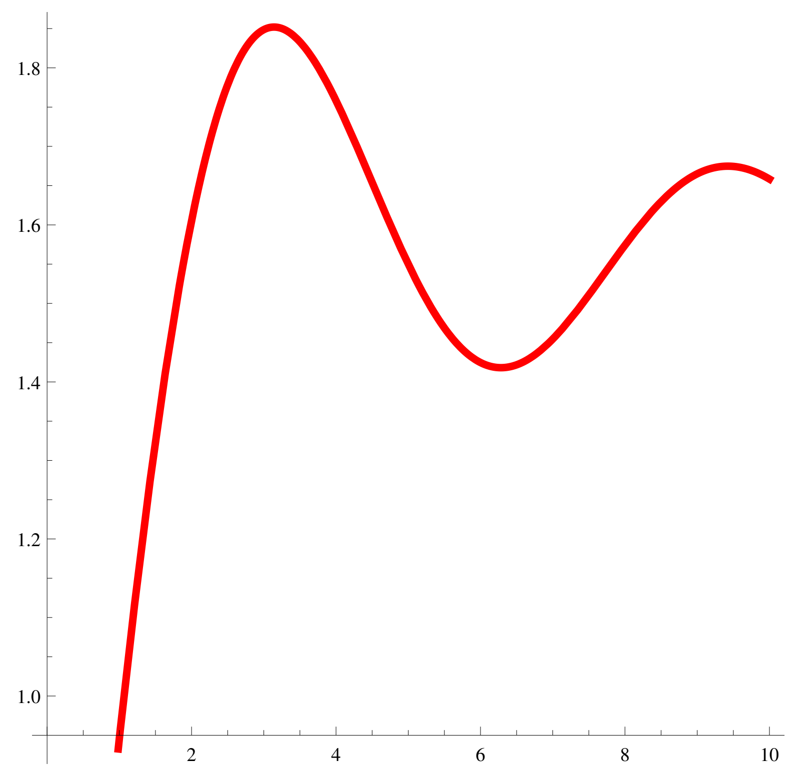
A function f is called **elementary**, if it can be constructed using addition, subtraction, multiplication, division, compositions from polynomials or roots. In other words, an elementary function is built up with functions like $x^3, \sqrt{\cdot}, \exp, \log, \sin, \cos, \tan$ and $\arcsin, \arccos, \arctan$.

6 The function $f(x) = \sin(\sin(\pi + \sqrt{x} + x^2)) + \log(1 + \exp((x^6 + 1)/(x^2 + 1)) + (\arctan(e^x))^{1/3}$ is an elementary function.

7 The anti derivative of the sinc function is called the **sine-integral**

$$Si(x) = \int_0^x \frac{\sin(t)}{t} \, dt \; .$$

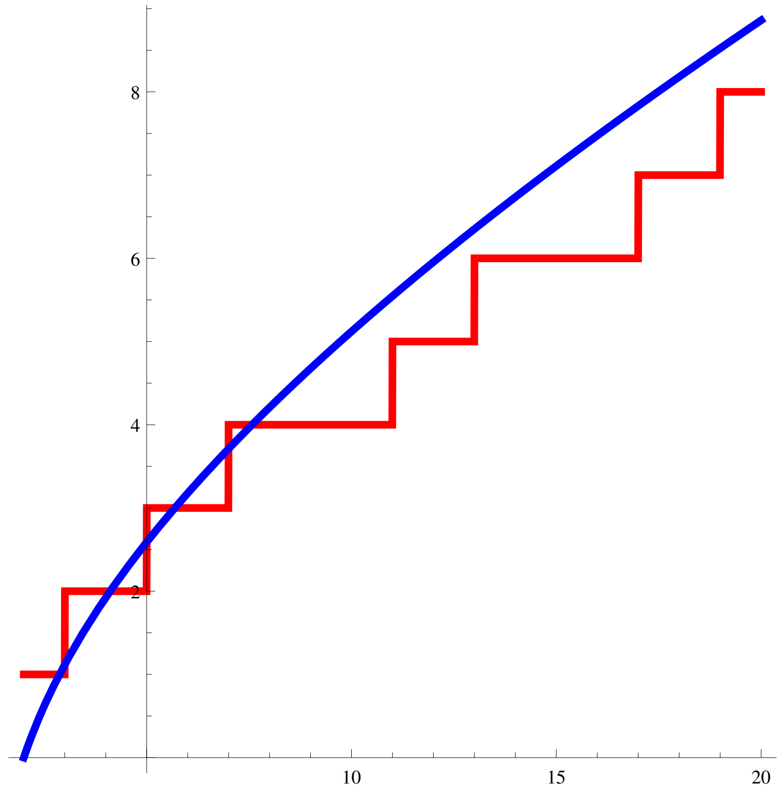
The function $Si(x)$ is not an elementary function.



8 The **offset logarithmic integral** is defined as

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}$$

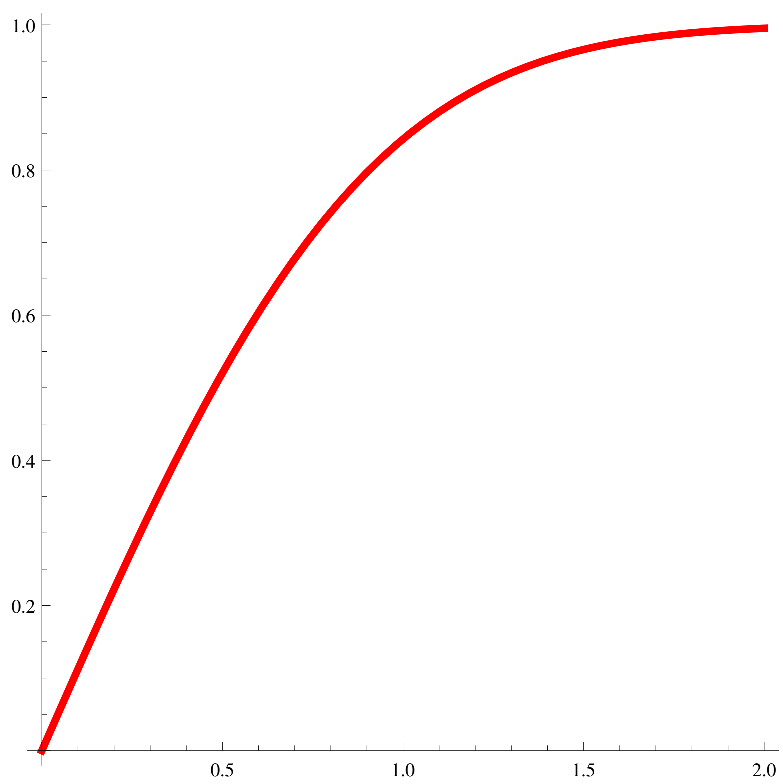
It is a specific anti-derivative. It is a good approximation of the number of prime numbers less than x. The graph below illustrates this. The second stair graph shows the number $\pi(x)$ of primes below x. For example, $\pi(10) = 4$ because $2, 3, 5, 7$ are the only primes below it. The function Li is not an elementary function.



9 The **error function**

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is important in statistics. It is not an elementary function.



The Mathematica command "Integrate" uses about 500 pages of Mathematica code and 600 pages of C code. [1] Before software was doing this, tables of integrals were used. These were thousands of pages thick books contains some integrals, which computer algebra systems have trouble with.

There are other integrals we can do, but Mathematica can not do. Especially definite integrals like $\int_0^{2\pi} \sin(\sin(\sin(x)))\, dx$.

Numerical evaluation

What do we do when we have can not find the integral analytically? We can still compute it numerically. Here is an example: the function $f(x) = \sin(\sin(x))$ also does not have an elementary anti-derivative. But you could compute the integral $\int_0^x f(x)\ dx$ numerically with a computer algebra system like Mathematica:

```
NIntegrate[ Sin[Sin[x]],{x,0,10}]
```

One can approximate such a function also using trigonometric Polynomials and then integrate those. In the case, $\sin(\sin(x))$, the function $0.88 Sin[x] + 0.04 Sin[3x]$ is already very close.

Pillow problems

There is no homework over spring break. Here are some integration riddles to ponder. We will learn techniques to deal with them. If you can not crack them, no problem. Maybe pick one or two and keep thinking about it over spring break. They make also good pillow problems, problems to think about while falling asleep. Try it. Sometimes, you might know the answer in the morning. Maybe you can guess a function which has $f(x)$ as a derivative.

1 $f(x) = \cos(\log(x))/x$.

2 $f(x) = \frac{1}{x^4-1}$.

3 $f(x) = \cot^2(x)$.

4 $f(x) = \cos^4(x)$.

5 $f(x) = \frac{1}{x\log(x)\log(\log(x))}$.

And a very tough one:

6 $f(x) = \cos(\log(x)) + \sin(\log(x))$.

[1] http://reference.wolfram.com/legacy/v3/MainBook/A.9.5.html

Lecture 20: Worksheet

Anti derivatives

Here are some trickier anti derivative puzzles. We still have no integration techniques and must rely on intuition and experiments to find the derivatives.

It is often a puzzle because we can try to combine derivatives of known functions to get the given function.

1 Find the anti-derivative of the function

$$f(x) = \frac{1+x}{1-x}$$

Hint. First compute the anti derivative of

$$g(x) = \frac{1}{1-x} \, .$$

Can you combine g and $\frac{1-x}{1-x}$ in some way to make it fit?

2 Find the anti derivative of the function

$$f(x) = \sin(x^3)3x^2 \ .$$

Hint. Think about the chain rule.

3 Find the anti-derivative of the function

$$f(x) = \sin(\sin(x))\cos(x) \ .$$

Hint. Think about the chain rule.

4 Find the anti-derivative of the function

$$f(x) = 2x\sin(x) + x^2\cos(x) \ .$$

Hint. Think about the product rule.

5 Find the anti-derivative of the function

$$f(x) = e^{e^{e^{e^{e^{e^x}}}}} \cdot e^{e^{e^{e^{e^x}}}} \cdot e^{e^{e^{e^x}}} \cdot e^{e^{e^x}} \cdot e^{e^x} \cdot e^x \ .$$

Hint. There is no hint.

Lecture 21: Area computation

If $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ is the **area under the graph** of $f(x)$ and above the interval $[a, b]$ on the x axes.

If the function is negative, then $\int_a^b f(x)\,dx$ is negative too and the integral is minus the area below the curve:

Therefore, $\int_a^b f(x)\,dx$ is the difference of the area above the graph minus the area below the graph.

More generally we can also look at areas sandwiched between two graphs f and g.

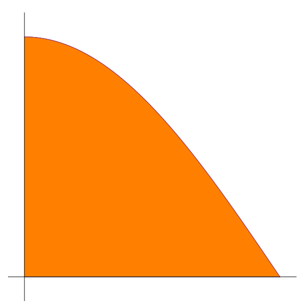
The area of a region G enclosed by two graphs $f \leq g$ and bound by $a \leq x \leq b$ is

$$\int_a^b g(x) - f(x)\,dx$$

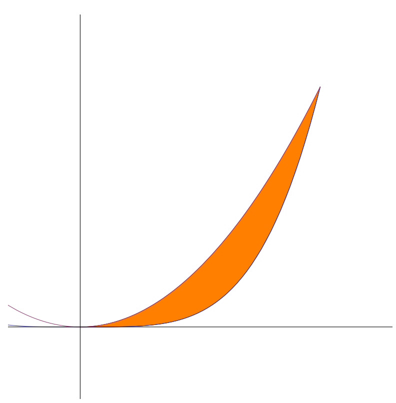
.

Make sure that if you have to compute such an integral that $g \geq f$ before giving it the interpretation of an area.

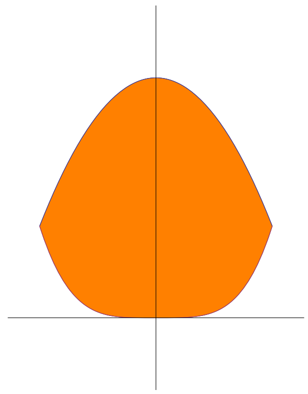
1 Find the area of the region enclosed by the x-axes, the y-axes and the graph of the cos function. **Solution**: $\int_0^{\pi/2} \cos(x)\,dx = 1$.



2 Find the area of the region enclosed by the graphs $f(x) = x^2$ and $f(x) = x^4$.



3 Find the area of the region enclosed by the graphs $f(x) = 1 - x^2$ and $g(x) = x^4$. **Solution**: The intersection points are $\pm(\sqrt{5} - 1)/2$ and called golden ratio. Now it is routine.

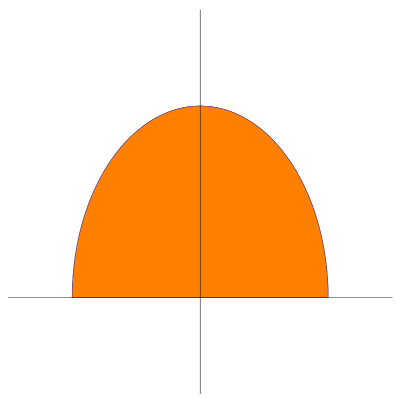


4 Find the area of the region enclosed by a half circle of radius 1. **Solution**: The half circle is the graph of the function $f(x) = \sqrt{1-x^2}$. The area under the graph is

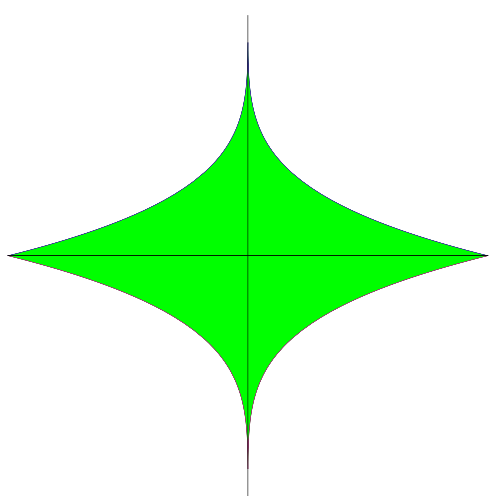
$$\int_{-1}^{1} \sqrt{1-x^2}\; dx \; .$$

Finding the anti-derivative is not so easy. We will find techniques to do so, for now we just are told to look at the derivative of $x\sqrt{1-x^2} + \arcsin(x)$ and see what happens. With this "inspiration", we find the anti derivative to be $(x\sqrt{1-x^2} + \arcsin(x))/2$. The area is therefore

$$\frac{x\sqrt{1-x^2} + \arcsin(x)}{2}|_{-1}^{1} = \frac{\pi}{2} \; .$$

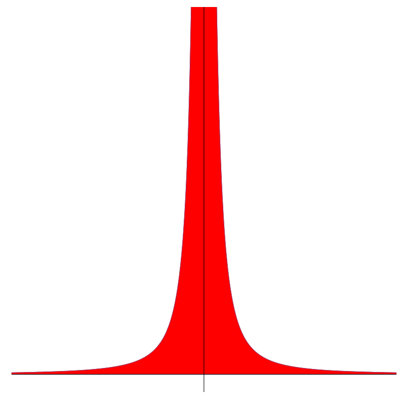


5 Find the area of the region between the graphs of $f(x) = 1-|x|^{1/4}$ and $g(x) = -1+|x|^{1/4}$.

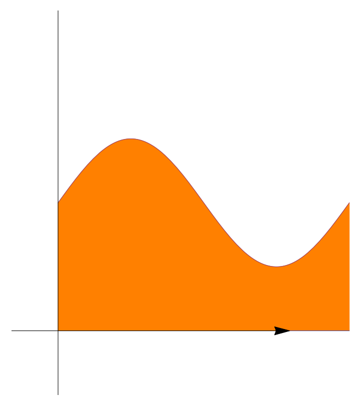


6 Find the area under the curve of $f(x) = 1/x^2$ between -6 and 6. Solution. $\int_{-6}^{6} x^{-2}\; dx = -x^{-1}|_{-6}^{6} = -1/6 - 1/6 = -1/3$. There is something fishy with this computation because

$f(x)$ is nonnegative so that the area should be positive. But we obtained a negative answer. What is going on?



7 Find the area between the curves $x = 0$ and $x = 2 + \sin(y)$, $y = 2\pi$ and $y = 0$. **Solution:** We turn the picture by 90 degrees so that we compute the area under the curve $y = 0, y = 2 + \sin(x)$ and $x = 2\pi$ and $x = 0$.



8 **The grass problem**. Find the area between the curves $|x|^{1/3}$ and $|x|^{1/2}$. **Solution**. This example illustrates how important it is to have a picture. This is good advise for any "word problem" in mathematics.

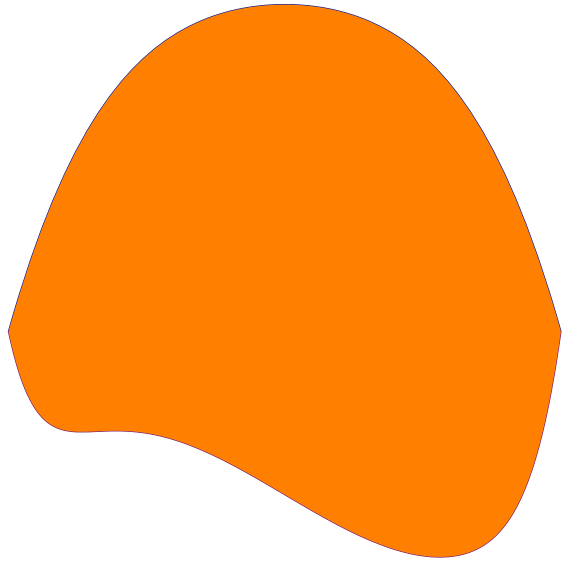
Use a picture of the situation while doing the computation.

Homework

1 Find the area of the bounded region enclosed by the graphs $f(x) = 2x^5 - 24x$ and $g(x) = 4x^2$.

2 Find the area of the region enclosed by the curves $x = 0, x = \pi/2, y = 4 + \sin(5x), y = 2 + \sin^2(2x)$.

3 Find the area of the region enclosed by the graphs $1 - x^4 - x^2 + 1$ and $x^{10} - 1 + x^3 - x$.



4 Find the area of the region enclosed by the four lines $y = x, y = 3 - 2x, y = -2x, y = 3x - 1$.

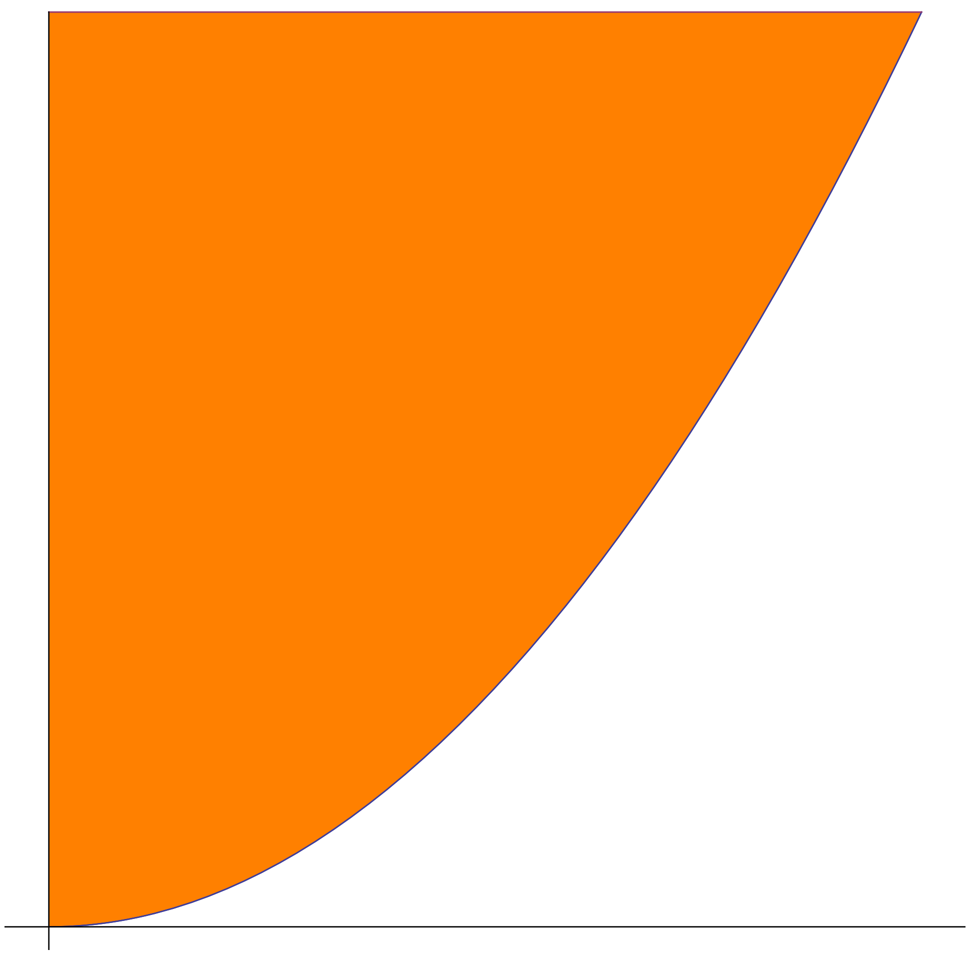
5 Write down an integral which gives the area of the **area 51** region $x^2 + |y|^{51} \leq 1$ by writing the region as a sandwich between two graphs. Evaluate the integral numerically using Wolfram alpha, Mathematica or any other software.

Lecture 21: Worksheet

Area Computation

1 Can you find the area enclosed by the graphs x^2 and $8 - x^2$?

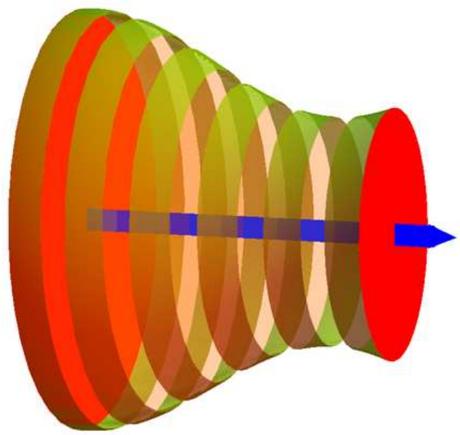
2 Lets compute the area of the region enclosed by the lines $x = 0, x = \sqrt{y}, y = 0$ and $y = 4$. In order to solve such an area problem, we have to draw a picture. We started doing that. Find ways to find the area.



3 Lets compute the area of the region enclosed by the lines $x = 0, x = y^2, y = 0$ and $y = 2$. Now its your turn to draw a picture and compute the area.

Lecture 22: Volume computation

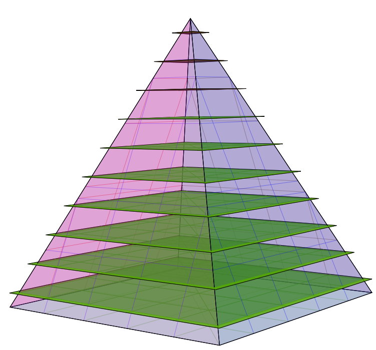
To compute the volume of a solid, we cut it into slices, where each slice is perpendicular to a given line x. If $A(x)$ is the area of the slice and the body is enclosed between a and b then $V = \int_a^b A(x)\, dx$ is the volume. Think of $A(x)dx$ as the volume of a slice. The integral adds them up.



1 Compute the volume of a pyramid with square base length 2 and height 2. **Solution**: we can assume the pyramid is built over the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. The cross section area at height h is $A(h) = (2-h)^2$. Therefore,

$$V = \int_0^2 (2-h)^2 \, dh = \frac{8}{3} \, .$$

This is base area 4 times height 2 divided by 3.



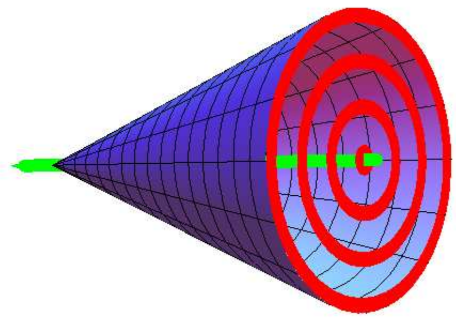
A **solid of revolution** is a surface enclosed by the surface obtained by rotating the graph of a function $f(x)$ around the x-axis.

The area of the cross section at x of a solid of revolution is $A(x) = \pi f(x)^2$. The volume of the solid is $\int_a^b \pi f(x)^2 \, dx$.

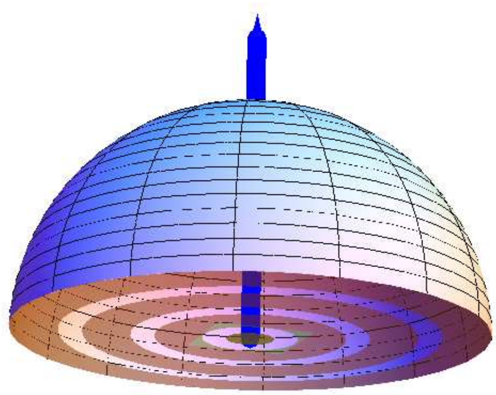
2 Find the volume of a round cone of height 2 and where the circular base has the radius 1. **Solution**. This is a solid of revolution obtained by rotation the graph of $f(x) = x/2$ around the x axes. The area of a cross section is $\pi x^2/4$. Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4 \; dx = \frac{x^3}{4 \cdot 3}|_0^2 = \frac{2\pi}{3} \; .$$

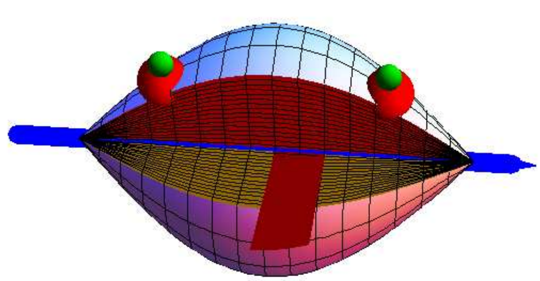
This is the height 2 times the base area π divided by 3.



3 Find the volume of a half sphere of radius 1. **Solution:** The area of the cross section at height h is $\pi(1 - h^2)$.



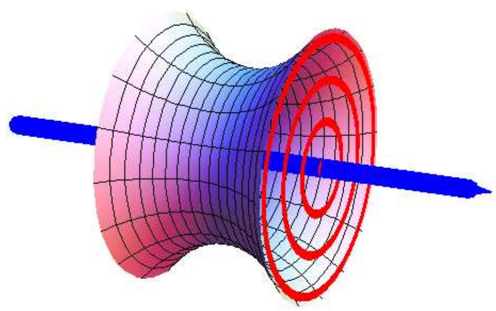
4 We rotate the graph of the function $f(x) = \sin(x)$ around the x axes. But now we cut out a slice of $60 = \pi/3$ degrees out. Find the volume of the solid.
Solution: The area of a slice without the missing piece is $\pi \sin^2(x)$. The integral $\int_0^\pi \sin^2(x) \; dx$ is $\pi/2$ as derived in the lecture. Having cut out 1/6'th the area is $(5/6)\pi \sin^2(x)$. The volume is $\int_0^\pi (5/6)\pi \sin(x)^2 \; dx = (5/6)\pi^2/2$.



Homework

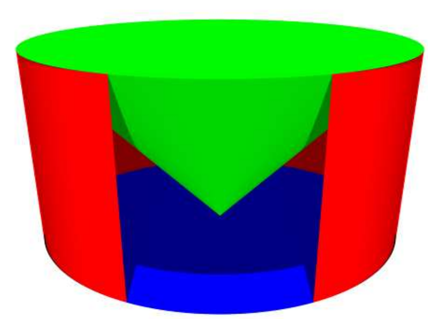
1 Find the volume of the paraboloid for which the radius at position x is $4-x^2$ and x ranges from 0 to 2.

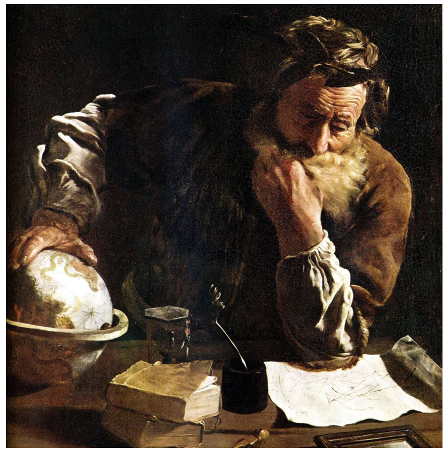
2 A **catenoid** is the surface obtained by rotating the graph of $f(x) = \cosh(x)$ around the x-axes. We have seen that the graph of f is the chain curve, the shape of a hanging chain. Find the volume of of the solid enclosed by the catenoid between $x = -2$ and $x = 2$.
Hint. You might want to check first the identity $2\cosh(x)^2 = 1 + \cosh(2x)$ using the definition $\cosh(x) = (\exp(x) + \exp(-x))/2$.



3 A **tomato** is given by $z^2 + x^2 + 4y^2 = 1$. If we slice perpendicular to the y axes, we get a circular slice $z^2 + x^2 \leq 1 - 4y^2$ of radius $\sqrt{1-4y^2}$.
a) Find the area of this slice.
b) Determine the volume of the tomato.
c) Fix yourself a tomato, cuecumber etc salad as we did in class. Add dressing and staple to to your homework paper as proof that you did it.

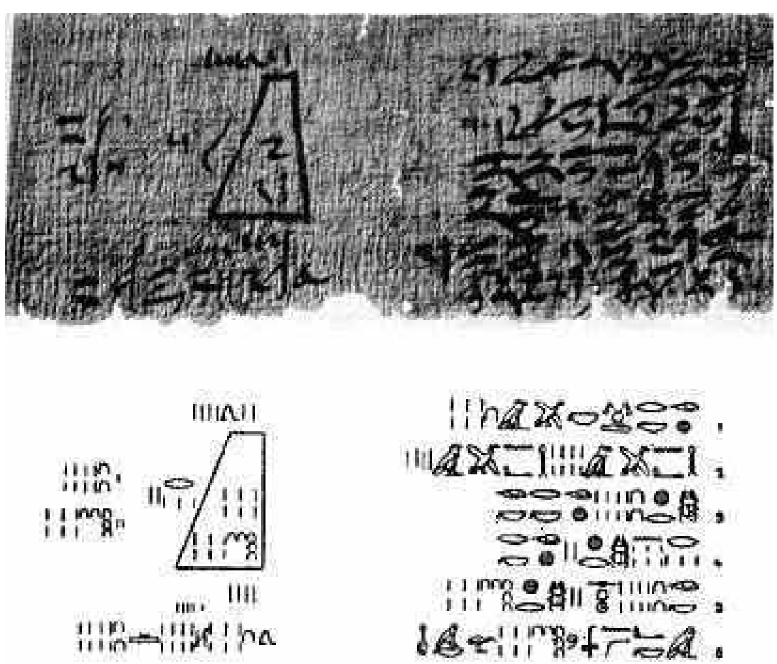
4 As we have seen in the movie of the first class, **Archimedes** was so proud of his formula for the volume of a sphere that he wanted the formula on his tomb stone. He wrote the volume of a half sphere of radius 1 as the difference between the volume of a cylinder of radius 1 and height 1 and the volume of a cone of base radius 1 and height 1. Relate the cross section area of the cylinder-cone complement with the cross section area of the sphere to recover his argument! If stuck, draw in the sand, soak in the bath tub or eat your tomato salad. No credit is given for streaking and screaming "Eureka".





5 Volumes were among the first quantities, Mathematicians wanted to measure and compute. One problem on **Moscow Egypt papyrus** dating back to 1850 BC explains the general formula $h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h.
a) Verify that if you slice the frustrum at height z, the area is $(a + (b - a)z/h)^2$.
b) Find the volume using calculus.
Here is the translated formulation from the papyrus: [1] [2]

"You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4 result 16. You are to double 4 result 8. You are to square 2, result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one-third of 6 result 2. You are to take 28 twice, result 56. See it is 56. You will find it right".



[1] Howard Eves, Great moments in mathematics, Volume 1, MAA, Dolciani Mathematical Expositions, 1980, page 10
[2] Image Source: http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian_papyri.html

Lecture 22: Worksheet

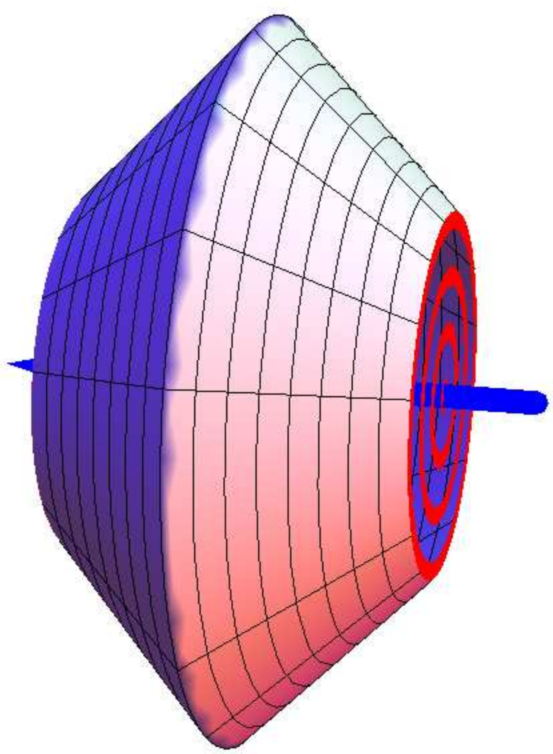
Volume Computation

1 Find the volume of the solid that is formed by rotating the graph of $y = 3x^2$ around the x-axis, for $1 \leq x \leq 2$.

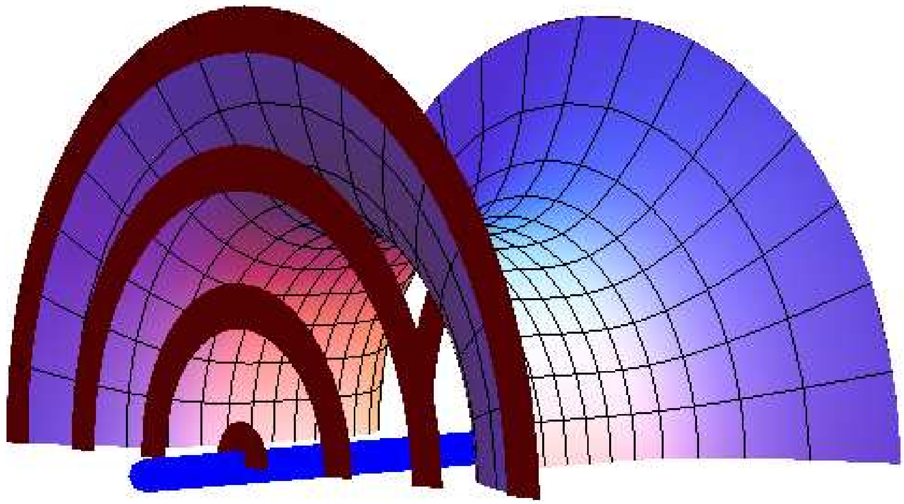
2 Find the volume of the solid that is formed by rotating the graph of $y = x^2$ around the y-axis, for $1 \leq y \leq 3$.

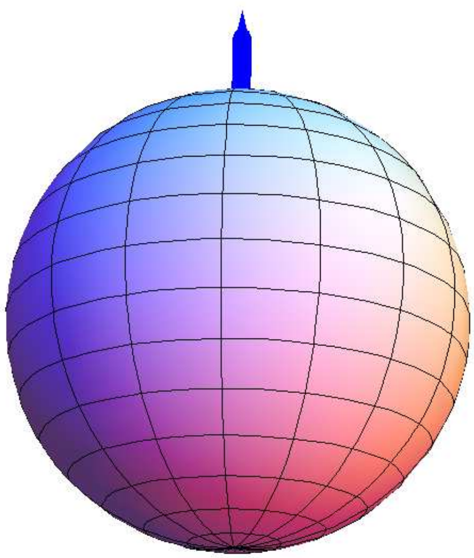
3 Derive the formula for the volume of a sphere ($\frac{4}{3}\pi r^3$).

4 Find the volume of the solid of revolution for which the radius at height z is $2 - |z|$ and $-1 \leq z \leq 1$.



5 The solid of revolution for which the radius at position x is x^4+1 and $x \in [-2,2]$ is taken only above the xy plane as in the picture. Find the volume.





Lecture 23: Improper integrals

In this lecture, we look at integrals on infinite intervals or integrals, where the function can get infinite at some point. These integrals are called **improper integrals**. The area under the curve can remain finite or become infinite.

1 What is the integral

$$\int_1^\infty \frac{1}{x^2}\, dx \ ?$$

Since the anti-derivative is $-1/x$, we have

$$\frac{-1}{x}|_1^\infty = -1/\infty + 1 = 1 \ .$$

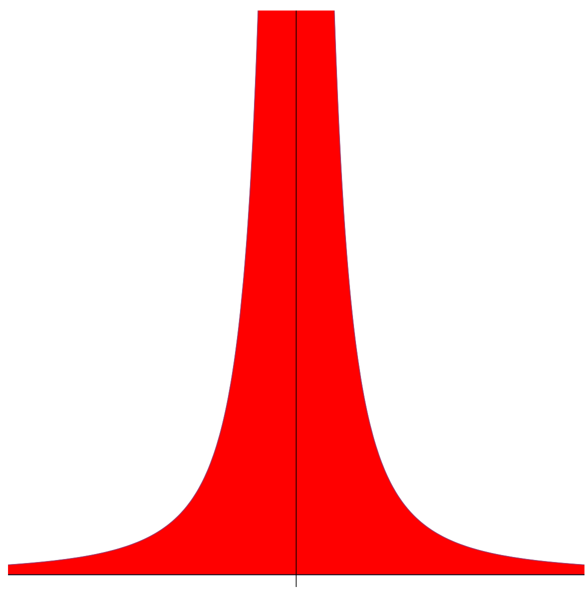
To justify this, compute the integral $\int_1^b 1/x^2 dx = 1 - 1/b$ and see that in the limit $b \to \infty$, the value 1 is achieved.

In a previous lecture, we have seen a chocking example similar to the following one:

2

$$\int_{-1}^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_{-1}^1 = -1 - 1 = -2 \ .$$

This does not make any sense because the function is positive so that the integral should be a positive area. The problem is this time not at the boundary $-1, 1$. The sore point is $x = 0$ over which we have carelessly integrated over.



The next example illustrates the problem with the previous example better:

3 The computation

$$\int_0^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_0^1 = -1 + \infty \ .$$

indicates that the integral does not exist. We can justify by looking at integrals

$$\int_a^1 \frac{1}{x^2}\, dx = -\frac{1}{x}|_a^1 = -1 + \frac{1}{a}$$

which are fine for every $a > 0$. But this does not converge for $a \to 0$.

Do we always have a problem if the function goes to infinity at some point?

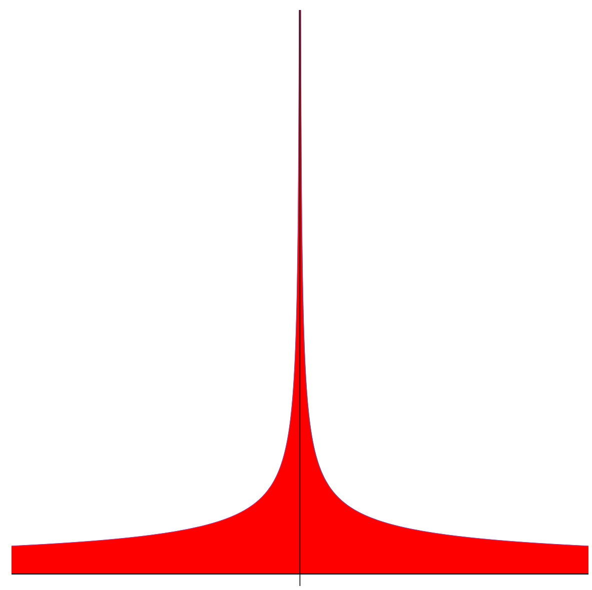
4 Find the following integral

$$\int_0^1 \frac{1}{\sqrt{x}} \, dx \; .$$

Solution: Since the point $x = 0$ is problematic, we integrate from a to 1 with positive a and then take the limit $a \to 0$. Since $x^{-1/2}$ has the antiderivative $x^{1/2}/(1/2) = 2\sqrt{x}$, we have

$$\int_a^1 \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x}|_a^1 = 2\sqrt{1} - 2\sqrt{a} = 2(1 - \sqrt{a}) \; .$$

There is no problem with taking the limit $a \to 0$. The answer is 2. Even so the region is infinite its area is finite. This is an interesting example. Imaging this to be a container for paint. We can fill the container with a finite amount of paint but the wall of the region has infinite length.



5 Evaluate the integral $\int_0^1 1/\sqrt{1-x^2} \, dx$. **Solution:** The antiderivative is $\arcsin(x)$. In this case, it is not the point $x = 0$ which produces the difficulty. It is the point $x = 1$. Take $a > 0$ and evaluate

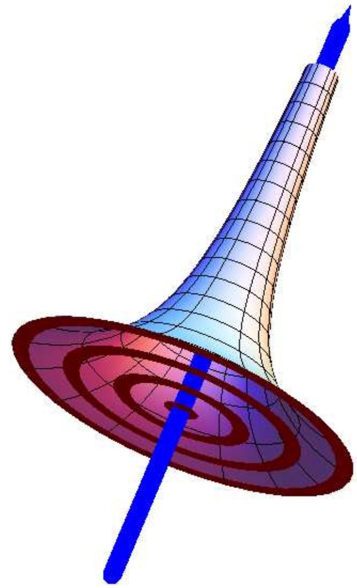
$$\int_0^{1-a} \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x)|_0^{1-a} = \arcsin(1-a) - \arcsin(0) \; .$$

Now $\arcsin(1-a)$ has no problem at limit $a \to 0$. Since $\arcsin(1) = \pi/2$ exists. We get therefore the answer $\arcsin(1) = \pi/2$.

6 Rotate the graph of $f(x) = 1/x$ around the x-axes and compute the volume of the solid between 1 and ∞. The cross section area is π/x^2. If we look at the integral from 1 to a fixed R, we get

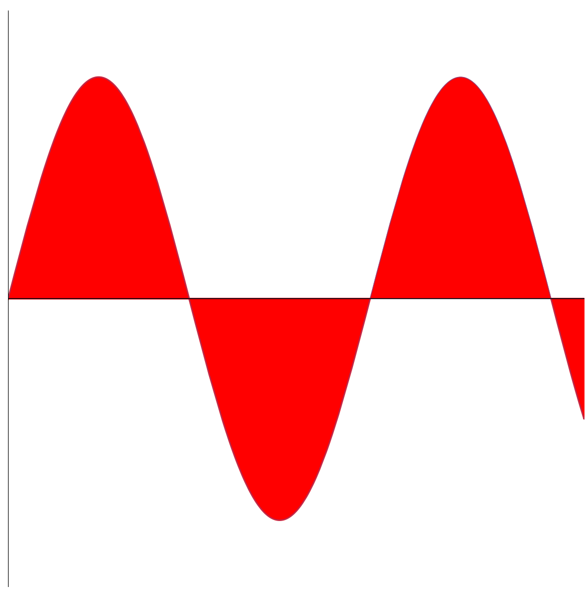
$$\int_1^R \frac{\pi}{x^2} \, dx = -\frac{\pi}{x}|_1^R = -\pi/R + \pi \; .$$

This converges for $R \to \infty$. The volume is π. This famous solid is called **Gabriels trumpet**. This solid is so prominent because if you look at the surface area of the small slice, then it is larger than $dx 2\pi/x$. The total surface area of the trumpet from 1 to R is therefore larger than $\int_1^R 2\pi/x \, dx = 2\pi(\log(R) - \log(1))$. which goes to infinity. We can **fill** the trumpet with a finite amount of paint but we can not **paint** its surface.



Finally, lets look at the following example

7 Evaluate the integral $\int_0^\infty \sin(x)\, dx$. **Solution.** There is no problem at the boundary 0 nor at any other point. We have to investigate however, what happens at ∞. Therefore, we look at the integral $\int_0^b \sin(x)\, dx = -\cos(x)|_0^b = 1 - \cos(b)$. We see that the limit $b \to \infty$ does not exist. The integral fluctuates between 0 and 2.

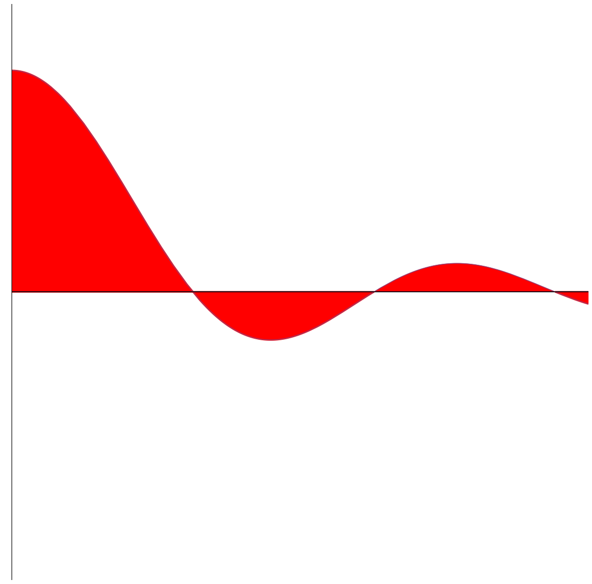


The next example leads to a topic in a follow-up course. It is not covered here, but could make you curious:

8 What about the integral

$$I = \int_0^\infty \frac{\sin(x)}{x}\, dx \ ?$$

Solution. The anti derivative is the Sine integral $Si(x)$ so that we can write $\int_0^b \sin(x)/x\, dx = Si(b)$. It turns out that the limit $b \to \infty$ exists and is equal to $\pi/2$ but this is a topic for a second semester course like Math 1b. The integral can be written as an alternating series, which converges and there are many ways to compute it: [1]



Lets summarize the two cases of improper integrals: infinitely long intervals and a point where the function becomes infinite.

1) To investigate the improper integral $\int_a^\infty f(x)\,dx$ we look at the limit $\int_a^b f(x)\,dx$ for $b \to \infty$.
1) To investigate improper integral $\int_0^b f(x)\,dx$ where $f(x)$ is not continuous at 0, we take the limit $\int_a^b f(x)\,dx$ for $a \to 0$.

Homework

1 Evaluate the integral $\int_1^2 \frac{5}{\sqrt{x-1}}\,dx$.

2 Evaluate the following integrals
a) $\int_0^1 2x/\sqrt{1-x^2}\,dx$.
b) $\int_0^1 4/\sqrt{1-x^2}\,dx$.
Hint: For a) think about the chain rule $d/dx f(g(x)) = f'(g(x))g'(x)$

3 Evaluate the integral $\int_{-3}^4 (x^2)^{1/3}\,dx$. To make sure that the integral is fine, check whether $\int_{-3}^0$ and $\int_0^4$ work.

4 The integral $\int_{-2}^1 1/x\,dx$ does not exist. We can however take a positive $b > 0$ and look at

$$\int_{-2}^{-b} 1/x\,dx + \int_b^1 1/x\,dx = \log|b| - \log|-2|) + (\log|1| - \log|b|) = \log(2)\ .$$

This value is called the **Cauchy principal value** of the integral. Find the principal value of

$$\int_{-4}^5 3/x^3\,dx$$

using the same process as before, by cutting out $[-a, a]$ and then taking the limit $a \to 0$.

5 a) Evaluate $\int_{-1}^1 \frac{1}{x^4}\,dx$ blindly, without worrying that the function is infinite.
b) Can we give a principal value integral value to $\int_{-1}^1 \frac{1}{x^4}\,dx$? If yes, find the value. If not, tell why not.

[1] Hardy, Mathematical Gazette, 5, 98-103, 1909.

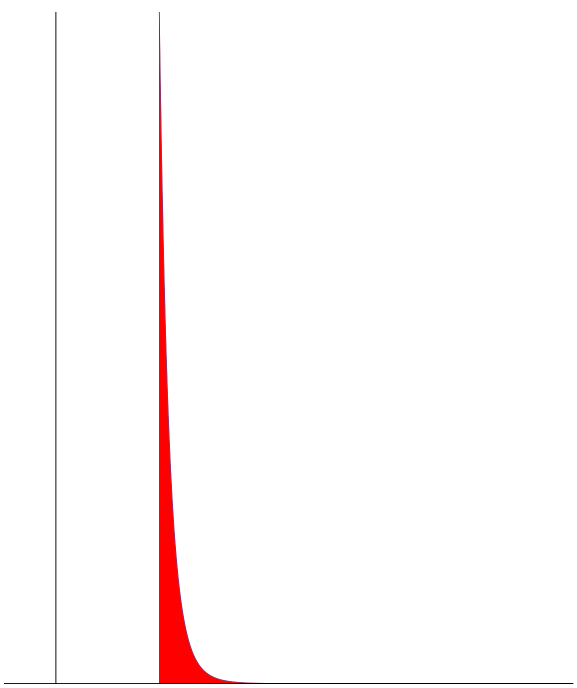
Lecture 23: Worksheet

28 3 2014 (week 12)
23 is the smallest odd prime that is not a prime twin.
365 log(2) is the number of pairs of 23 items
Humans have 23 chromosomes
Michael Jordans Tshirt had the number 23.

Improper integrals

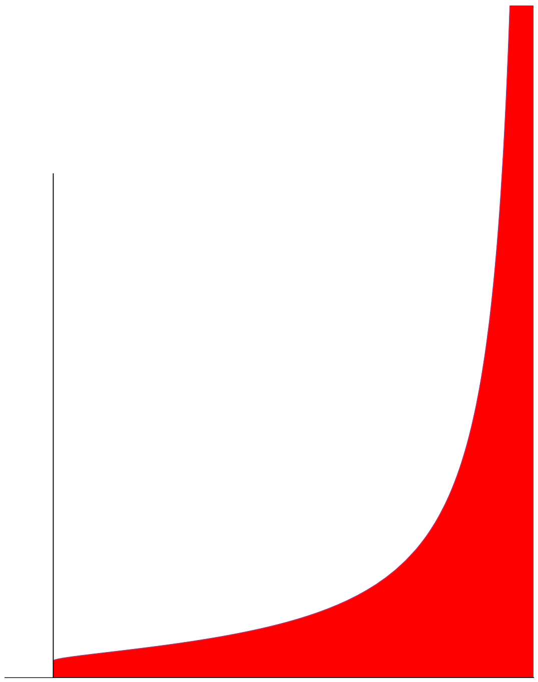
1 Find the value of the improper integral

$$\int_1^\infty \frac{1}{x^{17}}\, dx$$



2 Find the following improper integral

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,.$$



3 We have met the **Maria Agnesi** function

$$f(x) = \frac{1}{1+x^2}$$

early in the course already. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx \ .$$

The function $g(x) = \frac{1}{I}\frac{1}{1+x^2}$ is a probability distribution called **Cauchy distribution**. It is a nonzero function which has the property that $\int_{-\infty}^{\infty} g(x)\, dx = 1$.

Lecture 24: Applications of integration

Here is a list of topics, where integration is used for applications

- the computation of area
- the computation of volume
- position from acceleration
- cost from marginal cost

Here are some more:

- probabilities and distributions
- averages and expectations
- finding moments of inertia
- work from power

Probability

In **probability theory** functions are used as observables or to define probabilities.

Assuming our probability space to be the real line, an interval $[a, b]$ is called an **event**. Given a nonnegative function $f(x)$ which has the property that $\int_{-\infty}^{\infty} f(x)\, dx = 1$, we call

$$\mathrm{P}[A] = \int_0^b f(x)\, dx$$

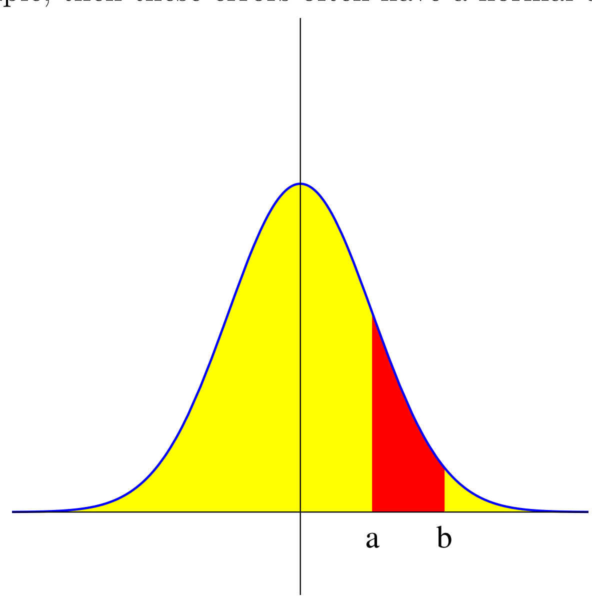
the **probability** of the event. The function $f(x)$ is called the **probability density function**.

The most important probability density is the normal distribution:

The **normal distribution** has the density

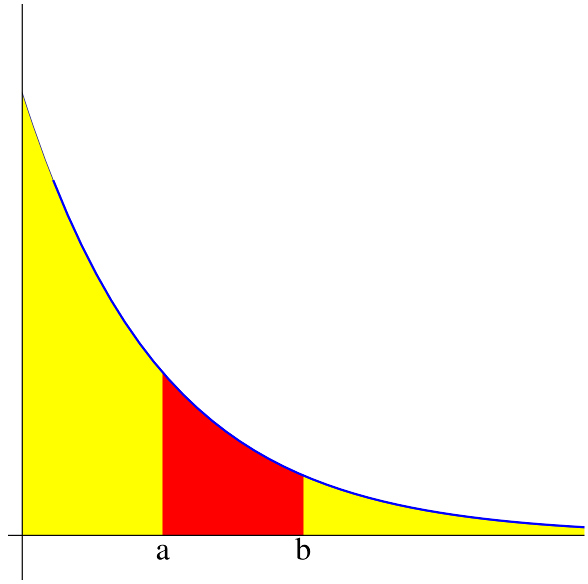
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

It is the distribution which appears most often if data can take both positive and negative values. The reason why it appears so often is that if one observes different unrelated quantities with the same statistical properties, then their sum, suitably normalized becomes the normal distribution. If we measure **errors** for example, then these errors often have a normal distribution.



1 The probability density function of the **exponential distribution** is defined as $f(x) = e^{-x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. It is used to used measure lengths of arrival times like the time until you get the next phone call. The density is zero for negative x because there

is no way we can travel back in time. What is the probability that you get a phone call between times $x = 1$ and times $x = 2$ from now? The answer is $\int_1^2 f(x)\, dx$.



Assume f is a probability density function (PDF). The antiderivative $F(x) = \int_{-\infty}^x f(t)\, dt$ is called the **cumulative distribution function** (CDF).

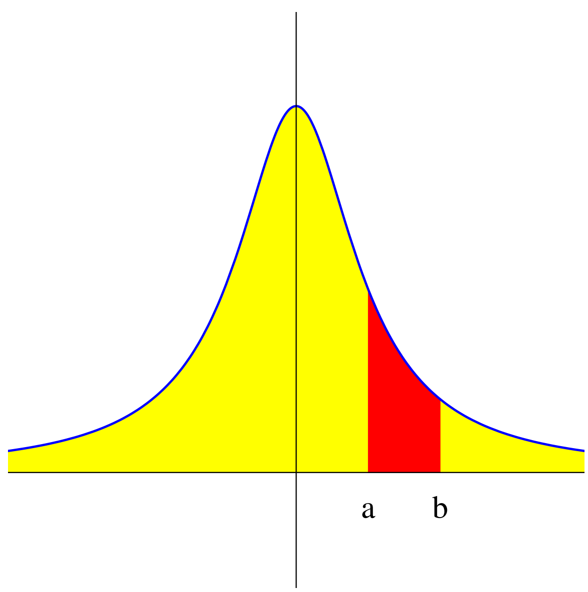
2 For the exponential function the cumulative distribution function is

$$\int_{-\infty}^x f(x)\, dx = \int_0^x f(x)\, dx = -e^{-x}|_0^x = 1 - e^{-x} .$$

The probability density function $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ is called the Cauchy distribution.

3 Find its cumulative distribution function. **Solution:**

$$F(x) = \int_{-\infty}^x f(t)\, dt = \frac{1}{\pi}\arctan(x)|_{-\infty}^x = (\frac{1}{\pi}\arctan(x) + \frac{1}{2}) .$$

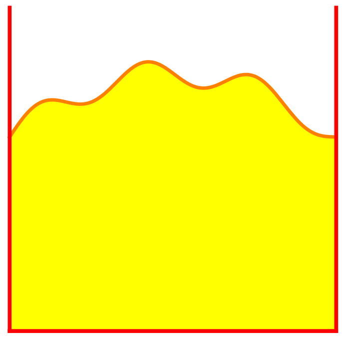


Average

Here is an example for computing the **average**.

4 Assume the level in a **honey jar** over $[0, 2\pi]$ containing crystallized honey is given by a function $f(x) = 3 + \sin(3x)/5 + x(2\pi - x)/10$. In order to restore the honey, it is placed into hot water. The honey melts to its normal state. What height does it have? **Solution:** The average height is $\int_0^{2\pi} f(x)\, dx/(2\pi)$ which is the area divided by the base length.

In probability theory we would call $f(x)$ a **random variable** and the average of f with $\mathrm{E}[f]$ the **expectation**.



Moment of inertia

If we spin a wire of radius L of mass density $f(x)$ around an axes, the **moment of inertia** is defined as $I = \int_0^L x^2 f(x)\, dx$.

The significance is that if we spin it with angular velocity w, then the energy is $Iw/2$.

5 Assume a wire has density $1 + x$ and length 3. Find its moment of inertia. **Solution:**

6 **Flywheels** have a comeback for **powerplants** to absorb energy. If there is not enough power, the flywheels are charged, in peak times, the energy is recovered. They work with 80 percent efficiency. Assume a flywheel is a cylinder of radius 1, density 1 and height 1, then the moment of inertia integral is $\int_0^1 z^2 f(z)\, dx$, where $f(z)$ is the mass in distance z.

Work from power

If $P(t)$ is the amount of power produced at time t, then $\int_0^T P(t)\, dt$ is the **work**=energy produced in the time interval $[0, T]$.

Energy is the anti-derivative of power.

7 Assume a power plant produces power $P(t) = 1000 + \exp(-t) + t^2 - t$. What is the energy produced from $t = 1$ to $t = 10$? **Solution.**

Wouldn't be nice to have one of those bikes with interactive training environments in the gym, allowing to ride in the Peruvian or Swiss Mountains, the California coast, in Tuscany or Cancún? Additionally, there should be some computer game features, racing other riders through beaches, deserts or Texan highways (could be on google earth). Training would be so much more entertaining. Business opportunities everywhere. The first offering such training equipment will make a fortune. Until then we are stuck with TV programs which really suck.

Homework

1 Assume the probability density for the time you have to wait for your next email is $f(x) = 5e^{-5x}$ where x is time in hours. What is the probability that you get your next email in the next 4 hours?

2 Assume the probability distribution for the waiting time to the next warm day is $f(x) = (1/4)e^{-x/4}$, where x has days as unit. What is the probability to get a warm day between tomorrow and after tomorrow that is between $x = 1$ and $x = 2$?

3 A cone of base radius 1 and height 2 has temperature z^2. What is the average temperature? Remember that if $A(z)$ is the area of a slice at height z then $V = \int_0^2 A(z)\, dz$ is the volume. You have to compute $\int_0^2 z^2 A(z)\, dz/V$.

4 A CD Rom has radius 6. If we would place the material at radius x onto one point, we get a density of $f(x) = 2\pi x$. Find the moment of inertia I of the disc. If we spin it with an angular velocity of $w = 20$ rounds per second. Find the energy $E = Iw^2/2$.

Without credit: Explode a CD: http://www.powerlabs.org/cdexplode.htm. Careful!

5 a) You are on a stationary bike at the Hemenway gym and pedal with power

$$P(t) = 200 + 100\sin(10\pi t) - \frac{t}{300} + \frac{t^2}{19440}$$

(in Watts=W). The periodic fluctuations come from a hilly route. The linear term is the "tiring effect" and the quadratic term is due to **endorphins** kicking in. What energy (Joules J=W s) have you produced in the time $t \in [0, 1800]$ (s=seconds)?

b) Since we do math not physics, we usually ignore all the units but this one is just too much fun. If you divide the result by 4184, you get **kilo calories** = **food calories**. An apple has about 80 food calories. How many apples can you eat after your half hour workout to compensate the spent energy?

Lecture 24: Worksheet

Applications of integration

1 Find the cumulative distribution function

$$F(x) = \int_{-\infty}^{x} f(t)\, dt \ .$$

of the exponential distribution in the case $f(x) = 2\exp(-2x)$.

2 Find the moment of inertia of a rod which has density $f(x) = x$ and length 10.

$$\int_0^L x^2 f(x)\, dx \ .$$

3 How much heat energy is in a sphere of radius 1 if the heat at height z is z? Remember that if $A(z)$ is the radius of the sphere at height z then $\int_{-1}^{1} A(z)\, dz$ was the volume. You have to compute

$$\int_{-1}^{1} zA(z)\, dz/V\ .$$

Lecture 25: Related rates

Before we continue with integration, we include a short flash-back on differentiation. which allows us to review the **chain rule**

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

a rule which will lead to the "substitution" integration technique. Since the chain rule is often perceived hard, it is good to review it before launching into integration techniques.
Related rates problem deal with a relation for variables. Differentiation gives a relation between the derivatives (rate of change). In all these problems, we have an **equation** and a **rate**. You can then solve for the rate which is asked for.

1 Hydrophilic **water gel spheres** have volume $\boxed{V(r(t)) = 4\pi r(t)^3/3}$ and expand at a rate $\boxed{V' = 30}$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.

2 A **wine glass** has a shape $y = x^2$ and volume $\boxed{V(y) = y^2\pi/2}$. Assume we slurp the wine with constant rate $\boxed{V' = -0.1}$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi y y'(t)$ so that $y'(t) = -1/(\pi y)$.

3 A **ladder** has length 1. Assume slips on the ground away with constant speed $\boxed{x' = 2}$. What is the speed of the top part of the ladder sliding down the wall at the time when

$x = y$ if $\boxed{x^2(t) + y^2(t) = 1}$. Differentiation gives $2x(t)x'(t) + 2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

4 A **kid slides** down a slide of the shape $\boxed{y = 2/x}$. Assume $\boxed{y' = -7}$. What is $x'(t)$? Evaluate it at $x = 1$. **Solution:** differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.

Image source: http://www.dmfco.com

5 A **canister of oil** releases oil so that the area grows at a constant rate $\boxed{A' = 5}$. With what rate does the radius increase? **Solution.** We have $\boxed{A(r) = r^2\pi}$ and so $5 = A'(r) = 2rr'\pi$. Solving for r' gives $r' = 5/(2r\pi)$.

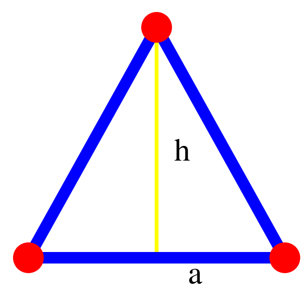
Related rates problems link quantities by a $\boxed{\text{rule}}$. These quantities can depend on time. To solve a related rates problem, differentiate the $\boxed{\text{rule}}$ with respect to time use the given $\boxed{\text{rate of change}}$ and solve for the unknown rate of change. Since related change problems are often difficult to parse. We have the $\boxed{\text{rule}}$ and given $\boxed{\text{rate of change}}$ boxed.

Homework

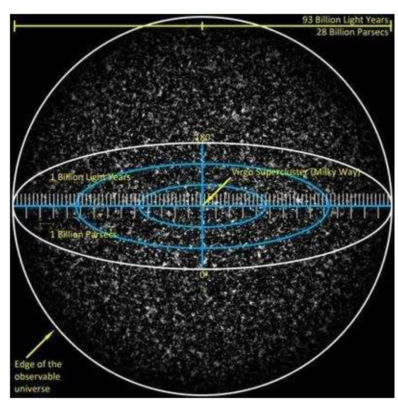
1 The **ideal gas law** $\boxed{pV = T}$ relates pressure p and volume V and temperature T. Assume the temperature $T = 50$ is fixed and $\boxed{V' = -5}$. Find the rate p' with which the pressure increases if $V = 10$ and $p = 5$.

2 Assume the **total production rate** P of an Oculus Rift is constant $P = 100$ and given by the **Cobb-Douglas formula** $\boxed{P = L^{1/3}K^{2/3}}$. Assume labor is increased at a rate $\boxed{L' = 2}$. What is the cost change K'? Evaluate this at $K = 125$ and $L = 64$.

3 You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\boxed{\phi' = 5\pi/180 = \pi/36}$ radiants per second (which means 5 degrees per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\boxed{\tan(\phi) = x/h}$ to get ϕ for $x = 0$.



4 An **isosceles triangle** with base $2a$ and height h has fixed area $\boxed{A = ah = 1}$. Assume the height is decreased by a rate $\boxed{h' = -2}$. With what rate does a increase if $h = 1/2$?



5 There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $\boxed{V = \pi^2 r^4/2}$ increases at a rate $\boxed{V' = 100\pi^2 r^2}$. What is r'? Evaluate it for $r = 47$ (billion light years).

Lecture 25: Worksheet

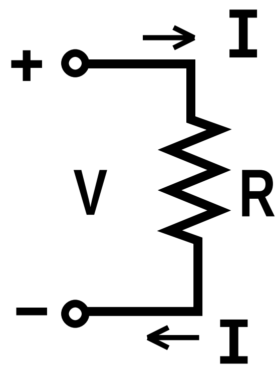
Related rates

1 An underwater oil spill releases oil at the constant amount. The area $\boxed{A(r) = \pi r^2}$ of the oil increases with $A'(r(t)) = 5$. What is the rate of change of r?

2 The resistance R, voltage U and current I are related by

$$U = RI \; .$$

Assume the voltage is $U = 4$ and the resistance $R(t)$ increases by a constant amount $R' = 2$. What is the rate of change of I?

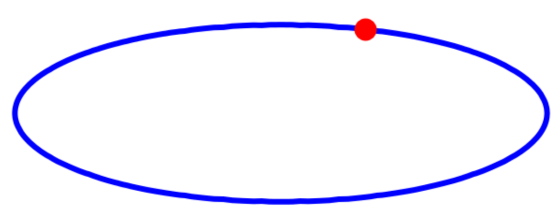


Lecture 26: Implicit differentiation

Implicit differentiation had bin crucial for finding the derivative of inverse functions. We will review this here because this will give us handy tools for integration.

The chain rule, related rates and implicit differentiation belong all to the same concept. But each sees it from a different angle. You can see implicit differentiation as a special case of related rates, where one of the quantities is "time" meaning that this is the variable with respect to which we differentiate. There is considerable overlap with what we do here with what we have done last time.

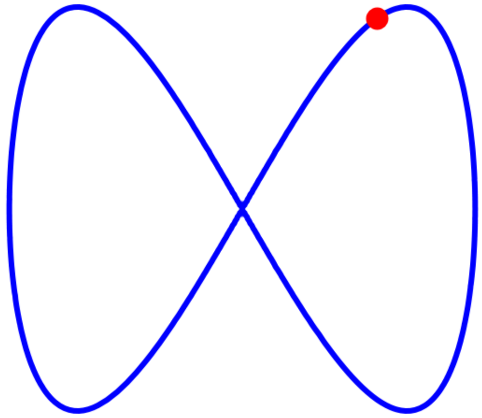
1 Points (x, y) in the plane which satisfy $x^2 + 9y^2 = 10$ form an ellipse. Find the slope y' of the tangent line at the point $(1, 1)$.
Solution: We want to know the derivative dy/dx. We have $2x + 18yy' = 0$. Using $x = 1, y = 1$ we see $y' = -2x/(18y) = -1/9$.
Remark. We could have looked at this as a related rates problem where $x(t), y(t)$ are related and $x' = 1$ Now $2xx' + 9 \cdot 2yy' = 0$ allows to solve for $y' = -2xx'/(9y) = -2/9$.



2 The points (x, y) which satisfy the equation $x^4 - 3x^2 + y^2 = 0$ forms a **figure 8** called **lemniscate of Gerono**. It contains the point $(1, \sqrt{2})$. Find the slope of the curve at that point. **Solution**: We differentiate the law describing the curve with respect to x. This gives

$$4x^3 - 6x + 2yy' = 0$$

We can now solve for $y' = (6x - 4x^3)/(2y) = 1/\sqrt{2}$.



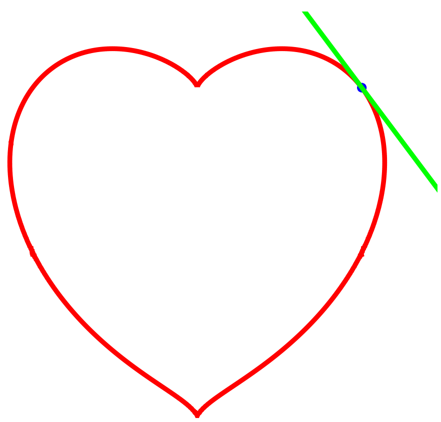
3 The **Valentine equation** $(x^2+y^2-1)^3 - x^2y^3 = 0$ contains the point $(1,1)$. Near $(1,1)$, we have $y = y(x)$ so that $(x^2+y(x)^2-1)^3 - x^2y(x)^3 = 0$. Find y' at the point $x = 1$. **Solution:** Take the derivative

$$0 = 3(x^2+y^2-1)^2(2x+2yy') - 2xy^3 - x^2 3y^2 y'(x)$$

and solve for

$$y' = -\frac{3(x^2+y^2-1)2x - 2xy^3}{3(x^2+y^2-1)2y - 3x^2y^2} .$$

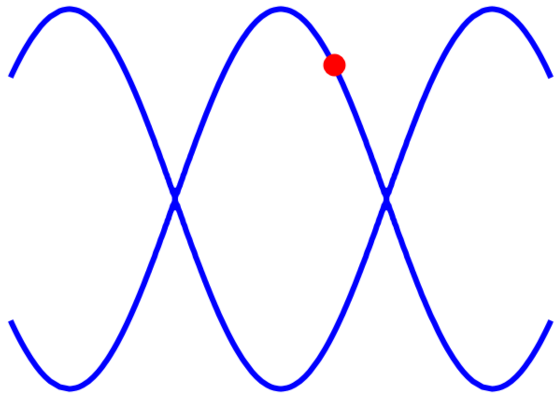
For $x = 1, y = 1$, we get $-4/3$.



4 The energy of a **pendulum** with angle x and angular velocity y is

$$y^2 - \cos(x) = 1$$

is constant. What is y'? We could solve for y and then differentiate. Simpler is to differentiate directly and get $yy' + \sin(x) = 0$ so that $y' = -\sin(x)/y$. At the point $(\pi/2, 1)$ for example we have $y' = -1$.



What is the difference between related rates and implicit differentiation?

Implicit differentiation is the **special case** of related rates where one of the variables is time.

Derivatives of inverse functions

Implicit differentiation has an important application: it allows us to compute the derivatives of inverse functions. It is good that we review this, because we can use these derivatives to find anti-derivatives. We have seen this already. Lets do it again.

5 Find the derivative of $\log(x)$ by differentiating $\exp(\log(x)) = x$.
Solution:

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{dx}\exp(\log(x)) \\ &= \exp(\log(x))\frac{d}{dx}\log(x) = x\log'(x)\ . \end{aligned}$$

Solve for $\log'(x) = 1/x$. Since the derivative of $\log(x)$ is $1/x$. The anti-derivative of $1/x$ is $\log(x) + C$.

6 Find the derivative of $\arccos(x)$ by differentiating $\cos(\arccos(x)) = x$.
Solution:

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{dx}\cos(\arccos(x)) \\ &= -\sin(\arccos(x))\arccos'(x) = -\sqrt{1-\cos^2(\arccos(x))}\arccos'(x) \\ &= -\sqrt{1-x^2}\arccos'(x)\ . \end{aligned}$$

Solving for $\arccos'(x) = -1/\sqrt{1-x^2}$. The anti-derivative of $\arccos(x)$ is $-1/\sqrt{1-x^2}$.

7 Find the derivative of $\arctan(x)$ by differentiating $\tan(\arctan(x)) = x$.
Solution: This is a derivative which we have seen several times by now. We use the identity $1/\cos^2(x) = \tan^2(x) + 1$ to get

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{dx}\tan(\arctan(x)) \\ &= \frac{1}{\cos^2(\arctan(x))}\arctan'(x) \\ &= (1+\tan^2(\arctan(x)))\arctan'(x)\ . \end{aligned}$$

Solve for $\arctan'(x) = 1/(1+x^2)$. The anti-derivative of $\arctan(x)$ is $1/(1+x^2)$.

8 Find the derivative of $f(x) = \sqrt{x}$ by differentiating $(\sqrt{x})^2 = x$.

Solution:

$$\begin{aligned} 1 &= \frac{d}{dx}x = \frac{d}{x}(\sqrt{x})^2 \\ &= 2\sqrt{x}f'(x) \end{aligned}$$

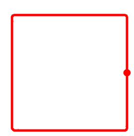
so that $f'(x) = 1/(2\sqrt{x})$.

Homework

1 The equation $y^2 = x^2 - x$ defines the graph of the function $f(x) = \sqrt{x^2 - x}$. Find the slope of the graph at $x = 2$ directly by differentiating f. Then use the implicit differentiation method and differentiate $y^2 = x^2 - x$ assuming $y(x)$ is a function of x and solving for y'.

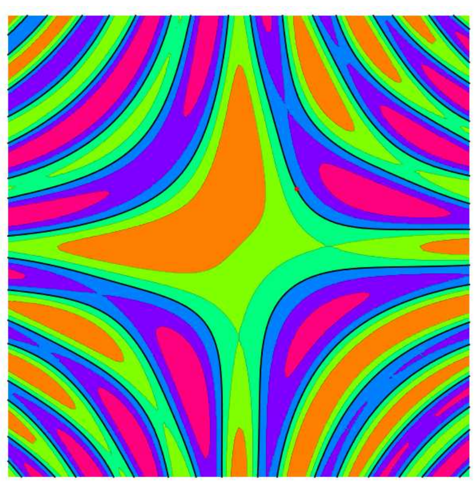
2 The equation $x^2 + 4y^2 = 5$ defines an ellipse. Find the slope of the tangent at $(1, 1)$.

3 The equation $x^{100} + y^{100} = 1 + 2^{100}$ defines a curve which looks close to a square. Find the slope of the curve at $(2, 1)$.



4 Derive the derivative of $\text{arctanh}(x)$ by using the identity $\tanh(\text{arctanh}(x)) = x$. You can use $\cosh^2(x) - \sinh^2(x) = 1$ which implies that $1 - \tanh^2(x) = 1/\cosh^2(x)$.

5 This is a related rates problem: the relation $\sin(x - y) - 2\cos((\pi/2)xy) = 0$ relates $x(t)$ and $y(t)$. What is $y' = 2$ at $(1, 1)$ what is x' at this point?



Lecture 26: Worksheet

Implicit differentiation

1 Find the slope of $y'(x)$ if $2x^3 - y^3 = y$ at the point $(1,1)$.

2 Find the derivative of $y(x) = x^{1/5}$ by differentiating $y^5 = x$.

Lecture 27: Review for second midterm

Major points

The **mean value theorem** assures that there is $x \in (a,b)$ with $f'(x) = (f(b) - f(a))/(b-a)$. A special case is Rolle's theorem, where $f(b) = f(a)$.

Catastrophes are parameter values where a local minimum disappears. To find the parameter look at the second derivative f'' at the critical point and find c which makes this zero.

Definite integrals $F(x) = \int_0^x f(t)\ dt$ are defined as a limit of Riemann sums $S_n/n = \frac{1}{n}[f(0/n) + f(1/n) + ... + f((n-1)/n)]$.

A function $F(x)$ satisfying $F' = f$ is called the anti-derivative of f. The general anti-derivative is $F + c$ where c is a constant.

The **fundamental theorem of calculus** tells $d/dx \int_0^x f(x)\ dx = f(x)$ and $\int_0^x f'(x)\ dx = f(x) - f(0)$.

The integral $\int_a^b g(x) - f(x)\ dx$ is the **signed area between the graphs** of f and g. Places, where $f < g$ are counted negative. When area is asked, split things up.

The integral $\int_a^b A(x)\ dx$ is a **volume** if $A(x)$ is the area of a slice of the solid perpendicular to a point x on an axes.

Write **improper integrals** as limits of definite integrals $\int_1^\infty f(x)\ dx = \lim_{R\to\infty} \int_1^R f(x)\ dx$. We similarly treat points, where f is discontinuous.

Besides **area**, **volume**, **total cost**, or **position**, we can compute **averages**, **inertia** or **work** using integrals.

If x, y are related by $F(x(t), y(t)) = 0$ and $x(t)$ is known we can compute $y'(t)$ using the chain rule. This is **related rates**.

If $f(g(t))$ is known we can compute $g'(x)$ using the chain rule. This works for inverse functions. This is **implicit differentiation**.

To determine the **catastrophes** for a family $f_c(x)$ of functions, determine the critical points in dependence of c and find values c, where a critical point changes from a local minimum to a local maximum.

Important integrals

Which one is the derivative which the integral?

$\sin(x)$	$-\cos(x)$.	$\log(x)$	$x\log(x) - x$
$\tan(x)$	$1/\cos^2(x)$.	$1/x$	$\log(x)$
$\arctan(x)$	$1/(1+x^2)$.	$-1/(1+x^2)$	$\operatorname{arccot}(x)$
$1/\sqrt{1-x^2}$	$\arcsin(x)$	$-1/\sqrt{1-x^2}$	$\arccos(x)$

Improper integrals

$\int_1^\infty 1/x^2 \, dx$ Prototype of first type improper integral which exists.
$\int_1^\infty 1/x \, dx$ Prototype of first type improper integral which does not exist.
$\int_0^1 1/x \, dx$ Prototype of second type improper integral which does not exist.
$\int_0^1 1/\sqrt{x} \, dx$ Prototype of second type improper integral which does exist.

The fundamental theorem

$\frac{d}{dx} \int_0^x f(t) \, dt = f(x)$

$\int_0^x f'(t) \, dt = f(x) - f(0)$.

This implies

$\int_a^b f'(x) \, dx = f(b) - f(a)$

Without limits of integration, we call $\int f(x) \, dx$ the **anti derivative**. It is defined up to a constant. For example $\int \sin(x) \, dx = -\cos(x) + C$.

Applications

Calculus applies directly if there are situations where one quantity is the derivative of the other.

function	anti derivative
acceleration	velocity
velocity	position
function	area under the graph
length of cross section	area of region
area of cross section	volume of solid
marginal prize	total prize
power	work
probability density function	cumulative distribution function

Tricks

Make a picture, whenever we deal with an area or volume computation! In related rates problems, we have to understand the variables and the constants.
For volume computations, find the area of the cross section $A(x)$ and integrate.
For area computations find the length of the slice $f(x)$ and integrate.

Most important integrals

The most important integral is the integral

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1}}$$

holds for all n different from 1.

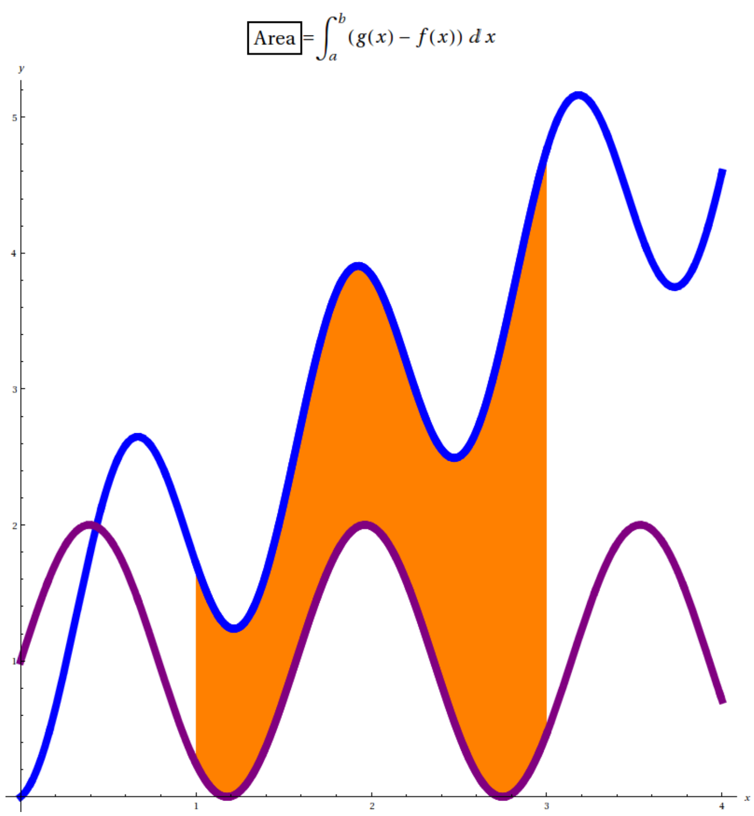
$$\boxed{\int \frac{1}{x} \, dx = \log(x)}$$

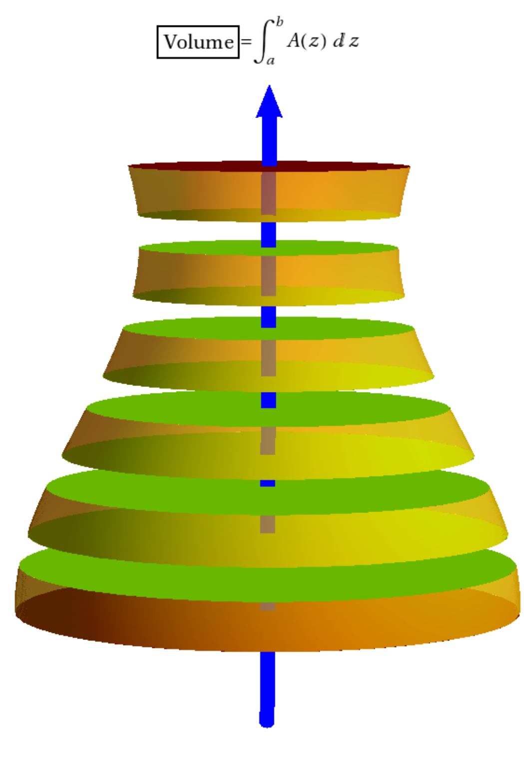
Example: $\int \sqrt{x+7} \, dx = \frac{2}{3}(x+7)^{3/2}$.
Example: $\int \frac{1}{x+5} \, dx = \log(x+5)$
Example: $\int \frac{1}{4x+3} \, dx = \log(4x+3)/4$

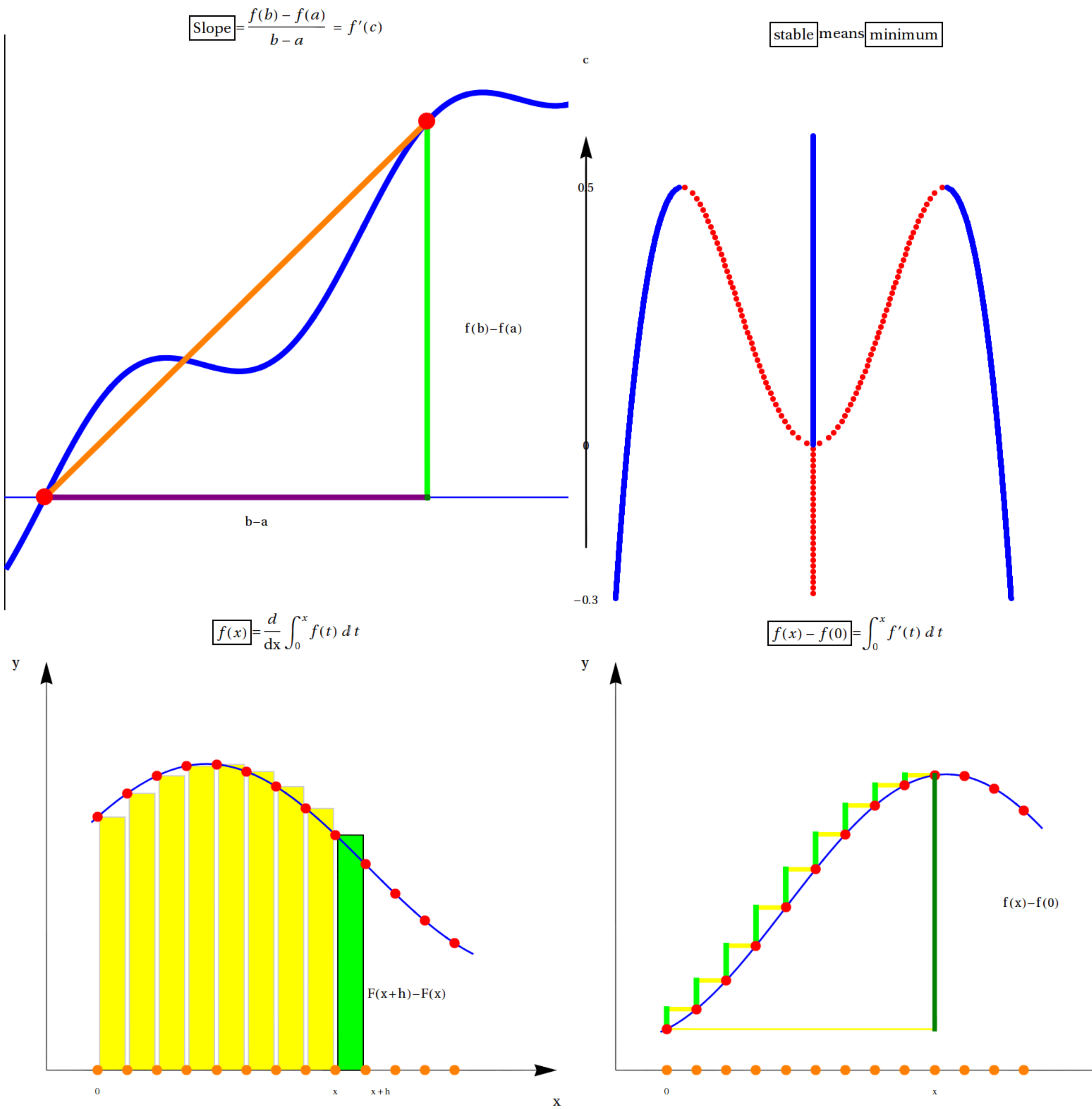
Related rates - Implicit differentiation

Assume $\cos(xy) + y^4 = 2y$, where x, y both change and $x' = 7$. Find y' at $x = 0, y = 1$. Given$\cos(xy) + y^4 = 2y$. Find $y'(x)$ at $x = 0$.

Key pictures







Some integration tricks

$\int f(ax+b)\,dx = F(ax+b)/a$. Example: $\int \frac{1}{1+(x+1)^2}\,dx = \arctan(1+x)$. We have learned to deal with this with integration.

Lecture 27: Review Problems

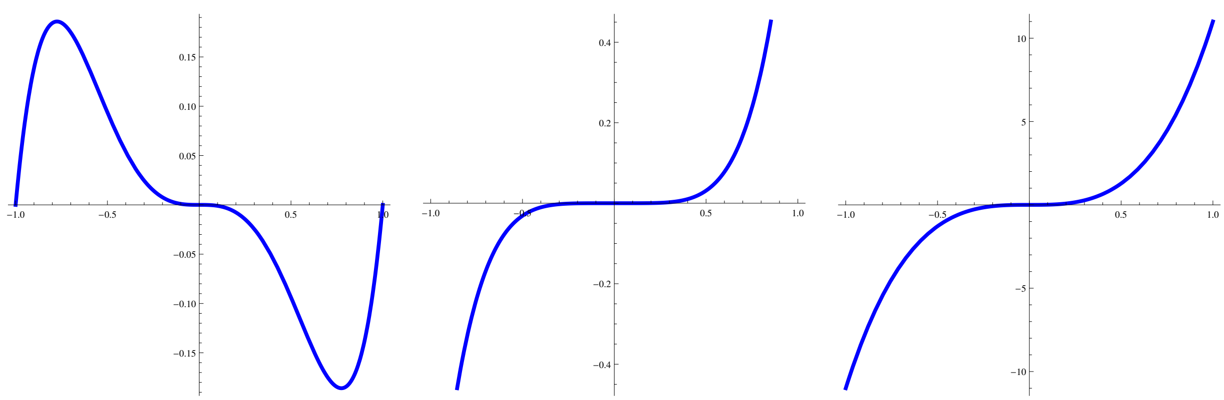
Definite integral

1 The following integral defines the area of a region. Draw it:

$$\int_{\pi/2}^{\pi} x - \sin(x)\, dx \ .$$

Catastrophes

2 Lets look at the family of functions $f_c(x) = x^5 + cx^3$. You see three graphs. They display the function for $c = -1$, $c = 0$ and $c = 1$. What can you say about catastrophes?

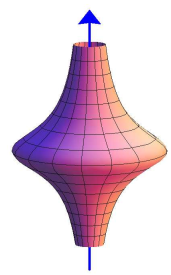


Area

3 Find the area of the region bound by $y = 2 - x, x = y, y = 0$ and $y = 1$.

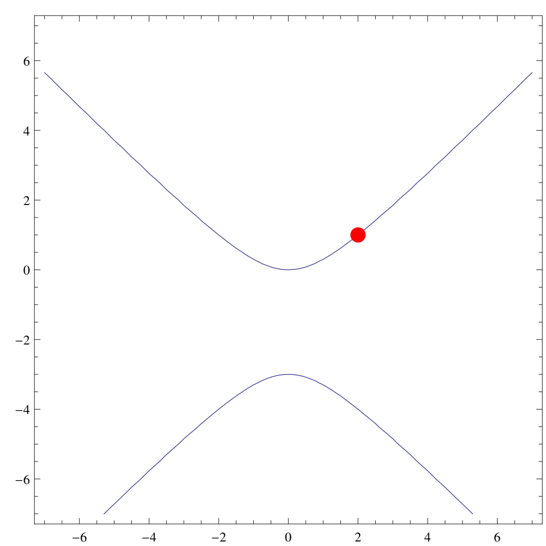
Volumes

4 If we rotate the curve $1/\sqrt{1+z^2}$ around the z axes, we obtain a solid. Find its volume if $-5 \leq z \leq 5$.



Related Rates

5 The curve $x^2 - y^2 = 3y$ is and example of a hyperbola. If $x(t) = 2 + t$. Find the related rate y' near $(2, 1)$.



Lecture 28: Substitution

You should by now also to be able to integrate functions like e^{6x} or $1/(1+x)$. Substitution makes this easier. If we differentiate the function $\sin(x^2)$ and use the chain rule, we get $\cos(x^2)2x$. By the fundamental theorem of calculus, the anti derivative of $\cos(x^2)2x$ is $\sin(x^2)$. We know therefore

$$\int \cos(x^2)2x \, dx = \sin(x^2) + C \, .$$

Spotting the chain rule

How can we see the integral without knowing the result already? Here is a very important case:

If we can spot that $f(x) = g(u(x))u'(x)$, then the anti derivative of f is $G(u(x))$ where G is the anti derivative of g.

1 Find the anti derivative of

$$e^{x^4+x^2}(4x^3+2x) \, .$$

Solution: The derivative of the inner function is to the right.

2 Find

$$\int \sqrt{x^5+1}x^4 \, dx \, .$$

Solution. The derivative of x^5+1 is $5x^4$. This is almost what we have there but the constant can be adapted. The answer is $(2/15)(x^5+1)^{3/2}$.

3 Find the anti derivative of

$$\frac{\log(x)}{x} \, .$$

Solution: The derivative of $\log(x)$ is $1/x$. The antiderivative is $\log(x)^2/2$.

4 Find the anti derivative of

$$\cos(\sin(x^2))\cos(x)2x \, .$$

Solution: We see the derivative of $\sin(x^2)$ appear on the right. Therefore, we have $\sin(\sin(x^2))$.

In the next three examples, substitution is actually not necessary. You can just write down the anti derivative, and adjust the constant. It uses the following "speedy rule":

If $\int f(ax+b) \, dx = F(ax+b)/a$ where F is the anti derivative of f.

5 $\int \sqrt{x+1}\, dx$. **Solution:** $(x+1)^{3/2}(2/3)$.
6 $\int \frac{1}{1+(5x+2)^2}\, dx$. **Solution:** $\arctan(5x+2)(1/5)$.

Doing substitution

Spotting things is sometimes not easy. The method of substitution helps to formalize this. To do so, identify a part of the formula to integrate and call it u then replace an occurrence of $u'dx$ with du.

$$\int f(\text{ u(x) })\text{ u'(x) dx} = \int f(\text{ u })\text{ du} .$$

Here is a more detailed description: replace a prominent part of the function with a new variable u, then use $du = u'(x)dx$ to replace dx with du/u'. We aim to end up with an integral $\int g(u)\, du$ which does not involve x anymore. Finally, after integration of this integral, replace the variable u again with the function $u(x)$. The last step is called **back-substitution**.

7 Find the anti-derivative

$$\int \log(\log(x))/x\, dx .$$

Solution: Replace $\log(x)$ with u and replace $u'dx = 1/xdx$ with du. This gives $\int \log(u)\, du = u\log(u) - u = \log(x)\log\log(x)) - \log(x)$.

8 Solve the integral

$$\int x/(1+x^4)\, dx .$$

Solution: Substitute $u = x^2, du = 2xdx$ to get $(1/2)\int du/(1+u^2)\, du = (1/2)\arctan(u) = (1/2)\arctan(x^2)$.

9 Solve the integral

$$\int \sin(\sqrt{x})/\sqrt{x} .$$

Here are some examples which are not so straightforward:

10 Solve the integral

$$\int \sin^3(x)\, dx .$$

Solution:. We replace $\sin^2(x)$ with $1-\cos^2(x)$ to get
$$\int \sin^3(x)\ dx = \int \sin(x)(1-\cos^2(x))\ dx = -\cos(x) + \cos^3(x)/3\ .$$

11 Solve the integral
$$\int \frac{x^2+1}{\sqrt{x+1}}\ dx\ .$$
Solution: Substitute $u = \sqrt{x+1}$. This gives $x = u^2-1, dx = 2udu$ and we get $\int 2(u^2-1)^2+1\ du$.

12 Solve the integral
$$\int \frac{x^3}{\sqrt{x^2+1}}\ dx\ .$$
Trying $u = \sqrt{x^2+1}$ but this does not work. Try $u = x^2+1$, then $du = 2xdx$ and $dx = du/(2\sqrt{u-1})$. Substitute this in to get
$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}}\ du = \int \frac{(u-1)}{2\sqrt{u}} = \int u^{1/2}/2 - u^{-1/2}/2\ du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2}\ .$$

Definite integrals

When doing definite integrals, we could find the antiderivative as described and then fill in the boundary points. Substituting the boundaries directly accelerates the process since we do not have to substitute back to the original variables:

$$\int_a^b g(u(x))u'(x)\ dx = \int_{u(a)}^{u(b)} g(u)\ du\ .$$

Proof. This identity follows from the fact that the right hand side is $G(u(b)) - G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has the anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b)) - G(u(a))$.

Top: To keep track which bounds we consider it can help to write $\int_{x=a}^{x=b} f(x)\ dx$.

13 Find the anti derivative of $\int_0^2 \sin(x^3-1)x^2\ dx$. **Solution**:
$$\int_{x=0}^{x=2} \sin(x^3+1)x^2\ dx\ .$$
Solution: Use $u = x^3+1$ and get $du = 3x^2dx$. We get
$$\int_{u=1}^{u=9} \sin(u)du/3 = (1/3)\cos(u)|_1^9 = [-\cos(9)+\cos(1)]/3\ .$$

Also here, we can see the integrals directly

To integrate $f(Ax+B)$ from a to b we get $[F(Ab+B) - F(Aa+B)]/A$, where F is the anti-derivative of f.

14 $\int_0^1 \frac{1}{5x+1}\ dx = [\log(u)]/5|_1^6 = \log(6)/5$.

15 $\int_3^5 \exp(4x-10)\ dx = [\exp(10) - \exp(2)]/4$.

Homework

1 Find the following anti derivatives.
a) $\int 20x \sin(x^2)\ dx$
b) $\int e^{x^6+x}(6x^5+1)\ dx$
c) $\cos(\cos^3(x))\sin(x)\cos^2(x)$
d) $e^{\tan(x)}/\cos^2(x)$.

2 Compute the following definite integrals.
a) $\int_2^5 \sqrt{x^5+x}(x^4+1/5)\ dx$
b) $\int_0^{\sqrt{\pi}} \sin(x^2)x\ dx$.
c) $\int_{1/e}^{e} \frac{\sqrt{\log(x)}}{x}\ dx$.
d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}}\ dx$.

3 a) Find the integral $\int_0^1 x^3\sqrt{1-x^4}\ dx$ using a substitution method.
b) Find the moment of inertia of a rod with density $f(x)=\sqrt{x^3+1}$ between $x=0$ and $x=4$. Remember that the moment of inertia is $\int_0^4 x^2 f(x)\ dx$.

4 a) Integrate

$$\int_0^1 \frac{\arcsin(x)}{\sqrt{1-x^2}}\ dx\ .$$

b) Find the definite integral

$$\int_e^{6e} \frac{dx}{\sqrt{\log(x)}x}\ .$$

5 a) Find the indefinite integral

$$\int \frac{x^5}{\sqrt{x^2+1}}\ dx\ .$$

b) Find the anti-derivative of

$$f(x) = \frac{1}{x(1+\log(x)^2)}\ .$$

Lecture 28: Worksheet

Substitution

1

$$\int xe^{x^2} \; ; dx$$

2

$$\int \sin(2x+3) \; dx$$

3

$$\int \frac{1}{(x+8)^5} \; dx$$

4

$$\int \frac{\log(5x)}{x} \; dx$$

5

$$\int \frac{x}{\sqrt{x^2+1}} \; dx$$

6

$$\int \frac{e^x}{(e^x+5)^2} \; ; dx$$

Lecture 29: Integration by parts

If we integrate the product rule $(uv)' = u'v + uv'$ we obtain an integration rule called **integration by parts**. It is a powerful tool, which complements substitution. As a rule of thumb, always try first to simplify a function and integrate directly, then give substitution a first shot before trying integration by parts.

$$\int \text{u(x)}\ \text{v'}\ (x)dx = u(x)v(x) - \int u'(x)v(x)\ dx.$$

1 Find $\int x\sin(x)\ dx$. **Solution.** Lets identify the part which we want to differentiate and call it u and the part to integrate and call it v'. The integration by parts method now proceeds by writing down uv and subtracting a new integral which integrates $u'v$:

$$\int \text{x}\ \sin(x)\ dx = \text{x}\ (-\cos(x)) - \int 1\ (-\cos(x)\ )\ dx = -x\cos(x) + \sin(x) + C\ dx\ .$$

2 Find $\int xe^x\ dx$. **Solution.**

$$\int \text{x}\ \exp(x)\ dx = x\exp(x) - \int 1\exp(x)\ dx = x\exp(x) - \exp(x) + C\ dx\ .$$

3 Find $\int \log(x)\ dx$. **Solution.** There is only one function here, but we can look at it as $\log(x)\cdot 1$

$$\int \log(x)\ 1\ dx = \log(x)x - \int \frac{1}{x}x\ dx = x\log(x) - x + C\ .$$

4 Find $\int x\log(x)\ dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed like this. We would get through but what if we do not know? Lets differentiate $\log(x)$ and integrate x:

$$\int \log(x)\ \text{x}\ dx = \log(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\ dx$$

which is $\log(x)x^2/2 - x^2/4$.

We see that it is better to differentiate log first.

5 **Marry go round:** Find $I = \int \sin(x)\exp(x)\ dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\ dx\ .$$

Lets do it again:

$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\ dx.$$

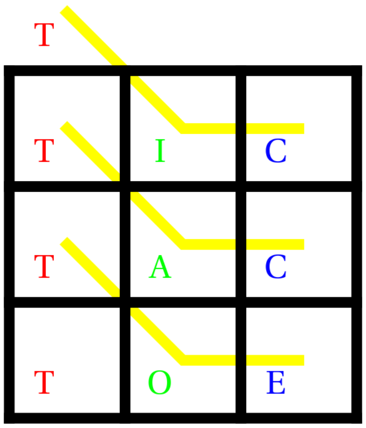
We moved in circles and are stuck! Are we really. We have derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

which we can solve for I and get

$$I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2\ .$$

Tic-Tac-Toe



Integration by parts can bog you down if you do it several times. Keeping the order of the signs can be daunting. This is why a **tabular integration by parts method** is so powerful. It has been called "Tic-Tac-Toe" in the movie Stand and deliver. Lets call it Tic-Tac-Toe therefore.

6 Find the anti-derivative of $x^2 \sin(x)$. **Solution:**

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The antiderivative is

$$-x^2 \cos(x) + 2x \sin(x) + 2\cos(x) + C \, .$$

7 Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

$(x-1)^3$	$\exp(2x)$	
$3(x-1)^2$	$\exp(2x)/2$	$\oplus$
$6(x-1)$	$\exp(2x)/4$	$\ominus$
6	$\exp(2x)/8$	$\oplus$
0	$\exp(2x)/16$	$\ominus$

The anti-derivative is

$$(x-1)^3 e^{2x}/2 - 3(x-1)^2 e^{2x}/4 + 6(x-1)e^{2x}/8 - 6e^{2x}/16 + C \, .$$

8 Find the anti-derivative of $x^2 \cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	$\oplus$
2	$-\cos(x)$	$\ominus$
0	$-\sin(x)$	$\oplus$

The anti-derivative is

$$x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C \, .$$

Ok, we are now ready for more extreme stuff.

9 Find the anti-derivative of $x^7\cos(x)$. **Solution:**

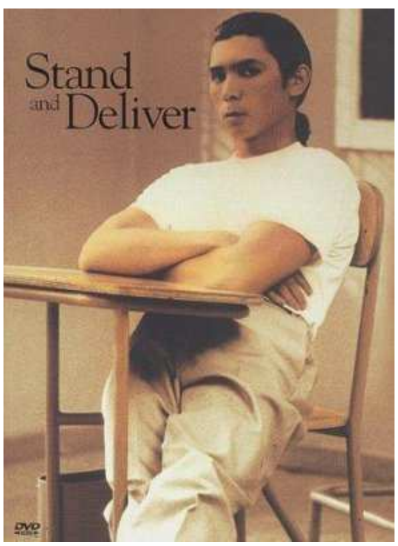
x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	$\oplus$
$42x^5$	$-\cos(x)$	$\ominus$
$120x^4$	$-\sin(x)$	$\oplus$
$840x^3$	$\cos(x)$	$\ominus$
$2520x^2$	$\sin(x)$	$\oplus$
$5040x$	$-\cos(x)$	$\ominus$
5040	$-\sin(x)$	$\oplus$
0	$\cos(x)$	$\ominus$

The anti-derivative is

$$\begin{aligned} F(x) &= x^7\sin(x) \\ &+ 7x^6\cos(x) \\ &- 42x^5\sin(x) \\ &- 210x^4\cos(x) \\ &+ 840x^3\sin(x) \\ &+ 2520x^2\cos(x) \\ &- 5040x\sin(x) \\ &- 5040\cos(x) + C\ . \end{aligned}$$

Do this without this method and you see the value of the method.

[1] [2] [3]:



I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to an article of V.N. Murty. The method realizes in a clever way an iterated integration by parts method:

$$\begin{aligned} \int fg\,dx &= fg^{(-1)} - f^{(1)}g^{-2} + f^{(2)}g^{(-3)} - \dots \\ &- (-1)^n \int f^{(n+1)}g^{(-n-1)}\,dx \end{aligned}$$

which can easily shown to be true by induction and justifies the method: the f function is differentiated again and again and the g function is integrated again and again. You see, where the alternating minus signs come from. You see that we always pair a k'th derivative with a $k+1$'th integral and take the sign $(-1)^k$.

Coffee or Tea?

[1]V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, pages 90-94.
[2]David Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, pages 307-311.
[3]K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, pages 542-543

When doing integration by parts, We want to try first to differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

For coffee lovers, there is an equivalent one: **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials which can be remembered as **LIATE** which is close to "latte" (the coffee).

Whether you prefer to remember it as a "coffee latte" or a "lipton tea" is up to you.

There is even a better method, the "opportunistic method":

> Just integrate what you can integrate and differentiate the rest.

An don't forget to consider integrating 1, if nothing else works.

Homework

1 Integrate $\int x^3 \log(x)\, dx$.

2 Integrate $\int x^5 \sin(x)\, dx$

3 Find the anti derivative of $\int 2x^6 \exp(x)\, dx$. (*)

4 Find the anti derivative of $\int \sqrt{x} \log(x)\, dx$.

5 Find the anti derivative of $\int \sin(x) \exp(-x)\, dx$.

(*) If you want to go for the record. Lets see who can integrate the largest $x^n \exp(x)$! It has to be done by hand, not with a computer algebra system although.

Lecture 29: Worksheet

Integration by parts

1 Find the anti-derivative of $\log(2x)\sqrt{x}$:

2 Stand and deliver!

Find the anti-derivative of $x^3 \sin(2x)$:

x^3	$\sin(2x)$	
		$\oplus$
		$\ominus$
		$\oplus$
		$\ominus$

Lecture 30: Numerical integration

We briefly look at some numerical techniques for computing integrals. There are variations of basic Riemann sums but speed up the computation.

Riemann sum with nonuniform spacing

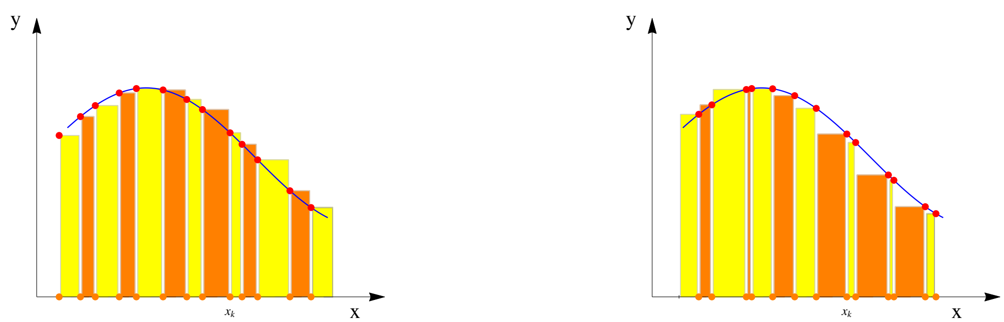
A more general Riemann sum is obtained by choosing n points $\{x_j\}$ in $[a,b]$ and then to look at

$$S_n = \sum f(y_j)(x_{j+1} - x_j) = \sum_{y_j} f(y_j)\Delta x_j \ ,$$

where y_j is in (x_j, x_{j+1}).

This generalization allows to use a small mesh size where the function fluctuates a lot. The function $f(x) = \sin(1/(x^2 + 0.1))$ for example fluctuates near the origin more and would need more division points there.

The sum $\sum f(x_j)\Delta x_j$ is called the **left Riemann sum**, the sum $\sum f(x_{j+1})\Delta x_j$ the **right Riemann sum**.



If $x_0 = a, x_n = b$ and $\max_j \Delta x_j \to 0$ for $n \to \infty$ then S_n converges to $\int_a^b f(x)\, dx$.

1 If $x_j - x_k = 1/n$ and $z_j = x_j$, then we have the Riemann sum as we defined it initially.

2 You numerically integrate $\sin(x)$ on $[0, \pi/2]$ with a Riemann sum. What is better, the left Riemann sum or the right Riemann sum? Look also at the interval $[\pi/2, \pi]$? **Solution:** you see that in the first case, the left Riemann sum is smaller than the actual integral. In the second case, the left Riemann sum is larger than the actual integral.

Trapezoid rule

The average between the left and right hand Riemann sum is called the **Trapezoid rule**. Geometrically, it sums up areas of trapezoids instead of rectangles.

The Trapezoid rule does not change things much in the case of equal spacing $x_k = a + (b-a)k/n$.

$$\frac{1}{2n}[f(x_0) + f(x_n)] + \frac{1}{n}\sum_{k=1}^{n-1} f(x_k) \ .$$

Simpson rule

The **Simpson rule** computes the sum

$$S_n = \frac{1}{6n}\sum_{k=1}^{n}[f(x_k) + 4f(y_k) + f(x_{k+1})] \ ,$$

where y_k are the midpoints between x_k and x_{k+1}.

The Simpson rule is good because it is exact for quadratic functions: for $f(x) = ax^2 + bx + c$, the formula

$$\frac{1}{v-u}\int_u^v f(x)\, dx = [f(u) + 4f((u+v)/2) + f(v)]/6$$

holds exactly. To prove it just run the following two lines in Mathematica: (== means "is equal")

```
f[x_] := a x^2 + b x + c;
Simplify[(f[u]+f[v]+4f[(u+v)/2])/6==Integrate[f[x],{x,u,v}]/(v-u)]
```

This actually will imply (as you will see in Math 1b) that the numerical integration for functions which are 4 times differentiable gives numerical results which are n^{-4} close to the actual integral. For 100 division points, this can give accuracy to 10^{-8} already.

There are other variants which are a bit better but need more function values. If x_k, y_k, z_k, x_{k+1} are equally spaced, then

The **Simpson 3/8 rule** computes

$$\frac{1}{8n}\sum_{k=1}^{n}[f(x_k) + 3f(y_k) + 3f(z_k) + f(x_{k+1})] \ .$$

This formula is again exact for quadratic functions: for $f(x) = ax^2 + bx + c$, the formula

$$\frac{1}{v-u}\int_u^v f(x)\, dx = [f(u) + 3f((2u+v)/3) + 3f((u+2v)/3) + f(v)]/6$$

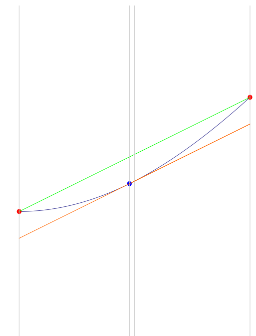
holds. If you are interested, run the two Mathematica lines:

```
f[x_] := a x^2 + b x + c; L=Integrate[f[x],{x,u,v}]/(v-u);
Simplify[(f[u]+f[v]+3f[(2u+v)/3]+3f[(u+2v)/3])/8==L]
```

This 3/8 method can be slightly better than the first Simpson rule.

Mean value method

The mean value theorem shows that for $x_k = k/n$, there are points $y_k \in [x_k, x_{k+1}]$ such that $f(y_j) = F'(y_j) = f(x_{j+1}) - f(x_j)$ and so
$\frac{1}{n}\sum_{k=1}^{n} f(y_k) = F(x_n) - F(x_0)$.
This is a version of the fundamental theorem of calculus which is exact in the sense that for every n, this is a correct formula. Lets call y_k the **Rolle points**.



The Rolle point is close to the interval midpoint.

For any partition x_k on $[a,b]$ with $x_0 = a, x_n = b$, there is a choice of Rolle points $y_k \in [x_k, x_{k+1}]$ such that the Riemann sum $\sum_k f(y_k)\Delta(x)_k$ is equal to $\int_a^b f(x)\, dx$.

For linear functions the Rolle points are the midpoints. In general, the deviation $g(t)$ from the midpoint is small if the interval is $[x_0 - t, x_0 + t]$. One can estimate $g(t)$ to be of the order $t^2 \frac{f'''(x_0)}{6f''(x_0)}$. We could modify the trapezoid rule and replace the line through the points by a Taylor polynomial. The Rolle point method is useful for functions which can have poles.

Monte Carlo Method

A powerful integration method is to chose n random points x_k in $[a,b]$ and look at the sum divided by n. Because it uses randomness, it is called **Monte Carlo method**.

The **Monte Carlo** integral is the limit S_n to infinity

$$S_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k) \ ,$$

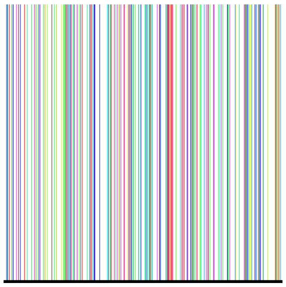
where x_k are n random values in $[a,b]$.

The law of large numbers in probability shows that the **Monte Carlo integral** is equivalent to the **Lebesgue integral** which is more powerful than the Riemann integral. Monte Carlo integration is interesting especially if the function is complicated.

3 Lets look at the **salt and pepper** function

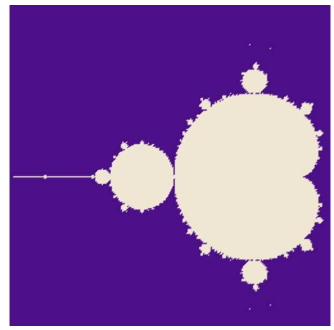
$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

The Riemann integral with equal spacing k/n is equal to 1 for every n. But this is only because we have evaluated the function at rational points, where it is 1.
The Monte Carlo integral gives zero because if we chose a random number in $[0,1]$ we hit an irrational number with probability 1.



The Salt and Pepper function and the Boston Salt and Pepper bridge (Anne Heywood).

4 The following two lines evaluate the **area of the Mandelbrot fractal** using Monte Carlo integration. The function F is equal to 1, if the parameter value c of the quadratic map $z \to z^2 + c$ is in the Mandelbrot set and 0 else. It shoots $100'000$ random points and counts what fraction of the square of area 9 is covered by the set. Numerical experiments give values close to the actual value around 1.51.... One could use more points to get more accurate estimates.



```
F[c_]:=Block[{z=c,u=1},Do[z=N[z^2+c];If[Abs[z]>3,u=0;z=3],{99}];u];
M=10^5; Sum[F[-2.5+3 Random[]+I(-1.5+3 Random[])],{M}]*(9.0/M)
```

Homework

1 Use a computer to generate 20 random numbers x_k in [0,1]. Sum up the square x_k^2 of these numbers and divide by 20. Compare your result with $\int_0^1 x^2 \, dx$. **Remark.** If using a program, increase the value of n as large as you can. Here is a Mathematica code:

```
n=20; Sum[ Random[]^2,{n}]/n
```

Here is an implementation in Perl. Its still possible to cram the code into one line:

```
#!/usr/bin/perl
$n=20;$s=0;for($i=0;$i<$n;$i++){$f=rand();$s+=$f*$f;} print $s/$n;
```

2 a) Use the Simpson rule to compute $\int_0^\pi \sin(x) \, dx$ using $n = 2$ intervals $[0, \pi/2]$ and $[\pi/2, \pi]$. On each of these intervals $[a, b]$ compute the Simpson sum $[f(a) + 4f((a + b)/2) + f(b)]/6$ with $f(x) = \sin(x)$. Compare with the actual integral.
b) Now use the 3/8 Simpson rule to estimate $\int_0^\pi f(x) \, dx$ using $n = 1$ intervals $[0, \pi]$. Again compare with the actual integral.

Instead of adding more numerical methods exercices, we want to practice a bit more integration. The challenge in the following problems is to find out which integration method is bets suited. This is good preparation for the final, where we will not reveal which integration method is the best.

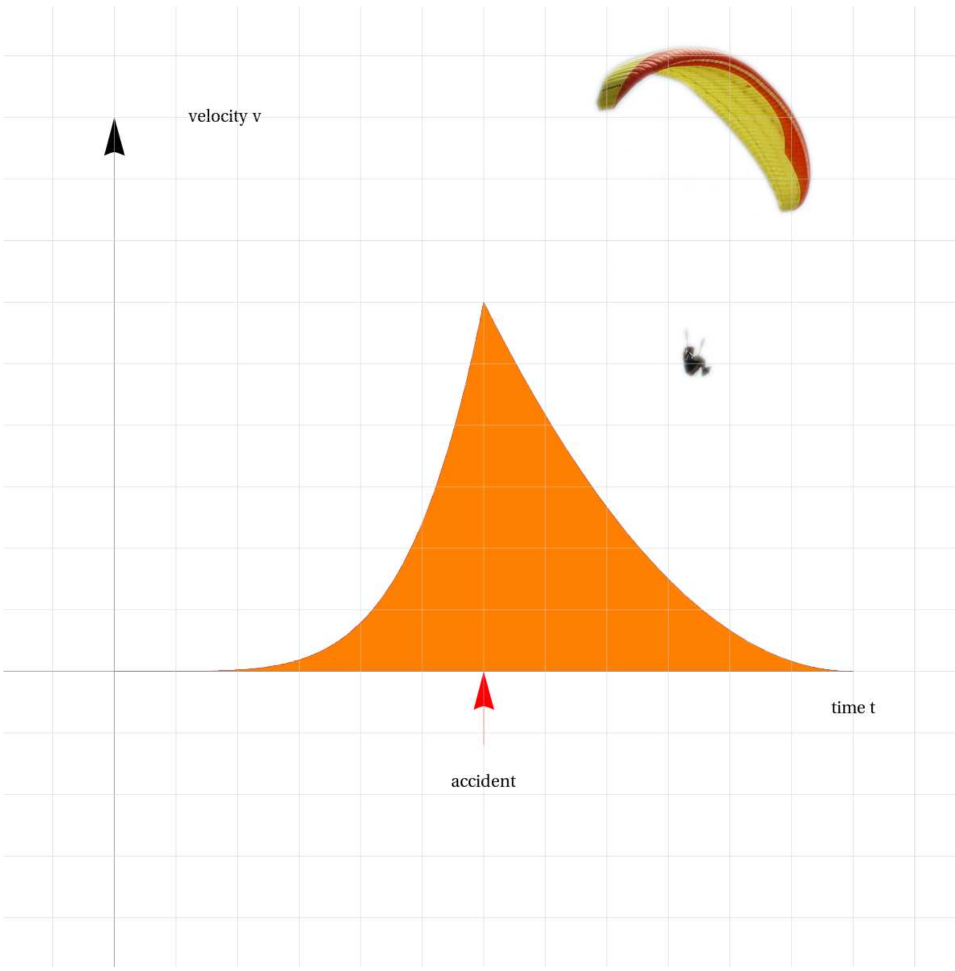
3 Integrate $\tan(x)/\cos(x)$ from 0 to $\pi/6$.

4 Find the antiderivative of $\sqrt{x} \log x$.

5 Find the antiderivative of $x/\sin(x)^2$.

Lecture 30: Worksheet

Numerical methods



1 A paraglider starts a flight in the mountain. The velocity is given in the above graph. Find out, whether the paraglider lands lower or higher than where it started.

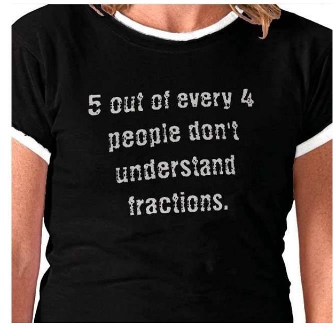
Hint To estimate integrals take the average of the number A of squares entirely below the graph and the number B of squares containing part of the region below the graph. The result $A + B$ is a good estimate for the area below the graph.

2 Review: Integrate $x^{1/3} \log(x)\, dx$

3 Review: Integrate $\log(x^5)(1/x)\, dx$

Lecture 31: Partial fractions

The partial fraction method will be covered in detail follow up calculus courses like Math 1b. Here we just look at some samples to see whats out there. We have learned how to integrate polynomials like $x^4 + 5x + 3$. What about rational functions? We will see here that they are a piece of cake - if you have the right guide of course ...



What we know already

Lets see what we know already:

- We also know that integrating $1/x$ gives $\log(x)$. We can for example integrate
$$\int \frac{1}{x-6}\, dx = \log(x-6) + C \; .$$
- We also have learned how to integrate $1/(1+x^2)$. It was an important integral:
$$\int \frac{1}{1+x^2}\, dx = \arctan(x) + C \; .$$
Using substitution, we can do more like
$$\int \frac{dx}{1+4x^2} = \int \frac{du/2}{1+u^2} = \arctan(u)/2 = \arctan(2x)/2 \; .$$
- We also know how to integrate functions of the type $x/(x^2+c)$ using substitution. We can write $u = x^2 + c$ and get $du = 2xdx$ so that
$$\int \frac{x}{x^2+c}\, dx = \int \frac{1}{2u}\, du = \frac{\log(x^2+c)}{2} \; .$$
- Also functions $1/(x+c)^2$ can be integrated using substitution. With $x + c = u$ we get $du = dx$ and
$$\int \frac{1}{(x+c)^2}\, dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{x+c} + C \; .$$

The partial fraction method

We would love to be able to integrate any rational function
$$f(x) = \frac{p(x)}{q(x)} \; ,$$
where p, q are polynomials. This is where **partial fractions come in**. The idea is to write a rational function as a sum of fractions we know how to integrate. The above examples have shown that we can integrate $a/(x+c), (ax+b)/(x^2+c), a/(x+c)^2$ and cases, which after substitution are of this type.

The partial fraction method writes $p(x)/q(x)$ as a sum of functions of the above type which we can integrate.

This is an algebra problem. Here is an important special case:

In order to integrate $\int \frac{1}{(x-a)(x-b)}\,dx$, write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \,.$$

and solve for A, B.

In order to solve for A, B, write the right hand side as one fraction again

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)} \,.$$

We only need to look at the nominator:

$$1 = Ax - Ab + Bx - Ba \,.$$

In order that this is true we must have $A + B = 0, Ab - Ba = 1$. This allows us to solve for A, B.

Examples

1 To integrate $\int \frac{2}{1-x^2}\,dx$ we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x) \,.$$

2 Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} \,.$$

Now multiply out and solve for A, B:

$$A(x-2) + B(x-3) = 5 - 2x \,.$$

This gives the equations $A + B = -2, -2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. We have not obtained

$$\frac{5-2x}{x^2-5x+6} = -\frac{1}{x-3} - \frac{1}{x-2}$$

and can integrate:

$$\int \frac{5-2x}{x^2-5x+6}\,dx = -\log(x-3) - \log(x-2) \,.$$

Actually, we could have got this one also with substitution. How?

3 Integrate $f(x) = \int \frac{1}{1-4x^2}\,dx$. **Solution.** The denominator is factored as $(1-2x)(1+2x)$. Write

$$\frac{A}{1-2x} + \frac{B}{1+2x} = \frac{1}{1-4x^2} \,.$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x)\,dx = \frac{1}{4}\log(1-2x) - \frac{1}{4}\log(1+2x) + C \,.$$

Hopital's method

There is a fast method to get the coefficients:

If a is different from b, then the coefficients A, B in

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} ,$$

are

$$A = \lim_{x \to a} (x-a) f(x) = p(a)/(a-b), \; B = \lim_{x \to b} (x-b) f(x) = p(b)/(b-a) .$$

Proof. If we multiply the identity with $x-a$ we get

$$\frac{p(x)}{(x-b)} = A + \frac{B(x-a)}{x-b} .$$

Now we can take the limit $x \to a$ without peril and end up with $A = p(a)/(x-b)$.

Cool, isn't it? This **Hopital method** can save you a lot of time! Especially when you deal with more factors and where sometimes complicated systems of linear equations would have to be solved. Remember

Math is all about elegance and does not use complicated methods if simple ones are available.

Here is an example:

4 Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = 11/12$, and $B = \frac{2*(-8)+3}{(-8-4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8} .$$

The integral is

$$\frac{11}{12} \log(x-4) + \frac{13}{12} \log(x+8) .$$

Here is an example with three factors:

5 Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2} \log(x-1) - 7 \log(x-2) + \frac{13}{2} \log(x-3) .$$

Homework

1 a) $\int \frac{1}{x^2-14x+45}\ dx$
b) $\int \frac{2}{x^2-9}\ dx$

2 $\int \frac{5dx}{4x^2+1}$.

3 $\int \frac{x^3-x+1}{x^2-1}\ dx$.

4 $\int \frac{23}{(x+2)(x-3)(x-2)(x+3)}\ dx$. Use Hopital.

5 $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)}\ dx$. Use Hopital.
Hint for 3). Subtract first a polynomial.

Lecture 31: Worksheet

Partial fractions

1 Integrate $\frac{1}{1+x}$.

2 Integrate $\frac{9}{(x-1)^2}$.

3 Integrate $\frac{7}{x^2+1}$.

4 Integrate $\frac{1}{1-x^4}$

Hint: write the last one first in the form

$$A/(x^2 - 1) + B/(1 + x^2)$$

Lecture 32: Trig substitutions

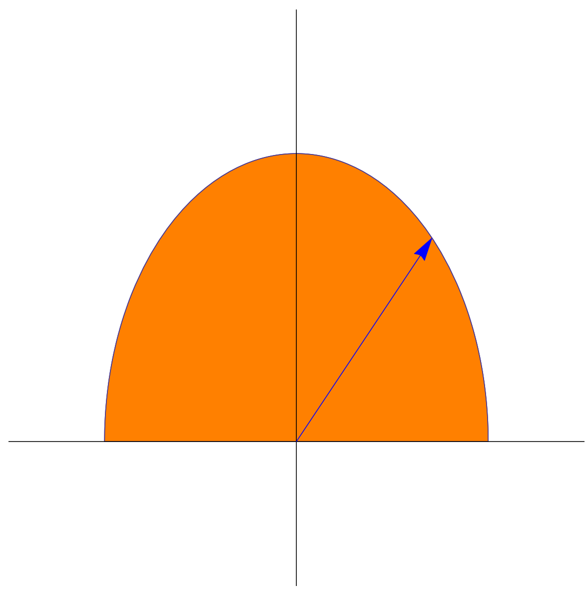
A **Trig substitution** is a special substitution, where x is a trigonometric function of u or u is a trigonometric function of x. Also this topic is covered more in follow up courses like Math 1b. This lecture allows us to practice more the substitution method. Here is an important example:

1 The area of a half circle of radius 1 is given by the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx .$$

Solution. Write $x = \sin(u)$ so that $\cos(u) = \sqrt{1-x^2}$. $dx = \cos(u)du$. We have $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$ the answer is

$$\int_{-\pi/2}^{\pi/2} \cos(u)\cos(u)du = \int_{-\pi/2}^{\pi/2} (1+\cos(2u))/2 = \frac{\pi}{2} .$$



Lets generalize this a bit and do the same computation for a general radius r:

2 Compute the area of a half disc of radius r which is given by the integral

$$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx .$$

Solution. Write $x = r\sin(u)$ so that $r\cos(u) = \sqrt{r^2-x^2}$ and $dx = r\cos(u)\, du$ and $r\sin(-\pi/2) = -r$ and and $r\sin(\pi/2) = r$. The answer is

$$\int_{-\pi/2}^{\pi/2} r^2\cos^2(u)\, du = r^2\pi/2 .$$

Here is an example, we know how to integrate

3 Find the integral

$$\int \frac{dx}{\sqrt{1-x^2}} .$$

We know the answer is $\arcsin(x)$. How can we do that without knowing? **Solution.** We can do it also with a trig substitution. Try $x = \sin(u)$ to get $dx = \cos(u)\, du$ and so

$$\int \frac{\cos(u)\, du}{\cos(u)} = u = \arcsin(x) + C \ .$$

Here is an example, where $\tan(u)$ is the right substitution. You have to be told that first. It is hard to come up with the idea:

4 Find the following integral:

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

by using the substitution $x = \tan(u)$. **Solution.** Then $1 + x^2 = 1/\cos^2(u)$ and $dx = du/\cos^2(u)$. We get

$$\int \frac{du}{\cos^2(u)\tan^2(u)(1/\cos(u))} = \int \frac{\cos(u)}{\sin^2(u)}\, du = -1/\sin(u) = -1/\sin(\arctan(x)) \ .$$

Trig substitution is based on the trig identity :

$\cos^2(u) + \sin^2(u) = 1$

Depending on whether you divide this by $\sin^2(u)$ or $\cos^2(u)$ we get

$1 + \tan^2(u) = 1/\cos^2(u)$, $1 + \cot^2(u) = 1/\sin^2(u)$

These identities are worth remembering. Lets look at more examples:

5 Evaluate the following integral

$$\int x^2/\sqrt{1-x^2}\, dx \ .$$

Solution: Substitute $x = \cos(u), dx = -\sin(u)\, du$ and get

$$\int -\frac{\cos^2(u)}{\sin(u)}\sin(u)du = -\int \cos^2(u)\, du = -\frac{u}{2} - \frac{\sin(2u)}{4} + C = -\frac{\arcsin(x)}{2} + \frac{\sin(2\arcsin(x))}{4} + C \ .$$

6 Evaluate the integral

$$\int \frac{dx}{(1+x^2)^2} \ .$$

Solution: we make the substitution $x = \tan(u), dx = du/(\cos^2(u))$. Since $1 + x^2 = \cos^{-2}(u)$ we have

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^2(u)\, du = (u/2) + \frac{\sin(2u)}{4} + C = \frac{\arctan(u)}{2} + \frac{\sin(2\arctan(u))}{4} + C \ .$$

Here comes an other prototype problem:

7 Find the anti derivative of $1/\sin(x)$. **Solution**: We use the substitution $u = \tan(x/2)$ which gives $x = 2\arctan(u), dx = 2du/(1+u^2)$. Because $1 + u^2 = 1/\cos^2(x/2)$ we have

$$\frac{2u}{1+u^2} = 2\tan(x/2)\cos^2(x/2) = 2\sin(x/2)\cos(x/2) = \sin(x) \ .$$

Plug this into the integral

$$\int \frac{1}{\sin(x)}\, dx = \int \frac{1+u^2}{2u}\frac{2du}{1+u^2} = \int \frac{1}{u}\, du = \log(u) + C = \log(\tan(\frac{x}{2})) + C \ .$$

Unlike before, where x is a trig function of u, now u is a trig function of x. This example shows that the substitution $u = \tan(x/2)$ is magic. Because of the following identities

0. $u = \tan(x/2)$
$\boxed{1}$ $dx = \frac{2du}{(1+u^2)}$
$\boxed{2}$ $\sin(x) = \frac{2u}{1+u^2}$
$\boxed{3}$ $\cos(x) = \frac{1-u^2}{1+u^2}$

It allows us to reduce any rational function involving trig functions to rational functions. [1]

Any function $p(x)/q(x)$ where p, q are trigonometric polynomials can be integrated using elementary functions.

It is usually a lot of work but here is an example:

8 To find the integral

$$\int \frac{\cos(x) + \tan(x)}{\sin(x) + \cot(x)}\, dx$$

for example, we replace $dx, \sin(x), \cos(x), \tan(x) = \sin(x)/\cos(x), \cot(x) = \cos(x)/\sin(x)$ with the above formulas we get a rational expression which involves u only This gives us an integral $\int p(u)/q(u)\, du$ with polynomials p, q. In our case, this would simplify to

$$\int \frac{2u\left(u^4 + 2u^3 - 2u^2 + 2u + 1\right)}{(u-1)(u+1)\left(u^2+1\right)\left(u^4 - 4u^2 - 1\right)} du$$

The method of partial fractions provides us then with the solution.

[1] Proofs: $\boxed{1}$ differentiate to get $du = dx/(2\cos^2(x/2)) = dx(1+u^2)/2$. $\boxed{2}$ use double angle $\sin(x) = 2\tan(x/2)\cos^2(x/2)$ and then $1/\cos^2(x/2) = 1 + \tan^2(x/2)$. $\boxed{3}$ use double angle $\cos(x) = \cos^2(x/2) - \sin^2(x/2) = (1 - \sin^2(x/2)/\cos^2(x/2))\cos^2(x/2)$ and again $1/\cos^2(x/2) = 1 + \tan^2(x/2)$.

Homework

1 Find the anti-derivative:

$$\int \sqrt{1-9x^2}\ dx\ .$$

2 Find the anti-derivative:

$$\int (1-x^2)^{3/2}\ dx\ .$$

3 Find the anti-derivative:

$$\int \frac{\sqrt{1-x^2}}{x^2}\ dx\ .$$

4 Integrate

$$\int \frac{dx}{1+\sin(x)}\ .$$

Use the substitution $u = \tan(x/2)$.

5 Compute

$$\int_0^{\pi/3} \frac{dx}{\cos(x)}$$

using the substitution $u = \tan(x/2)$. Instead of backsubstitution, you can also substitute the bounds.

Lecture 32: Worksheet

Trig Substitutions

1 Integrate $\sqrt{1+x^2}$. Hint. Use $x = \tan(u)$.

2 Integrate $\sqrt{1-x^2}$. Hint. Use $x = \cos(u)$.

3 Integrate $\sqrt{x^2-1}$. Hint. Use $x = 1/\cos(u)$.

4 Integrate $\frac{\arccos(x)}{1-x^2}$. Hint. Use $x = \cos(u)$.

Lecture 33: Calculus and Music

A music piece is a function

Calculus is relevant in music because every music piece just is a **function**. If you feed a loudspeaker the function $f(t)$ by displacing the membrane by $f(t)$ you can the music. The pressure variations in the air are sound waves then reach your ear, where your ear drum oscillates with the function $f(t-T)+g(t)$ where $g(t)$ is background noise and T is a time delay for the sound to reach your ear. Plotting and playing works the same way. In Mathematica, we can play a function with

```
Play[ Sin[2Pi 1000 x^2],{x,0,10}]
```

This function contains all the information about the music piece. A music ".WAV" file contains sampled values of the function. A sample rate of 44'100 per second is usual. Since our ear does not hear frequencies larger than 20'000 KHz, a sampling rate of 44 K is good enough by a **theorem of Nyquist-Shannon**. In .MP3 files encodes the function in a compressed way. To get from the sample values $f(n)$ the function back, the sinc function is used. The **Whittaker-Shannon interpolation formula**

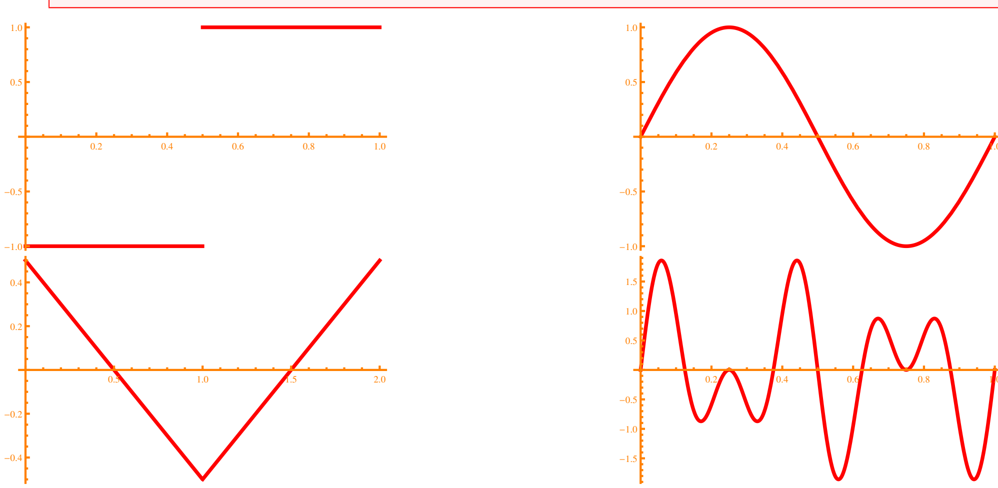
$$f(t) = \sum_n f(n)\text{sinc}(t+n)$$

is especially good and mention this because $\text{sinc}(x) = \sin(x)/x$ is one of our most beloved functions. We take this lecture as an opportunity to review some facts about functions. We especially see that log, exp and trigonometric functions play an important role in music.

The wave form and hull

A periodic signal is the building block of sound. Assume $g(x)$ is a 2π periodic function, we can generate a sound of 440 Hertz when playing the function $f(x) = g(440 \cdot 2\pi x)$. If the function does not have a smaller period, then we hear the A tone with 440 Hertz.

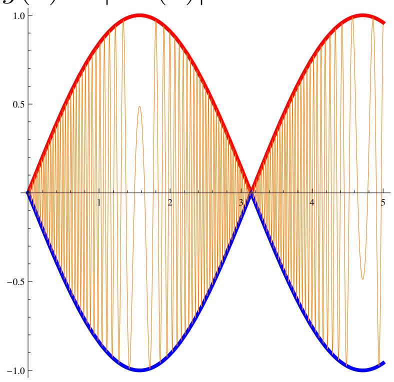
A periodic function g is called a **wave form**.



The wave form makes up the **timbre** of a sound which allows to model music instruments with macroscopic terms like "attack, vibrato, coloration, noise, echo, reverberation and other characteristics.

The upper **hull function** is defined as the interpolation of successive local maxima of f. The lower **hull function** is the interpolation of the local minima.

For the function $f(x) = \sin(100x)$ for example, the upper hull function is $g(x) = 1$ and the lower hull function is $g(x) = -1$. For $f(x) = \sin(x)\sin(100x)$ the upper hull function is approximately $g(x) = |\sin(x)|$ and the lower hull function is approximately $g(x) = -|\sin(x)|$.



We can not hear the actual function because the function changes too fast that we can notice individual vibrations. But we can hear the hull function. Simplest examples are change of dynamics in music like **creshendi** or **diminuendi** or a vibrato. We can generate a beautiful hull by playing two frequencies which are close. You hear **interference**.

The scale

Western music uses a discrete set of frequencies. This scale is based on the exponential function. The frequency f is an exponential function of the scale s. On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

The **Midi numbering** of musical notes is
$$s = 69 + 12 \cdot \log_2(f/440)$$

1 What is the frequency of the Midi tone 100? **Solution**. We have to solve the above equation for f and get the **piano scale function**

$$f(s) = 440 \cdot 2^{(s-69)/12} .$$

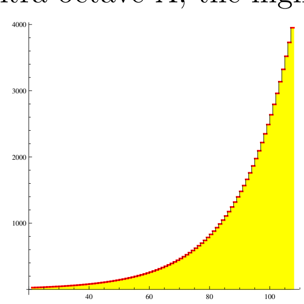
Evaluated at 100 we get 2637.02 Hz.

The piano scale function
$$f(s) = 440 \cdot 2^{(s-69)/12} .$$
is an exponential function $f(s) = be^{as}$ which satisfies $f(s + 12) = 2f(s)$.

2 Find the discrete derivative $Df(x) = f(x+1)-f(x)$ of the Piano scale function. **Solution:** The function is of the form $f(x) = A2^{ax}$. We have $f(x+1) = 2^a f$ and so $Df(x) = (2^a-1)f$ with $a = 1/12$. Lets get reminded that such discrete relations lead to the important property $\boxed{\frac{d}{dx}\exp(ax) = a\exp(x)}$ for the exponential function.

```
midifrequency[m_] := N[440 2^((m - 69)/12)]
```

The classical piano covers the 88 Midi tone scale from 21 to 108. The lowest frequency is $27.5Hz$, the sub-contra-octave A, the highest $4186.01Hz$, the 5-line octave C.



Decomposition in overtones: low and high pass filter Every wave form can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can here melodies. Here is an example of a decomposition: $f(x) = \sin(x) + \sin(2x)/2 + sin(3x)/3 + \sin(4x)/4 + \sin(5x)/5$. With infinitely many terms, one can also describe discontinuous functions.

Filtering and tuning: pitch and autotune An other advantage of a decomposition of a function into basic building blocks is that one can leave out frequencies which are not good. Examples are **low pass** or **high pass** filters. A popular filter is **autotune** which does not filter but moves the frequencies around so that you can no more sing wrong. If 440 Herz (A) and 523.2 Herz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4\cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4\cos(2\pi 523.2x)$. This filtering is done on the wave form scale.

Mixing different functions: rip and remix If f and g are two functions which represent songs, we can look at $(f+g)/2$ which is the **average** of the two songs. In real life this is done using **tracks**. Different instruments can be recorded independently for example and then mixed together. One can for example get guitar $g(t)$, voice $v(t)$ and piano $p(t)$ and form $f(t) = ag(t) + bv(t) + c(p(t)$, where the constants a, b, c are chosen.

Differentiate functions: reverberate and echo If f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x+h) - f(x)]/h$. Such a differentiation is easy to achieve with a real song. It turns out that for small h, like of order of $h = 1/1000$, the song does not change much. The reason is that a frequency $\sin(kx)$ or hearing the derivative $\cos(kx)$ produces the same song. However, if we allow h to be larger, then a **reverberate** or **echo** effect is produced.

Other relations with math

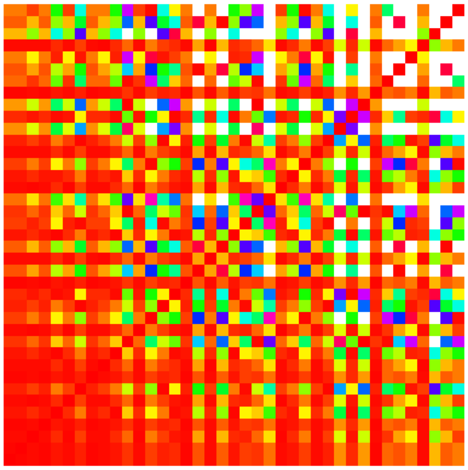
We might not have time for this during the lecture.

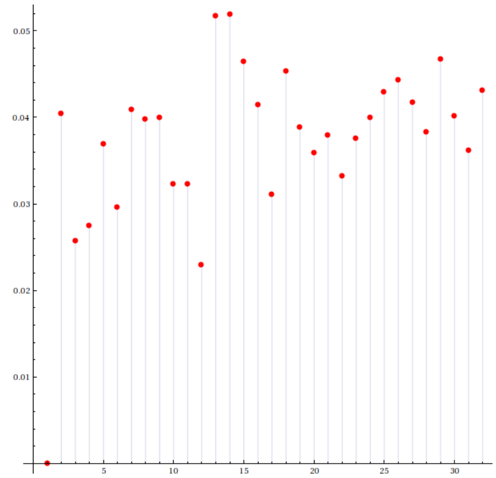
Symmetries. Symmetries play an important role in art and science. In geometry we know rotational, translational symmetries or reflection symmetries. Like in geometry, symmetries play a role both in Calculus as well as in Music. We see some examples in the presentation.

Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**, **Diophantine problems** like how good frequency rations are approximated by rationals: Why is the **chromatic scale** based on the twelfth root of 2 so good? Indian music for example uses **microtones** and a scale of 22. The 12 tone scale is good because many powers $2^{k/12}$ are close to rational numbers. I once defined the "scale fitness" function

$$M(n) = \sum_{k=1}^{n} \mathrm{min_{p,q}} |2^{k/n} - \frac{p}{q}| G(p,q)$$

which is a measure on how good a music scale is. It uses Euler's **gradus suavis** (="degree of pleasure") function $G(n,m)$ of a fraction n/m which is $G(n,m) = 1 + E(nm/\gcd(n,m))$, where the **Euler gradus** function $E(n) = \sum_{p|n} e(p)(p-1)$ and p runs over all prime factors p of n and $e(p)$ is the multiplicity. The picture to the left shows Euler's function $G(n,m)$, the right hand side the scale fitness function in dependence on n. You see that $n = 12$ is clearly the winner. This analysis could be refined to include scales like Stockhausen's $5^{k/25}$ scale. You can listen to the Stockhausen's scale with $f(t) = \sin(2\pi t 100 \cdot 5^{[t]/25})$, where $[t]$ is the largest integer smaller than t. Our familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 100 \cdot 2^{[t]/12})$.





3 The perfect fifth 3/2 has the gradus suavis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth 4/3 for which $1 + E(12) = 1 + (2-1)(3-1)$. You can listen to the perfect fifth $f(x) = \sin(1000x) + \sin(1500x)$ or the perfect fourth $\sin(1000x) + \sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x) + \sin(1333x) + \sin(1500x) + \sin(2000x)$.

Homework

1 **Modulation.** Draw the hull function of the following functions.

a) $f(x) = \cos(200x) - \cos(201x)$

b) $f(x) = \cos(x) + \cos(\tan(1000\sqrt{x}))$

c) $f(x) = \sqrt{x}\cos(10000x)$

d) $f(x) = \cos(x)\sin(e^{2x})/2$

Here is how to play a function with Mathematica. It will play for 9 seconds:

```
Play[Cos[x] Sin[Exp[2 x]]/x, {x, 0, 10}]
```

Hint. You can play functions online with Wolfram Alpha. Here is an example:

```
play sin(1000 x)
```

2 **Amplitude modulation (AM)**: If you listen to $f(x) = \cos(x)\sin(1000x)$ you hear an amplitude change. Draw the hull function. How many increase in amplitudes to you hear in 10 seconds?

3 **Frequency Modulation (FM)**: If we play $f(x) = x\sin(1000\sin(x))$ we see frequency changes. Draw the hull function. Try first without computer.

4 **Smoothness**: If we play the function $f(x) = \tan(\sin(3000\sin(x)))$, the sound sounds pretty nice. If we change that to $f(x) = \tan(2\sin(3000\sin(x)))$, the sound is awful. Can you see why? To answer this, you might want to plot a similar function where 3000 is replaced by 3. Think about the early part of the course.

5 **A mystery sound**: How would you describe the sound $f(x) = \cos(1/\sin(2\pi 3x))$? Our ear can not hear frequencies below 20 Hertz. Why can one still hear something? To answer this, plot first the function from $x = 0$ to $x = 10$.

Lecture 33: Worksheet

Calculus in Music

1 How do you think the function

$$f(x) = \sin(10000\sqrt{x})$$

sounds?

2 What about

$$f(x) = \sin(10000x^2)?$$

3 And finally

$$\sin(x)\sin(1000x)?$$

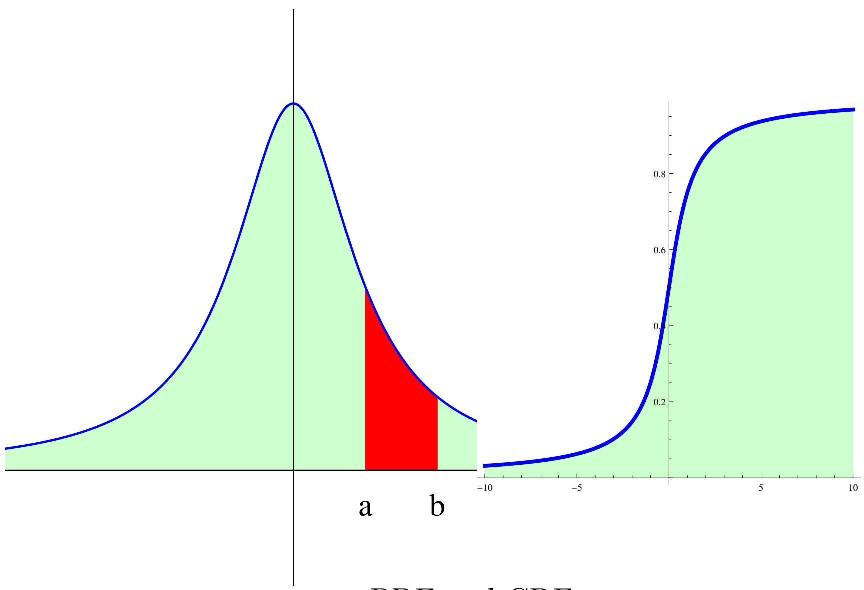
Lecture 34: Calculus and Statistics

In this lecture, we look at an application of calculus to statistics. We have already defined the probability density function f called PDF and its anti-derivative, the cumulative distribution function CDF.

Probability density

Recall that a probability density function is a function f satisfying $\int f(x)\, dx = 1$ and which has the property that it is ≥ 0 everywhere. We say f is a probability density function on an interval $[a, b]$ if $\int_a^b f(x)\, dx = 1$ and $f(x) \geq 0$ there. In such a case, we assume that f is zero outside the interval.

Recall also that we called the antiderivative of f the cumulative distribution function $F(x)$ (CDF).



PDF and CDF

Expectation

The **expectation** of probability density function f is

$$m = \int_{-\infty}^{\infty} x f(x)\, dx \; .$$

In the case, when the probability density function is zero outside some interval, we have

The **expectation** of probability density function f defined on some interval $[a, b]$ is

$$m = \int_a^b x f(x)\, dx \; .$$

As the name tells, the expectation tells what is the average value we expect to get.

Variance and Standard deviation

The **variance** of probability density function f is

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx - m^2 \ ,$$

where m is the expectation.

Again, if the probability density function is defined on some interval $[a,b]$ then

The **variance** of probability density function f is

$$\int_a^b x^2 f(x)\, dx - m^2 \ ,$$

where m is the expectation of f.

The square root of the variance is called the **standard deviation**.

The standard deviation tells us what deviation we expect from the mean.

Examples

In the lecture, we will compute this in some examples. Here is some sample.

1 The expectation of the geometric distribution $f(x) = e^{-x}$

$$\int x e^{-x}\, dx = 1 \ .$$

The variance of the geometric distribution $f(x) = e^{-x}$ is 1 and the standard deviation 1 too.

Remember that we can compute also with Tic-Tac-Toe:

$$\int x^2 e^{-x}\, dx$$

x^2	e^{-x}	
$2x$	$-e^{-x}$	$\oplus$
2	e^{-x}	$\ominus$
0	e^{-x}	$\oplus$

2 The expectation of the standard Normal distribution $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$

$$\int_0^{\infty} x (2\pi)^{-1/2} e^{-x^2/2}\, dx = 0 \ .$$

3 The variance of the standard Normal distribution $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$

$$\int_0^{\infty} x (2\pi)^{-1/2} x^2 e^{-x^2/2}\, dx = 0 \ .$$

We can do that by partial integration too. Its a bit more tricky.

The next example is for trig substitution:

4 The distribution on $[-1,1]$ with function $(1/\pi)(1-x^2)^{-1/2}$ is called the arcsin distribution. What is the cumulative distribution function? What is the mean m? What is the standard deviation σ? We will compute this in class. The answers are $m = 0, \sigma = 1/\sqrt{2}$.

Homework

1 The function $f(x) = \cos(x)/2$ on $[-\pi/2, \pi/2]$ is a probability density function. Its mean is 0. Find its variance

$$\int_{-\pi/2}^{\pi/2} x^2 \cos(x)\, dx \, .$$

2 The **uniform distribution on** $[a, b]$ is a distribution, where any real number between a and b is equally likely to occur. The probability density function is $f(x) = 1/(b-a)$ for $a \leq x \leq b$ and 0 elsewhere. Verify that $f(x)$ is a valid probability density function.

3 Verify that the function which is 0 for $x < 0$ and equal to

$$f(x) = \frac{1}{\log(2)} \frac{e^{-x}}{1+e^{-x}}$$

for $x \geq 0$ is a probability density function.

4 A particular **Cauchy distribution** has the probability density

$$f(x) = \frac{1}{\pi} \frac{1}{(x-1)^2+1} \, .$$

Verify that $f(x)$ is a valid probability density function.

5 Find the cumulative distribution function (CDF) $F(x)$ of f in the previous problem.

Lecture 34: Worksheet

Calculus in Statistics

1 Verify that the function $f(x) = \frac{1}{x}$ is a probability density function on $[1, e]$. As usual we assume that f is zero outside the interval $[1, e]$.

2 Find the expectation

$$m = \int_1^e x f(x)\, dx$$

of this distribution function f.

3 Find the variance

$$\int_1^e x^2 f(x)\, dx - m^2$$

of f.

Lecture 35: Calculus and Economics

In this lecture we look at applications of calculus to **economics**. This is an opportunity to review extrema problems and synchronize some jargon between the different disciplines. Here is a random dictionary:

calculus	economics
problem	program
extrema problem	programming problem
the derivative of	marginal
C'	marginal cost
R'	marginal revenue
other things kept constant	ceteris paribus
$C(x)/x$	average cost
$x^\alpha y^\beta$	Cobb-Douglas function
f'' changes sign	non-convexity
mean	average
$f' > 0$	growth, boom
$f' < 0$	decline, recession
horizontal asymptote	stagnation
vertical asymptote	crash
discontinuity	inelastic behavior
catastrophe	catastrophe

Marginal and total cost

Recall that the **marginal cost** was defined as the derivative of the **total cost**. Both, the marginal cost and total cost are functions of the quantity of goods produced.

1 Assume the total cost function is $C(x) = 10x - 0.01x^2$. Find the marginal cost and the place where the total cost is minimal. **Solution**. Differentiate $C' = 10 + 0.02x$ Now find x which makes this vanish. We have $x = 50$.

2 You sell spring water. The marginal cost to produce it is given by $f(x) = 10000 - x^4$. For which x is the total cost maximal?

3 The following example is adapted from the book "Dominik Heckner and Tobias Kretschmer: Don't worry about Micro, 2008", where the following strawberry story appears: (verbatim citation in italics):

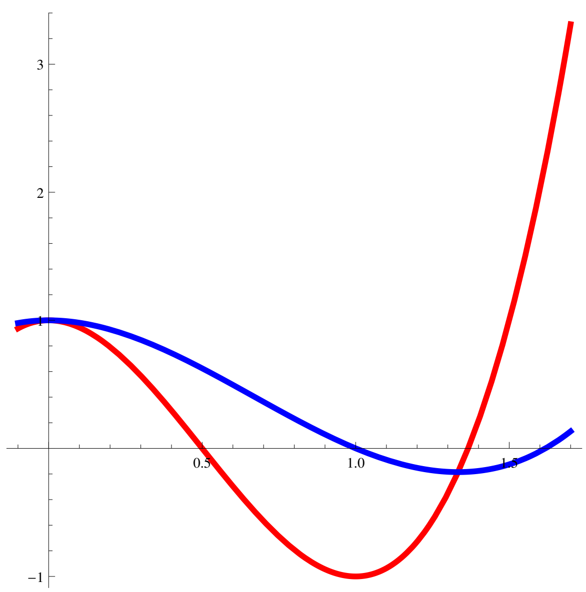
Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

Again, if $F(x)$ is the **total cost function** in dependence of the quantity x, then $F' = f$ is called the **marginal cost**.

The function $g(x) = F(x)/x$ is called the **average cost**.

A point where $f = g$ is called a **break even point**.

4 If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, lets call it the "strawberry theorem":

Strawberry theorem: We have $g'(x) = 0$ if an only if $f = g$.

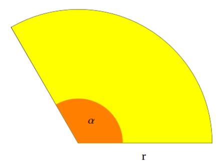
Proof.

$$g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g) .$$

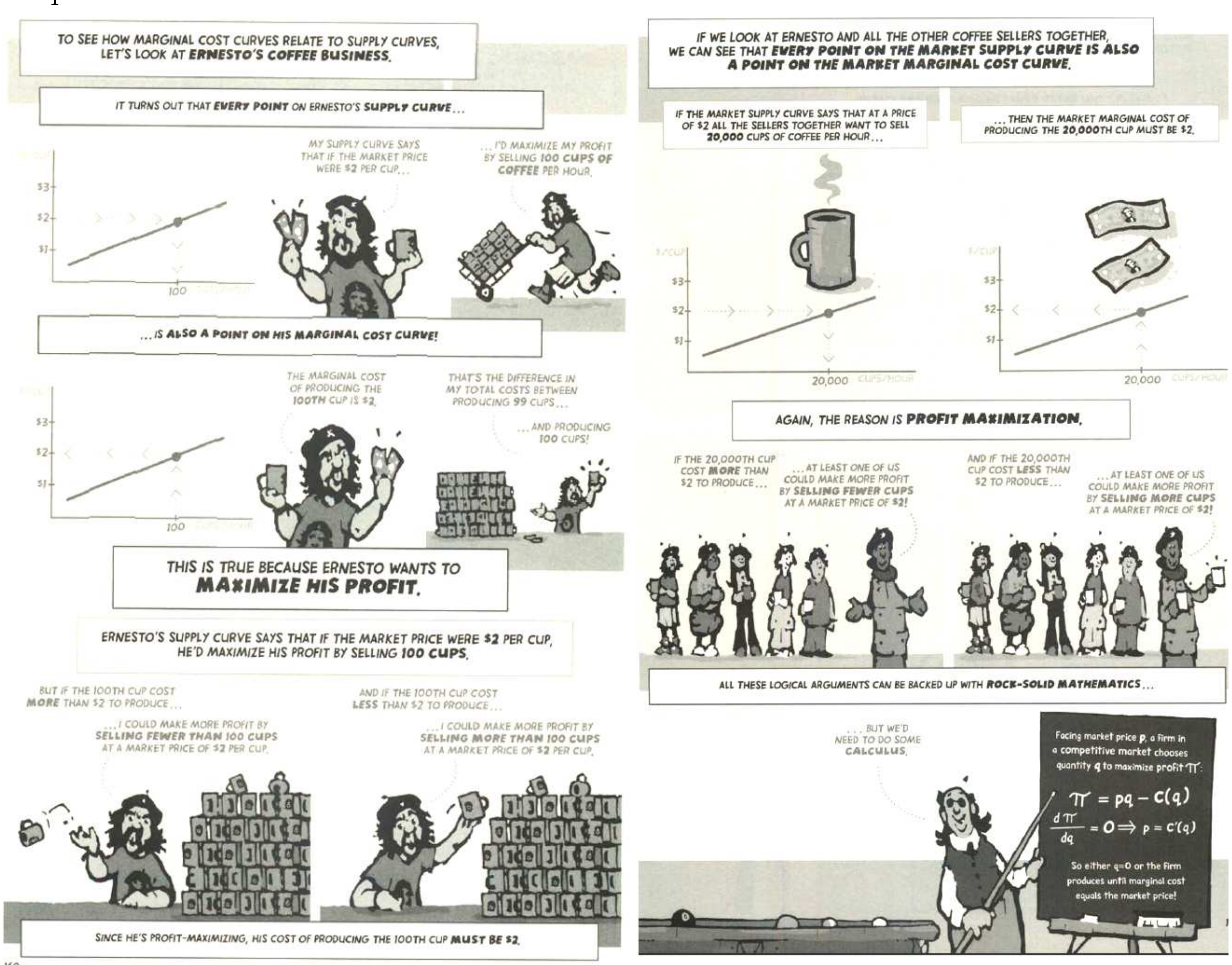
More extremization problems

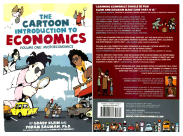
1 Find the rhomboid with side length 1 which has maximal area. Use an angle α to extremize.

2 Find the sector of radius $r = 1$ and angle α which has minimal circumference $f = 2r + r\alpha$ if the area $r^2\alpha/2 = 1$ is fixed. **Solution.** Find $\alpha = 2/r^2$ from the second equation and plug it into the first equation. We get $f(r) = 2r + 2/r$. Now the task is to find the places where $f'(r) = 0$.



3 Find the ellipse of length $2a$ and width $2b$ which has fixed area $\pi ab = \pi$ and for which the sum of diameters $2a + 2b$ is maximal. **Solution.** Find $b = 1/a$ from the first equation and plug into the second equation. Again we have to extremize $f(a) = 2a + 2/a$. The same problem as before.





Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang. You can detect the strawberry theorem ($g' = 0$ is equivalent to $f = g$) can be seen on the blackboard.

Homework

1 a) Find the break-even point for an economic system if the marginal cost is $f(x) = 1/x$. b) Assume the marginal cost is $f(x) = x^7$. Verify that the average cost $g(x) = F(x)/x$ satisfies $8g(x) = f(x)$ and that $x = 0$ is the only break even point.

2 Let $f(x) = \cos(x)$. Compute $F(x)$ and $g(x)$ and verify that $f = g$ agrees with $g' = 0$.

3 The **production function** in an office gives the production $Q(L)$ in dependence of labor L. Assume $Q(L) = 5000L^3 - 3L^5$. Find L which gives the maximal production.

This can be typical: For smaller groups, production usually increases when adding more workforce. After some point, bottle necks occur, not all resources can be used at the same time, management and bureaucracy is added, each person has less impact and feels less responsible, meetings slow down production etc. In this range, adding more people will decrease the productivity.

4 **Marginal revenue** f is the rate of change in **total revenue** F. As total and marginal cost, these are functions of the **cost** x. Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the points, where the total revenue has a local maximum.

5 Find a line $y = mx$ through the points $(3, 4), (6, 3), (2, 5)$. which minimize the function

$$f(m) = (3m - 4)^2 + (6m - 3)^2 + (2m - 5)^2 .$$

Lecture 35: Worksheet

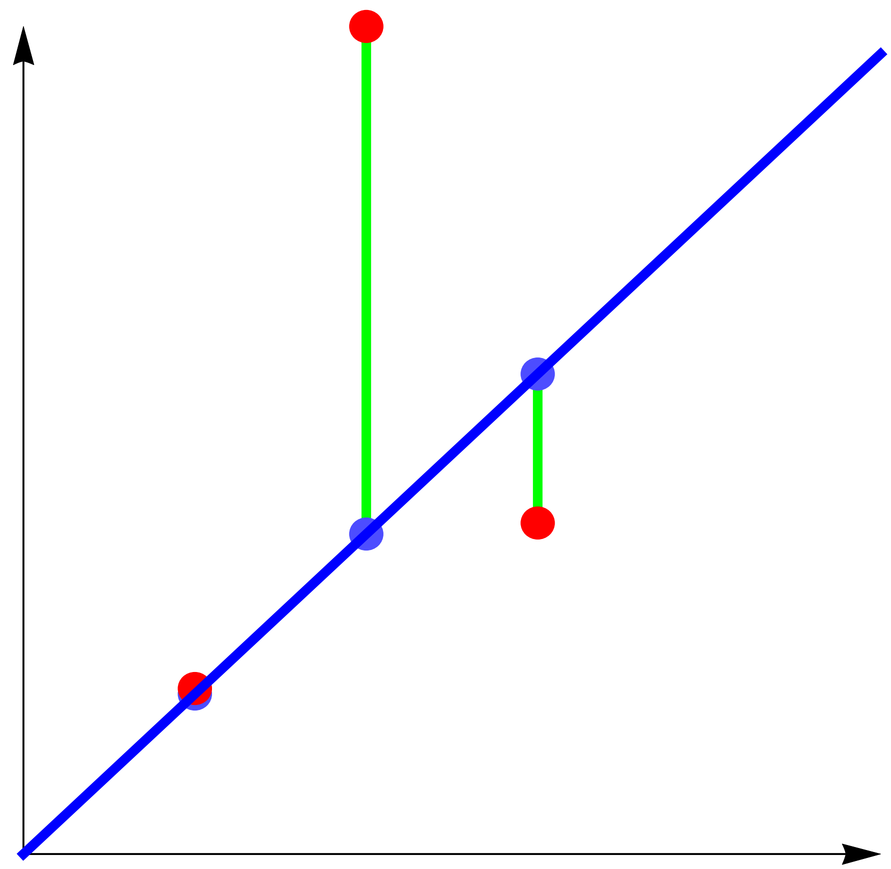
Calculus in Economics

Extremisation is by definition a big deal in economics. We look at more examples. Assume we have a couple of data points and we want to find the best line $y = mx$ through this in the sense that the sum of the squares of the points to the line is minimal. This leads to an extremal problem which is a special case of a **data fitting problem**. It would be more adequate to fit with lines $y = mx + b$ or more generally with functions but then we have more variables and run into **multivariable calculus** or **linear algebra problems** much outside the scope of this course. But if we have only one parameter, we get a single variable calculus problem.

1 Find the best line $y = mx$ through the points $(1, 1), (3, 2), (2, 5)$. We have to minimize the function.

$$f(m) = (m - 1)^2 + (3m - 2)^2 + (2m - 5)^2$$

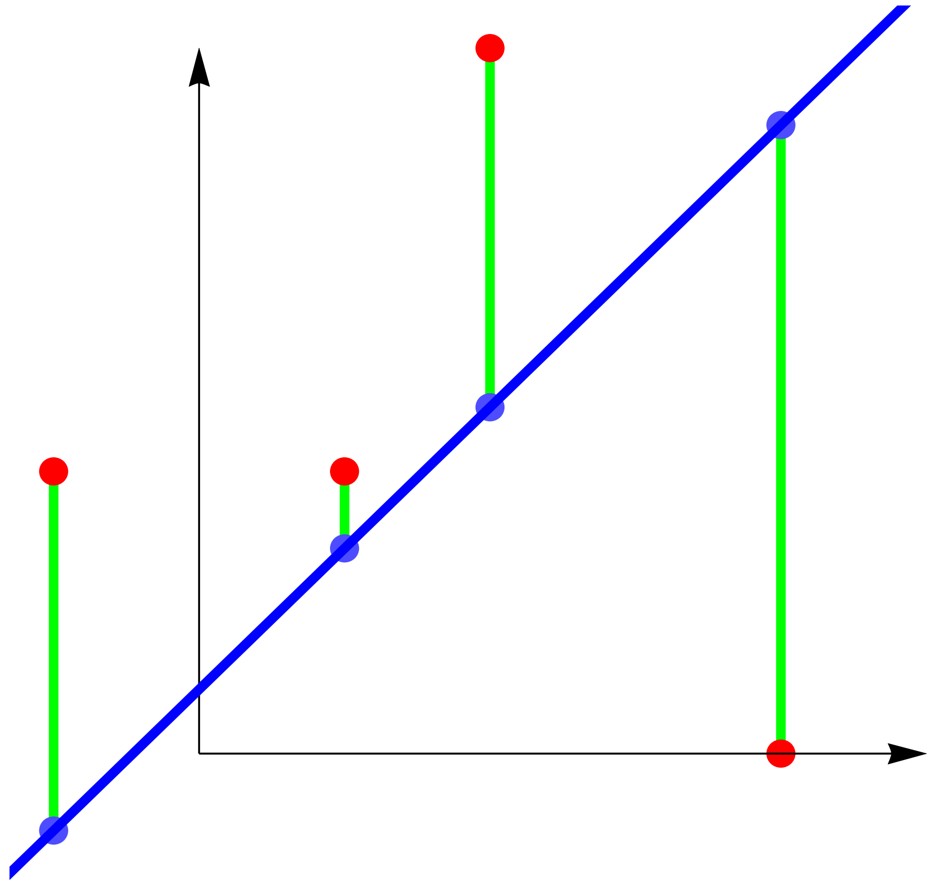
Find the minimum. **Solution**. 17/14



Lets take a different set of data points and look at the problem to fit functions of the form $y = x + b$.

2 Find the best line $y = x + b$ through the points

$$(1,2),(2,5),(-1,2),(4,0),(3,1)\ .$$



9.5 The Connection Between Marginal and Average Costs

This section is slightly more technical than the rest of this chapter. The subject of our analysis at this point is the connection between different cost curves. To be more precise, we investigate how the MC curve cuts through the average cost curves at their respective minima. This is shown in Fig. 9.4.

Shutdown and Breakeven Points

Before we commence with our analysis, let us link the graph back to some of the discussion above. Looking at Fig. 9.4 you may have noticed that the two quantities for the intersection of MC with AVC and MC with ATC are labelled Q_S and Q_B. These are the shutdown and breakeven points, respectively.

> The MC curve crosses the AVC and ATC curves at their respective minima.

As we discussed previously, when employing the profit-maximising condition and when AVC is equal to MC, we would be just about indifferent between shutting down or producing in the short run. Secondly, when we set price equal to MC and when at this point MC is also equal to ATC, the firm is just breaking even.

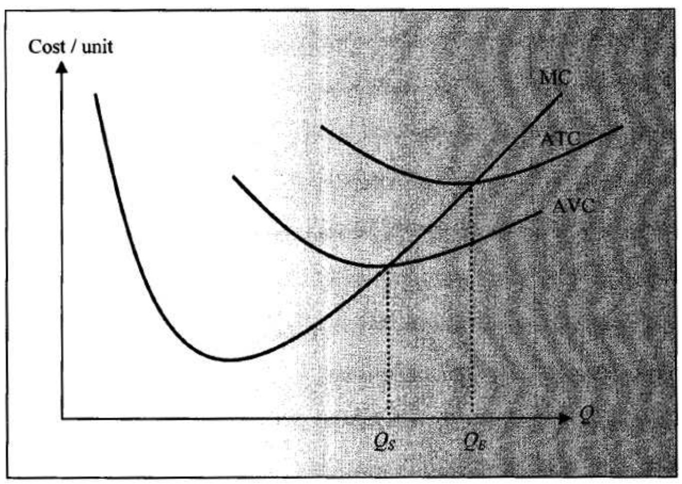


Fig. 9.4. The marginal cost MC curve cuts through average variable cost AVC and average total cost ATC curves at their respective minima. These points are the shutdown and break-even points, respectively

You take one strawberry after another and place them on a scale that tells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller (until you reach the smallest one). Because of the literal "weight" of the heavier ones, average weight is larger than marginal weight (i.e. the weight of each strawberry you handle). Average weight still decreases, although less steeply than marginal weight.

Once you reach the smallest strawberry, every subsequent strawberry will be larger, which means that the rate of decrease of the average weight becomes smaller and smaller until eventually it stands still. At this point, the marginal weight is just equal to the average weight.

This logic is an analogy of why MC cuts through the average cost curves at their minimums. The reasoning is identical for both AVC and ATC.

Mathematical Proof

Rather than blindly trusting the intuition above, we can also prove our analysis mathematically. Let us perform this proof for the intersection of MC and ATC. Our first step is to compute the derivative of ATC with respect to Q and set this equal to zero to find the curve's critical point, here the minimum:

$$\frac{d\text{ATC}}{dQ} = 0 \tag{9.14}$$

In order to make Equation 9.14 usable, let us substitute TC/Q for ATC. Therefore, we get:

$$\frac{d(\frac{\text{TC}}{Q})}{dQ} = 0 \tag{9.15}$$

To avoid complicated calculus, let us reformulate the numerator as a product:

$$\frac{d(\text{TC}\cdot Q^{-1})}{dQ} = 0 \tag{9.16}$$

Remembering the **product rule**, we differentiate TC $\cdot\, Q^{-1}$ with respect to Q by taking the derivative of the first term and multiplying it by the second, and adding the derivative of the second term and multiplying it by the first. This gives us:

$$\frac{d\text{TC}}{dQ}Q^{-1} - \text{TC}\cdot Q^{-2} = 0 \tag{9.17}$$

Notice the negative sign between the terms, which is a result of the -1 "brought down" from Q^{-1} of Equation 9.16 in the process of differentiation. When looking at Equation 9.17, we notice that the very first term is MC and so we can write:

$$\text{MC}\cdot Q^{-1} - \text{TC}\cdot Q^{-2} = 0 \tag{9.18}$$

As a final step we multiply both sides by Q and write the second term as a fraction:

$$\text{MC} - \frac{\text{TC}}{Q} = 0 \tag{9.19}$$

Since, by definition, TC/Q is equal to ATC, we finalise our equation to become:

$$\text{MC} - \text{ATC} = 0 \tag{9.20}$$

Now our task of proving that ATC is equal to MC when ATC is at its minimum is easy. Having taken the derivative of ATC in Equation 9.14 to show its minimum, we have worked all the way to Equation 9.20. This last Equation will hold true, i.e. will correspond to a minimum of the ATC curve when we set MC and ATC equal to each other. Hence, when MC is equal to ATC, ATC is at its minimum. The same mathematical steps can be followed to prove the intersection of AVC and MC at the minimum of AVC.

Source: "Dominik Heckner and Tobias Kretschmer: Don't worry about Micro, 2008", pages 271-274.

Lecture 36: Calculus and AI

What would it take to build an artificial calculus teacher?

Machines assist us already in many domains: heavy work is done by **machines and robots**, accounting by **computers** and fighting by **drones**. Lawyers and doctors are assisted by artificial intelligence. There is no reason why teaching is different. The **web** has become a "gigantic brain" to which virtually any question can be asked or googled: "Dr Know" in Spielberg's movie "AI" is humbled: enter symptoms for an illness and get a diagnosis, enter a legal question and find previous cases. Enter a calculus problem and get an answer. Building an **artificial calculus teacher** involves calculus itself: such a bot must connect dots on various levels: understand questions, read and grade papers and exams, write good and original exam questions, know about learning and pedagogy. Ideally, it should also have "ideas" like to "make a lecture on artificial intelligence". But first of all, our AI friend needs to know calculus and be able to generate and solve calculus problems. [1]

Generating calculus problems

Having been involved in a linear algebra book project once, helping to generating solutions to problems, I know that some calculus books are written with help of computer algebra systems. They generate problems and solutions. This applies mostly to drill problems. In order to generate problems, we first must build **random functions**. Our AI engine "sofia" knew how to generate random problems with solutions. probably in one day as much content as our Sofia group could do in a week for our "pet project". Random functions are involved when asked "give me an example of a function". This is easy: the system would generate functions of reasonable complexity:

Call the 10 functions $\{\mathrm{sin}, \mathrm{cos}, \mathrm{log}, \mathrm{exp}, \mathrm{tan}, \mathrm{sqt}, \mathrm{pow}, \mathrm{inv}, \mathrm{sca}, \mathrm{tra}\}$ **basic functions**.

Here $\mathrm{sqt}(x) = \sqrt{x}$ and $\mathrm{inv}(x) = 1/x^k$ for a random integer k between -1 and -3, $\mathrm{pow}(x) = x^k$ for a random integer k between 2 and 5. $\mathrm{sca}(x) = kx$ is a scalar multiplication for a random nonzero integer k between -3 and 3 and $\mathrm{tra}(x) = x+k$ translates for a random integer k between -4 and 4.

Second, we use addition, subtraction multiplication, division and composition to build more complicated functions:

A **basic operation** is an operation from the list $\{f \circ g, f + g, f * g, f/g, f - g\}$.

The operation x^y is not included because it is equivalent to $\exp(x \log(y)) = \exp \circ (x \cdot \log)$. We can now build functions of various complexities:

[1] In the academic year of 2003/2004, thanks to a grant from the Harvard Provost, I could work with undergraduates **Johnny Carlsson**, **Andrew Chi** and **Mark Lezama** on a "calculus chat bot". We spent a couple of hours per week to enter mathematics and general knowledge, build interfaces to various computer algebra systems like Pari, Mathematica, Macsyma and build a web interface. We fed our knowledge to already known chat bots and newly built ones and even had various bots chat with each other. We explored the question of automated learning of the bots from the conversations as well as to add context to the conversation, since bots needs to remember previous topics mentioned to understand some questions. We learned how immense the task is. In the mean time it has become business. Companies like **Wolfram research** have teams of mathematicians and computer scientists working on content for the "Wolfram alpha" engine.

A **random function** of complexity n is obtained by taking n random basic functions $f_1, \dots, f_n$, and n random basic operators $\oplus_1, \dots, \oplus_n$ and forming $f_n \oplus_n f_{n-1} \oplus_{n-1} \cdots \oplus_2 f_1 \oplus_1 f_0$ where $f_0(x) = x$ and where we start forming the function from the right.

1 **Visitor:** "Give me an easy function": Sofia looks for a function of complexity one: like $x\tan(x)$, or $x + \log(x)$, or $-3x^2$, or $x/(x-3)$.

2 **Visitor:** "Give me a function": Sofia returns a random function of complexity two: $x\sin(x) - \tan(x)$, or $-e^{\sqrt{x}} + \sqrt{x}$ or $x\sin(x)/\log(x)$ or $\tan(x)/x^4$.

3 **Visitor:** "Give me a difficult function": Sofia builds a random function of complexity four like $x^4 e^{-\cos(x)}\cos(x) + \tan(x)$, or $x - \sqrt{x} - e^x + \log(x) + \cos(x)$, or $(1+x)(x\cot(x) - \log(x))/x^2$, or $(-x + \sin(x+3) - 3)\csc(x)$

Now, we can build a random calculus problem. To give you an idea, here are some templates for integration problems:

A **random integration problem** of complexity n is a sentence from the sentence list { "Integrate $f(x) = F(x)$", "Find the anti derivative of $F(x)$", "What is the integral of $f(x) = F(x)$?", "You know the derivative of a function is $f'(x) = F(x)$. Find $f(x)$." }, where F is a random function of complexity n.

4 **Visitor** "Give me a differentiation problem". **Sofia:** Differentiate $f(x) = x\sin(x) - \frac{1}{x^2}$. The answer is $\frac{2}{x^3} + \sin(x) + x\cos(x)$.

5 **Visitor**: "Give me a difficult integration problem". **Sofia**: Find f if $f'(x) = \frac{1}{x} + \left(3\sin^2(x) + \sin(\sin(x))\right)\cos(x)$. The answer is $\log(x) + \sin^3(x) - \cos(\sin(x))$.

6 **Visitor**: "Give me an easy extremization problem". **Sofia**: Find the extrema of $f(x) = x/\log(x)$. The answer is $x = e$.

7 **Visitor**: Give me an extremization problem". **Sofia**: Find the maxima and minima of $f(x) = x - x^4 + \log(x)$. The extrema are

$$-\frac{\sqrt{\left(9+\sqrt{3153}\right)^{2/3} - 8\sqrt[3]{6}} + \sqrt{8\sqrt[3]{6} - \left(9+\sqrt{3153}\right)^{2/3}\left(1 + 6\sqrt{\frac{2}{9+\sqrt{3153}-8\sqrt[3]{6\left(9+\sqrt{3153}\right)}}}\right)}}{2 2^{5/6}\sqrt[3]{3}\sqrt[6]{9+\sqrt{3153}}}.$$

The last example shows the perils of random generation. Even so the function had decent complexity, the solution was difficult. Solutions can even be transcendental. This is not a big deal: just generate a new problem. By the way, all the above problems and solutions have been generated by Sofia. The dirty secret of calculus books is that there are maybe a thousand different type of questions which are usually asked. This is a reason why textbooks have become boring clones of each other and companies like "Aleks", "demidec" etc exist which constantly mine the web and course sites like this and homework databases like "webwork" which contain thousands of pre-compiled problems in which randomness is already built in.

Automated problem generation is the "fast food" of teaching and usually not healthy. But like "fast food" has evolved, we can expect more and more computer assisting in calculus teaching.

Be assured that for this course, the problems have been written by hand (I sometimes use Mathematica to see whether answers are reasonable). Handmade problems can sometimes a bit "rough" but hopefully some were more interesting. I feel that it is not fair to feed computer generated problems to humans. It is possible to write a program giving an answer to "Write me a final exam", but the exam would be uninspiring.

Here is an 1A exam completely written by a bot:

Homework

1 Lets build a differentiation problem by combining log and sin and exp. Differentiate all of the 6 combinations $\log(\sin(\exp(x)))$, $\log(\exp(\sin(x)))$,$\exp(\log(\sin(x)))$, $\exp(\sin(\log(x)))$, $\sin(\log(\exp(x)))$ and $\sin(\exp(\log(x)))$.

2 Four of the 6 combinations of log and sin and exp can be integrated as elementary functions. Do these integrals.

3 From the 10 functions f and 10 functions g and 5 operations, we can build 500 functions. Statistics shows that 5 can not be integrated. An example is $\exp(\sin(x))$. Find 4 more.

4 Lets be creative! Build an extremization problem which is applied. For example: a common theme for extrema are area and length. Invent an extremum problem involving an isoscele triangle. It should be of the form: "Maximize the area ... ". Now solve the problem.

5 Now lets be creative and build a volume problem for a surface of revolution similarly than the Hershey Kiss problem in the second midterm. The problem should not have appeared yet in lecture, homework or exams. Now solve your problem.

Lecture 36: Worksheet

This worksheet as well as the solutions was generated by Sofia, a bot written in the academic year 2003/2004 using grant from the Harvard Provost together with Harvard students **Johnny Carlsson**, **Andrew Chi** and **Mark Lezama**. At that time, people have laughed at the chat bot idea. Now it is big business: Google, Siri, Cortana, Wolfram alpha: these are all AI bots which constantly become more and more sophisticated.

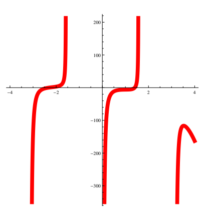
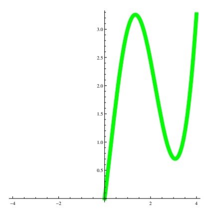
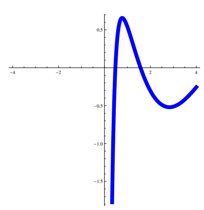
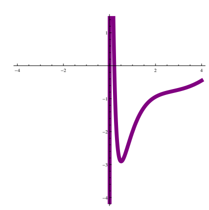
1 Differentiate the following functions:
a) $f(x) = -3x$
b) $f(x) = 2\cos(x)$
c) $f(x) = 0$

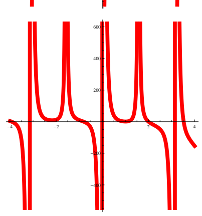
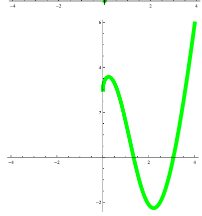
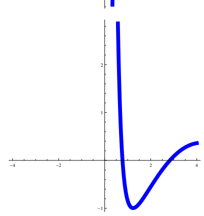
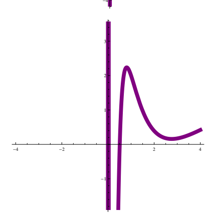
2 Integrate the following functions:
a) $f(x) = 2(\log(x) + 1)$
b) $f(x) = 3\left(\sec^2(x) + 1\right)$
c) $f(x) = 2\cos(x)$

3 Differentiate the following functions:
a) $f(x) = 4e^{-x}x\cot(x)$
b) $f(x) = 3(\tan(x) + 3)$
c) $f(x) = 4x^4\cos(x)$

4 Integrate the following functions:
a) $f(x) = 4\tan^2(x)$
b) $f(x) = \frac{3}{2\sqrt{x}} - 3\sec^2(x) + 3$
c) $f(x) = \frac{x+3\log(x)+3}{(x+3)^2}$

5 Match the following functions with derivatives:

a)  b)  c)  d) 

1)  2)  3)  4) 

6 Find the critical points of the following functions:
a) $f(x) = (x-9)^2(x-4)$
b) $f(x) = (x-9)(x-6)$
c) $f(x) = (x-5)^2(x-4)$

Lecture 36: Worksheet

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1 Differentiate the following functions:
a) $f(x) = -3x$
b) $f(x) = 2\cos(x)$
c) $f(x) = 0$

Solution:
a) $f'(x) = -3$
b) $f'(x) = -2\sin(x)$
c) $f'(x) = 0$

2 Integrate the following functions:
a) $f(x) = 2(\log(x) + 1)$
b) $f(x) = 3\left(\sec^2(x) + 1\right)$
c) $f(x) = 2\cos(x)$

Solution:
a) $\int f(x) = 2x\log(x) + C$
b) $\int f(x) = 3(x + \tan(x)) + C$
c) $\int f(x) = 2\sin(x) + C$

3 Differentiate the following functions:
a) $f(x) = 4e^{-x}x\cot(x)$
b) $f(x) = 3(\tan(x) + 3)$
c) $f(x) = 4x^4\cos(x)$

Solution:
a) $f'(x) = -4e^{-x}\left((x-1)\cot(x) + x\csc^2(x)\right)$
b) $f'(x) = 3\sec^2(x)$
c) $f'(x) = -4x^3(x\sin(x) - 4\cos(x))$

4 Integrate the following functions:

a) $f(x) = 4\tan^2(x)$

b) $f(x) = \frac{3}{2\sqrt{x}} - 3\sec^2(x) + 3$

c) $f(x) = \frac{x+3\log(x)+3}{(x+3)^2}$

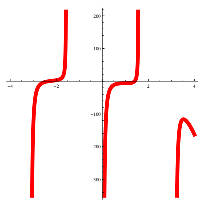
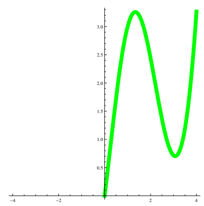
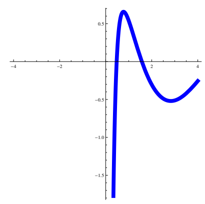
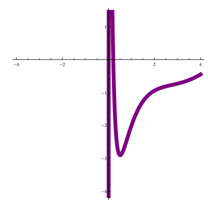
Solution:

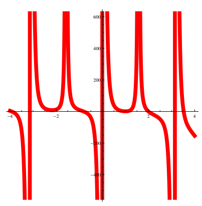
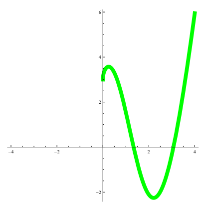
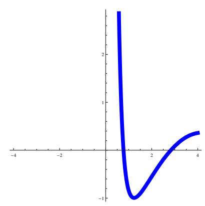
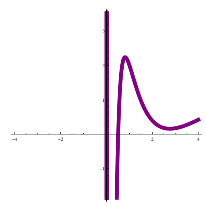
a) $\int f(x) = 4(-x + \tan(x) + 2) + C$

b) $\int f(x) = 3\left(x + \sqrt{x} - \tan(x)\right) + C$

c) $\int f(x) = \frac{x\log(x)}{x+3} + C$

5 Match the following functions with derivatives:

a)  b)  c)  d) 

1)  2)  3)  4) 

Solution:

a $\to$ 1, b $\to$ 2, c $\to$ 3, d $\to$ 4

6 Find the critical points of the following functions:

a) $f(x) = (x-9)^2(x-4)$

b) $f(x) = (x-9)(x-6)$

c) $f(x) = (x-5)^2(x-4)$

Solution:

a) $f'(x) = \left\{\left\{x \to \frac{17}{3}\right\}, \{x \to 9\}\right\}$

b) $f'(x) = \left\{\left\{x \to \frac{15}{2}\right\}\right\}$

c) $f'(x) = \left\{\left\{x \to \frac{13}{3}\right\}, \{x \to 5\}\right\}$

Lecture 37: Final remarks

The holographic picture

How can one tweet calculus in 140 characters? Here it is:

Calculus links rate-of-change=derivative of f with area-under-the-curve=integral F. The derivative F' of the integral F is equal to f.

In one paragraph:

The development of calculus from Archimedes, over Newton and Leibniz to Riemann is a major cultural achievement from the past millennium. Calculus consists of differential and integral calculus. The former studies the "rate of change" f' of a function, the later treats "accumulation" $\int_0^x f(x)\ dx$ which can be interpreted as "area under the curve". The fundamental theorem of calculus links the two: it tells that

$$\int_0^x f'(t)\ dt = f(x) - f(0), \frac{d}{dx}\int_0^x f(t)\ dt = f(x)\ .$$

The subject can be applied to problems from other scientific disciplines like economics where total, average and marginal costs can be related. It allows to estimate the speed of an airplane flying over your head, or explain catastrophic changes in society. It allows to compute volume and area in geometry, is pivotal in statistics for PDF and CDF. It is also useful in everyday life as it allows to fix a wobbly table.

or

1. Calculus relates two fundamental operations, the derivative measuring the rate of change of a function and the integral, which measures the area under the graph.
2. Taking derivatives is done with the chain, product and quotient rules, taking intervals with substitution including trig substitution, integration by parts and partial fraction rules.
3. Basic functions are polynomials and $\exp, \log, \sin, \cos, \tan$. We can add, subtract, multiply, divide and compose functions, we can differentiate and integrate them.
4. A function is continuous at p if $f(x) \to f(b)$ for $x \to p$. It is differentiable at p if $(f(x+h) - f(x))/h$ has a limit for $h \to 0$. Limits from the right and left should agree.
5. To extremize a function, we look at points where $f'(x) = 0$. If $f''(x) > 0$ we have a local minimum, if $f''(x) < 0$ we have a local maximum. There can be critical points $f'(x) = 0$ without being extremum like x^3.
6) To relate how different quantities change in time, we differentiate the formula relating the quantities using the chain rule. If there is a third time variable, then this is the story of related rates, if one of the variables is the parameter, then this is implicit differentiation.

One could look at math from a historical perspectives or read original things. One could do projects, use more computer algebra systems, practice visualization and visualize things. You have studied maybe 300 hours for this course including homework, reading, and discussing the material. Years would be needed to study it more on a research level. New calculus is constantly developed. I myself have been working mostly on more probabilistic versions of calculus which allows to bypass some of the difficulties when discretizing calculus. The loss of symmetries obtained by discretization can be compensated differently.

The future of calculus

Calculus will without doubt look different in 50 years. Many changes have already started, not only on the context level, also from outside: Calculus books will be gone, electronic paper which will be almost indistinguishable from real paper has replaced it. Text, computations, graphics are all fluid in that we can at any point adjust the amount of details. Similarly than we can zoom into a map or picture by pinching the screen, we can triple pinch a text or proof or picture. As we do so, more details are added, more steps of a calculation added, more information included into a graph etc. Every picture is interactive can turn in a movie, an animation, parameters can be changed, functions deformed with the finger. Every picture is a little laboratory. Questions can be asked directly to the text and answers provided. The text can at any time be set back to an official textbook version of the course. The teacher has the possibility to set global preferences and toss around topics. Examples, homework problems and exam problems will be adjusted automatically disallowing for example to treat integration by parts before the product rule. Much of this is not science fiction, there are electronic interactive books already now available for tablet computers which have impressive experimental and animation features. Impossible because it is too difficult to achieve? Remember the last lecture 36. We will have AI on our side and much of this grunt work to compress and expand knowledge can be done computer assisted.

Calculus courses after 1a

To prepare for this course, I set myself the task to formulate the main topic in one short sentence and then single out 4 major goals for the course, then build titles for each lecture etc Here are 4 calculus courses at Harvard drawn out at the level of a "4 point summary". At other schools of higher education, there are similar courses.

The course 1A	from extremization to the fundamental theorem
functions	polynomials, exp, log, trig functions
limits	velocity, tangents, infinite limits
derivatives	product, chain rule with related rates, extremization
integrals	techniques, area, volume, fundamental theorem
The course 1B	from series and integration to differential equations
integration	integration: parts, trig substitution, partial fractions, indefinite
series	convergence, Power, Taylor and Dirichlet series
diff equations	separation of variables, systems like exponential and logistic equations
systems diff eq	equilibria, nullclines, analysis
The course 21A	geometry, extremization and integral theorems in space
geometry	analytic geometry of space, geometric objects, distances
differentiation	curves and surfaces, gradient, curl, divergence
integration	double and triple integrals, other coordinate systems
integral theorems	line and flux integrals, Green, Stokes and Gauss
The course 21B	matrix algebra, eigensystems, dynamical systems and Fourier
equations and maps	Gauss-Jordan elimination, kernel, image, linear maps
matrix algebra	determinants, eigenvalues, eigenspaces, diagonalization
dynamical systems	difference and differential equations with various techniques
fourier theory	Fourier series and dynamical systems on function spaces

There is also a 19a/19b track. The 19a course focuses on models and applications in biology, the 19b course replaces differential equations from 21b with probability theory. The Math 20 course covers linear algebra and multivariable calculus for economists in one semester but covers less material than the 21a/21b track.

The lighter side of calculus

Sofia, our bot had also to know a lot of jokes, especially about math. Here are some relevant to calculus in some way. I left out the inappropriate ones.

1 Why do you rarely find mathematicians at the beach? Because they use sine and cosine to get a tan.

2 **Theorem:** The less you know, the more you make. **Proof:** We know Power = Work/Time. Since Knowledge = Power and Time = Money we know Knowledge = Work/Money. Solve for Money to get Money = Work/Knowledge. If Knowledge goes to zero, money approaches infinity.

3 Why do they never serve beer in a calculus class? Because you can't drink and derive.

4 Descartes comes to a bar. Barmen: An other beer? Descartes: I think not. And disappears.

5 If it's zero degrees outside today. Tomorrow it will be twice as cold. How cold will it be?

6 There are three types of calculus teachers: those who can count and those who can not.

7 Calculus is like love; a simple idea, but it can be complicated.

8 A mathematician and an engineer are on a desert island with two palm trees and coconuts. The engineer climbs up, gets its coconut gets down and eats. The mathematician climbs up the other, gets the coconut, climbs the first tree and deposits it. "I've reduced the problem to a solved one".

9 Pickup line: You are so x^2. Can I be $x^3/3$, the area under your curves?

10 The Evolution of calculus teaching:
1960ies: A peasant sells a bag of potatoes for 10 dollars. His costs are 4/5 of his selling price. What is his profit?
1970ies: A farmer sells a bag of potatoes for 10 dollars. His costs are 4/5 of his selling price, that is, 8 dollars. What is his profit?
1980ies: A farmer exchanges a set P of potatoes with a set M of money. The cardinality of the set M is equal to 10, and each element of M is worth one dollars Draw ten big dots representing the elements of M. The set C of production costs is composed of two big dots less than the set M. Represent C as a subset of M and give the answer to the question: What is the cardinality of the set of profits?
1990ies: A farmer sells a bag of potatoes for 10 dollars. His production costs are 8 dollars, and his profit is 2 dollars. Underline the word "potatoes" and discuss it with your classmates.
2000ies: A farmer sells a bag of potatoes for 10 dollars. His or her production costs are 0.80 of his or her revenue. On your calculator, graph revenue vs. costs and run the program POTATO to determine the profit. Discuss the result with other students and start blog about other examples in economics.
2010ies: A farmer sells a bag of potatoes for 10 dollars. His costs are 8 dollars. Use the Potato theorem to find the profit. Then watch the wobbling potato movie.

11 **Q:** What is the first derivative of a cow? **A:** Prime Rib!

12 **Q:** What does the zero say to the eight? **A:** Nice belt!

13 **Theorem.** A cat has nine tails. **Proof.** No cat has eight tails. Since one cat has one more tail than no cat, it must have nine tails.

14 **Q:** How can you tell that a mathematician is extroverted? **A:** When talking to you, he looks at your shoes instead of at his.

15 **Q:** What does the little mermaid wear? **A:** An algae-bra.

16 In a dark, narrow alley, a function and a differential operator meet: "Get out of my way - or I'll differentiate you till you're zero!" "Try it - I'm e^x ..." Same alley, same function, but a different operator: "Get out of my way - or I'll differentiate you till you're zero!" "Try it - I'm e^x..." "Too bad... I'm d/dy."

17 **Q:** How do you make 1 burn? **A:** Fire differentiation at a log.

18 An investment firm hires. In the last round, a mathematician, an engineer, and a business guy are asked what starting salary expectations they had: mathematician: "Would 30,000 be too much?" engineer: "I think 60,000 would be OK." Finance person: "What about 300,000?" Officer: "A mathematician will do the same work for a tenth!" Business guy: "I thought of 135,000 for me, 135,000 for you and 30,000 for the mathematician to do the work.

19 Theorem. Every natural number is interesting. Proof. Assume there is an uninteresting one. Then there is smallest one. But as the smallest, it is interesting. Contradiction!

Lecture 37: Calculus and the world

Last Lecture

Calculus and the world

There would have more to tell. One could make an entire course filled with applications of calculus. We have seen lectures on **music**, **statistics**, **economics** and **computer science**. Here are more ideas. It would be nice to have a few weeks to work them out. The number of applications explodes even more when doing multivariable calculus or linear algebra.

Calculus and Sports

Optimization and analysis of motion. Which path needs least energy? Calculus of motion in various sports.

Calculus and Biology

Exponential growth and decay. Populations grow exponentially. Radioactive particles decay fast.

Calculus and Physics

Chaos theory. How far into the future can we predict a system. Take a map an iterate it. Take a calculator and iterate.

Calculus and Art

We can use functions to generate new art forms using functions.

Calculus and Cosmology

How did the universe evolve. The Lorentz contraction. Is it realistic that we will ever meet an other civilisation.

Calculus and Medicine

Catastrophes happen also in our body. An example is the story of "Period doubling" in the heart.

Calculus and Finance

The mathematics of Finance is complex and is done with stochastic differential equations, chaos theory and power law heuristics.

Calculus and Romance

When is the optimal time to marry? If you choose too early, you don't know what is out there. If you chose too late, you will have to compare with too many previous cases.

Calculus and Friendship

Book by Steven Strogatz or Clio Cresswell: (Romance dynamics)

Calculus and Psychology

We have seen the catastrophic change of perception. Psychology needs a lot of statistics.

Calculus and Politics

Game theory and Equilibria. The calculus of conflict.

Calculus and Philosophy

Is calculus consistent. Can calculus be built in different ways? What is truth? Can we take limits?

Calculus and Architecture

The topic is much linked that most calculus books feature architecture on their book covers.

Calculus and History

The calculus wars between Newton and Leibniz. Or the story of Archimedes.

Calculus and business

Calculus is lucrative business today and the reason for a gold rush in many industries: It currently makes is responsible for billions of profit. Geography (i.e. google earth technology), computer vision (i.e. autonomous driving of cars), Photography (panoramas), artificial intelligence (i.e. optical character recognition), robots (i.e. space exploration), computer games (i.e. world of warcraft), Movies (i.e. CGI used in avatar), network visualization (i.e. in social networks). We live in a time where calculus is made to gold. If linear algebra is added to calculus (page rank example), then applications get even bigger.

Final Review

Please merge with older review sheets which can still found on the website.

Related rates

Implicit differentiation and related rates are applications of the chain rule.
A) Related rates: we have an equation $\boxed{F(x,y)=c}$ relating two variables $x(t), y(t)$ which depend on time t. Differentiate the equation with respect to t using the chain rule and solve for y'.
B) Implicit differentiation: we have an equation $\boxed{F(x,y(x))=c}$ relating y with x. Differentiate the equation with respect to x using the chain rule and solve for y'.

Examples:
A) $x^3+y^3=1$, $x(t)=\sin(t)$, then $3x^2x'+3y^2y'=0$ so that $y'=-x^2x'/y^2=-\sin^2(t)\cos(t)/(1-\sin^3(t))^{1/3}$.
B) Find y' for $x^4+3y^3=x+3y^2$ at $x=1, y=1$: $4x^3+9y^2y'=1+6y'y$ gives $4+9y'=1+6y'$ so that $y'=-1$.

Substitution

Substitution replaces $\int f(x)\,dx$ with $\int g(u)\,du$ with $u=u(x), du=u'(x)dx$. Special cases:

A) The antiderivative of $f(x)=g(u(x))u'(x)$, is $G(u(x))$ where G is the anti derivative of g.
B) $\int f(ax+b)\,dx = F(ax+b)/a$ where F is the anti derivative of f.

Examples:
A) $\int \sin(x^5)x^4\,dx = \int \sin(u)\,du/5 = -\cos(u)/5+C = -\cos(x^5)/5+C$.
B) $\int \log(5x+7)\,dx = \int \log(u)\,du/5 = (u\log(u)-u)/5+C = (5x+7)\log(5x+7)-(5x+7)+C$.

Integration by parts

A) Direct:

$$\int x \sin(x)\ dx = x\ (-\cos(x)) - \int 1\ (-\cos(x)\ )\ dx = -x\cos(x)+\sin(x)+C\ dx\ .$$

B) Tic-Tac-Toe: to integrate $x^2\sin(x)$

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The anti-derivative is

$$-x^2\cos(x)+2x\sin(x)+2\cos(x)+C\ .$$

C) Merry go round: Example $I=\int \sin(x)e^x\,dx$. Use parts twice and solve for I.

Partial fractions

A) Make a common denominator on the right hand side $\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$.
and compare coefficients $1 = Ax - Ab + Bx - Ba$ to get $A + B = 0, Ab - Ba = 1$ and solve for A, B.
B) If $f(x) = p(x)/(x-a)(x-b)$ with different a, b, the coefficients A, B in $\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ can be obtained from

$$A = \lim_{x \to a}(x-a)f(x) = p(a)/(a-b),\ B = \lim_{x \to b}(x-b)f(x) = p(b)/(b-a)\ .$$

Examples:
A) $\int \frac{1}{(x+1)(x+2)}\ dx = \int \frac{A}{x+1}\ dx + \int \frac{B}{x+2}\ dx$. Find A, B by multiplying out and comparing coefficients in the nominator.
B) Directly write down $A = 1$ and $B = -1$, by plugging in $x = -2$ after multiplying with $x - 2$. or plugging in $x = -1$ after multiplying with $x - 1$.

Improper integrals

A) Integrate over infinite domain.
B) Integrate over singularity.

Examples:
A) $\int_0^\infty 1/(1+x^2) = \arctan(\infty) - \arctan(0) = \pi/2 - 0 = \pi/2$.
B) $\int_0^1 1/x^{2/3}\ dx = (3/1)x^{1/3}|_0^1 = 3$.

Trig substitution

When integrating function like $\sqrt{1-x^2}$, replace x by $\sin(u)$.
Example:
$\int_{-1}^1 \sqrt{1-x^2}\ dx = \int_{-\pi/2}^{\pi/2} \cos(u)\cos(u)du = \int_{-\pi/2}^{\pi/2}(1+\cos(2u))/2 = \frac{\pi}{2}$.

Terminology in Application

Music: hull function, piano function
Economics: average cost, marginal cost and total cost. Strawberry theorem
Operations research: find extrema, critical points, 2. derivative test
Computer science: random function from given function
Statistics: PDF, cumulative distribution function, expectation, variance.
Distributions: normal distribution, geometric distribution, Cauchy distribution.
Geometry: area between two curves, volume of solid
Numerical integration: Riemann sum, trapezoid rule, Simpson rule, Monte Carlo
Root finding: Bisection method, Newton method $T(x) = x - f(x)/f'(x)$.
Psychology: critical points and catastrophes, hate of related rate.
Physics: position, velocity and acceleration, work and power
Gastronomy: turn table to prevent wobbling, bottle calibration.

Checklists:

Integral techniques to consider

Try in the following order:

- [ ] Knowing the integral
- [ ] Substitution
- [ ] Trig substitution
- [ ] Integration by parts
- [ ] Partial fractions

Especially:

- [ ] Tic-Tac-Toe for integration by parts
- [ ] Hopital Method for partial fractions
- [ ] Merry go round method for parts

Integrals to know well

- [ ] $\sin(x)$
- [ ] $\cos(x)$
- [ ] $\tan(x)$
- [ ] $\log(x)$
- [ ] $\exp(x)$
- [ ] $1/x$
- [ ] $1/x^n$
- [ ] x^n
- [ ] $\sqrt{x}$
- [ ] $1/\cos^2(x)$
- [ ] $1/\sin^2(x)$
- [ ] $1/(1+x^2)$
- [ ] $1/(1-x^2)$
- [ ] $1/\sqrt{1-x^2}$
- [ ] $\sqrt{1-x^2}$

Applications you have to know

- **Derivative:** Limit of differences $D_h f = [f(x+h) - f(x)]/h$ for $h \to 0$
- **Integral:** Limit of Riemann sums $S_h f = [f(0) + f(h) + \cdots + f(kh)]h$.
- **Newton step:** $T(x) = x - f(x)/f'(x)$.
- **Marginal cost:** the derivative F' of the total cost F.
- **Average cost:** F/x where F is the total cost.
- **Velocity:** Derivative of the position.
- **Acceleration:** Derivative of the velocity.
- **Curvature:** $f''(x)/(1 + f'(x)^2)^{3/2}$.
- **Probability distribution function:** nonnegative function with total $\int f(x)dx = 1$.
- **Cumulative distribution function:** anti-derivative of the PDF.
- **Expectation:** $\int x f(x)\, dx$, where f is the probability density function.
- **Piano function:** frequencies $f(k) = 440 \cdot 2^{k/12}$ for integer k.
- **Hull function:** $\sin(x)\sin(10000x)$ has hull $|\sin(x)|$
- **Catastrophe:** A parameter c at which a local minimum disappears.

Core concepts

- **Fundamental:** The fundamental theorem of calculus
- **Extrema:** Second derivative test
- **Derivatives:** slope rate of change
- **Integrals:** area, volume
- **Limits:** Hôpital!
- **Continuity:** know the enemies of continuity
- **Numerics:** Riemann sum, Trapezoid and Simpson rule
- **Rules:** Differentiation and integration rules.
- **Methods:** Integration by parts, Substitution, Partial fraction.

Not needed on your fingertips

- **Entropy:** $-\int f(x)\log(f(x))\, dx$.
- **Moment of inertia:** $\int x^2 f(x)\, dx$.
- **Monte Carlo integration:** $S_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$, where x_k are random in $[a, b]$.
- **Weierstrass function:** A function which is continuous but nowhere differentiable.
- **Bart Simpson rule:** $S_n = \frac{1}{6n}\sum_{k=1}^{n}[f(x_k) + 4f(y_k) + f(x_{k+1})]$.
- **Cocktail party stuff:** Eat,integrate and love, the story of exp in practice exam 2.
- **Bottles:** How to calibrate bottles. The calibration formula.
- **Sofia:** The name of a calculus bot once roaming the math department.
- **Wobbly chair:** One can turn a chair on any lawn to stop it from wobbling.
- **Song:** The hit: ”everybody hates, related rates”

Lecture 39: Checklists

This checklist complements the previous checklists for the midterms.

Integrals to know well

- [ ] $\sin(x)$
- [ ] $\cos(x)$
- [ ] $\tan(x)$
- [ ] $\log(x)$
- [ ] $\exp(x)$
- [ ] $1/x$
- [ ] x^n
- [ ] $1/\cos^2(x)$
- [ ] $1/\sin^2(x)$
- [ ] $1/(1+x^2)$
- [ ] $1/(1-x^2)$
- [ ] $1/\sqrt{1-x^2}$

Applications you have to know

Since there are few questions on what has to be known about applications and definitions (this list only covers application parts):

- [ ] **Derivative:** Limit of differences $D_h f = [f(x+h) - f(x)]/h$ for $h \to 0$
- [ ] **Integral:** Limit of Riemann sums $S_h f = [f(0) + f(h) + ...f(kh)]h$.
- [ ] **Newton step:** $T(x) = x - f(x)/f'(x)$.
- [ ] **Marginal cost:** the derivative F' of the total cost F.
- [ ] **Average cost:** F/x where F is the total cost.
- [ ] **Velocity:** Derivative of the position.
- [ ] **Acceleration:** Derivative of the velocity.
- [ ] **Curvature:** $f''(x)/(1 + f'(x)^2)^{3/2}$.
- [ ] **Probability distribution function:** nonnegative function with total $\int f(x)dx = 1$.
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Core concepts

- **Fundamental:** The fundamental theorem of calculus
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- **Continuity:** know the enemies of continuity
- **Numerics:** Riemann sum, Trapezoid and Simpson rule
- **Rules:** Differentiation and integration rules.
- **Methods:** Integration by parts, Substitution, Partial fraction.

Not needed on your fingertips

The following concepts have appeared but do not need to be learned by heart:

- **Entropy:** $-\int f(x)\log(f(x))\, dx$.
- **Moment of inertia:** $\int x^2 f(x)\, dx$.
- **Monte Carlo integration:** $S_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$, where x_k are random in $[a,b]$.
- **Weierstrass function:** A function which is continuous but nowhere differentiable.
- **Bart Simpson rule:** $S_n = \frac{1}{6n}\sum_{k=1}^{n}[f(x_k) + 4f(y_k) + f(x_{k+1})]$.
- **Chaikin step:** $R_{2i} = \frac{3}{4}P_i + \frac{1}{4}P_{i+1}, R_{2i+1} = \frac{1}{4}P_i + \frac{3}{4}P_{i+1}$.
- **Cocktail party stuff:** Eat,integrate and love, the story of exp in practice exam 2.
- **Bottles:** How to calibrate bottles. The calibration formula.
- **Sofia:** The name of a calculus bot once roaming the math department.
- **Wobbly chair:** One can turn a chair on any lawn to stop it from wobbling.
- **Warthog:** "Tuk", the name of the warthog which appears in practice exams.

Lecture 39: Checklists

This checklist complements the previous checklists for the midterms.

Integrals to know well

- ☐ $\sin(x)$
- ☐ $\cos(x)$
- ☐ $\tan(x)$
- ☐ $\log(x)$
- ☐ $\exp(x)$
- ☐ $1/x$
- ☐ x^n
- ☐ $1/\cos^2(x)$
- ☐ $1/\sin^2(x)$
- ☐ $1/(1+x^2)$
- ☐ $1/(1-x^2)$
- ☐ $1/\sqrt{1-x^2}$

Applications you have to know

Since there are few questions on what has to be known about applications and definitions (this list only covers application parts):

- ☐ **Derivative:** Limit of differences $D_h f = [f(x+h) - f(x)]/h$ for $h \to 0$
- ☐ **Integral:** Limit of Riemann sums $S_h f = [f(0) + f(h) + ...f(kh)]h$.
- ☐ **Newton step:** $T(x) = x - f(x)/f'(x)$.
- ☐ **Marginal cost:** the derivative F' of the total cost F.
- ☐ **Average cost:** F/x where F is the total cost.
- ☐ **Velocity:** Derivative of the position.
- ☐ **Acceleration:** Derivative of the velocity.
- ☐ **Curvature:** $f''(x)/(1+f'(x)^2)^{3/2}$.
- ☐ **Probability distribution function:** nonnegative function with total $\int f(x)dx = 1$.
- ☐ **Cumulative distribution function:** anti-derivative of the probability distribution function.
- ☐ **Expectation:** $\int x f(x)\, dx$, where f is the probability density function.
- ☐ **Piano function:** frequencies $f(k) = 440 \cdot 2^{k/12}$ for integer k.
- ☐ **Hull function:** The interpolation of local maxima.
- ☐ **Catastrophe:** A parameter c at which a local minimum disappears.

Core concepts

- **Fundamental:** The fundamental theorem of calculus
- **Extrema:** Second derivative test
- **Derivatives:** slope rate of change
- **Integrals:** area, volume
- **Limits:** Hôpital!
- **Continuity:** know the enemies of continuity
- **Numerics:** Riemann sum, Trapezoid and Simpson rule
- **Rules:** Differentiation and integration rules.
- **Methods:** Integration by parts, Substitution, Partial fraction.

Not needed on your fingertips

The following concepts have appeared but do not need to be learned by heart:

- **Entropy:** $-\int f(x)\log(f(x))\,dx$.
- **Moment of inertia:** $\int x^2 f(x)\,dx$.
- **Monte Carlo integration:** $S_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$, where x_k are random in $[a,b]$.
- **Weierstrass function:** A function which is continuous but nowhere differentiable.
- **Bart Simpson rule:** $S_n = \frac{1}{6n}\sum_{k=1}^{n}[f(x_k) + 4f(y_k) + f(x_{k+1})]$.
- **Chaikin step:** $R_{2i} = \frac{3}{4}P_i + \frac{1}{4}P_{i+1}, R_{2i+1} = \frac{1}{4}P_i + \frac{3}{4}P_{i+1}$.
- **Cocktail party stuff:** Eat,integrate and love, the story of exp in practice exam 2.
- **Bottles:** How to calibrate bottles. The calibration formula.
- **Sofia:** The name of a calculus bot once roaming the math department.
- **Wobbly chair:** One can turn a chair on any lawn to stop it from wobbling.
- **Warthog:** "Tuk", the name of the warthog which appears in practice exams.