3/4/2014: First Midterm Exam

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F

The function $f(x) = \exp(-x^2) - 1$ has the root $x = 0$.

Solution:
The exp function is 1 at $x = 0$.

2) T F

If f is continuous function and odd then 0 is a root of f.

Solution:
Odd means that the graph can be reflected at the origin and stays the same. This forces the graph to go through the origin.

3) T F

$\log(\log(e)) = 0$, if log is the natural log.

Solution:
Yes, $\log(e) = 1$ and $\log(1) = 0$.

4) T F

The chain rule assures that $\frac{d}{dx} \sin^2(x) = 2\cos(x)$.

Solution:
This is not the correct application.

5) T F

The function $f(x) = x^2/(1-x^2)$ is continuous everywhere on the real axes.

Solution:
It has a pole at $x = 1$ and $x = -1$.

6) T F

The function $\arctan(x)$ is the inverse function of the function $\tan(x)$.

Solution:
Yes, by definition

7) T F

The Newton method is $T(x) = x - f'(x)/f''(x)$.

Solution:
Too many derivatives

8) T F $\cos(3\pi/2) = 0$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees which is indeed a root

9) T F If a function f is continuous on $[-1, 1]$ and $f(1) = 1, f(-1) = -1$, then there is $-1 < x < 1$, where $f(x) = 0$.

Solution:
By the intermediate value theorem

10) T F The chain rule assures that $\frac{d}{dx} g(1/x) = -g'(1/x)/x^2$.

Solution:
This is the correct application of the formula!

11) T F We have $\lim_{x\to\infty}(2x + 1)/(3x - 1) = 2/3$.

Solution:
This is a consequence of l'Hopital's rule when applied twice.

12) T F If 1 is a root of f, then $f'(x)$ changes sign at 1.

Solution:
It is a point, where the second derivative changes sign

13) T F If $f''(0) < 0$ and $f''(1) > 0$ then there is a point in $(0, 1)$, where f has an inflection point.

Solution:
By the intermediate number theorem.

14) T F The intermediate value theorem assures that the equation $f(x) = x^2 - \cos(x) = 0$ has a root.

Solution:
$f(0) = -1, f(\pi) > 0$.

15) T F The function $f(x) = x/\sin(x)$ is continuous everywhere if $f(0)$ is suitably defined.

Solution:
We can define the value to be 1.

16) T F $f'(x) = 0$ and $f'''(0) < 0$ at $x = 0$ assures that f has a maximum at $x = 0$.

Solution:
It is the second derivative test, not the third one

17) T F If f is constant, then $f(x+h) - f(x)/h = 0$ for all $h > 0$.

Solution:
$Df(x) = 2x - 1$.

18) T F The quotient rule is $\frac{d}{dx}(f/g) = (f'(x)g'(x) - f(x)g(x))/(g'(x))^2$.

Solution:
No, it is not related to l'Hopital.

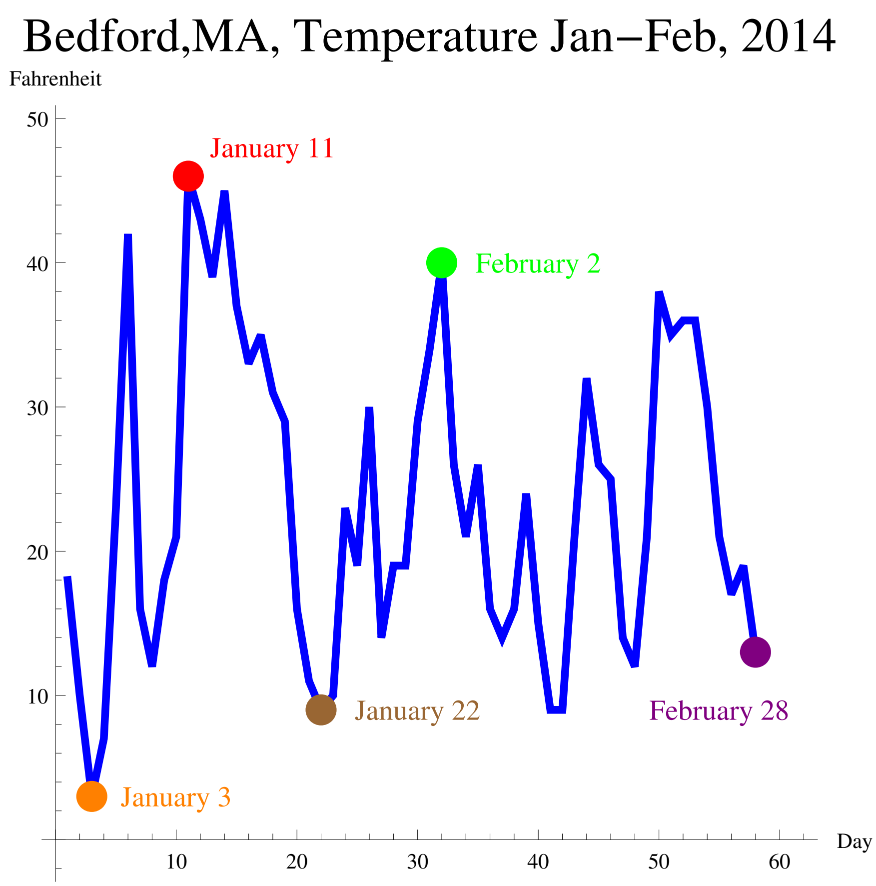
19) T F $\sin(2\pi) + \tan(2\pi) = 0$.

Solution:
$\sin(0) = \sin(2\pi) = 0$.

20) T F It is true that $e^{x\log(5)} = x^5$.

Solution:
This is no identity.

Problem 2) Matching problem (10 points) No justifications are needed.



In this winter, the **polar vortex** ruled the weather in Boston. The above graph shows the temperatures of the first two months of 2014 measured at the **Hanscom field** in Bedford, MA. While temperatures are measured hourly, you can assume that temperature is a continuous function of time. Remember that "global maximum" includes being local too so that only one entry in each line of the table below needs to be checked.

a) (5 points) Check what applies, by checking one entry in each of the 5 dates.

Date	local maximum	local minimum	global maximum	global minimum
January 3				
January 11				
January 22				
February 2				
February 28				

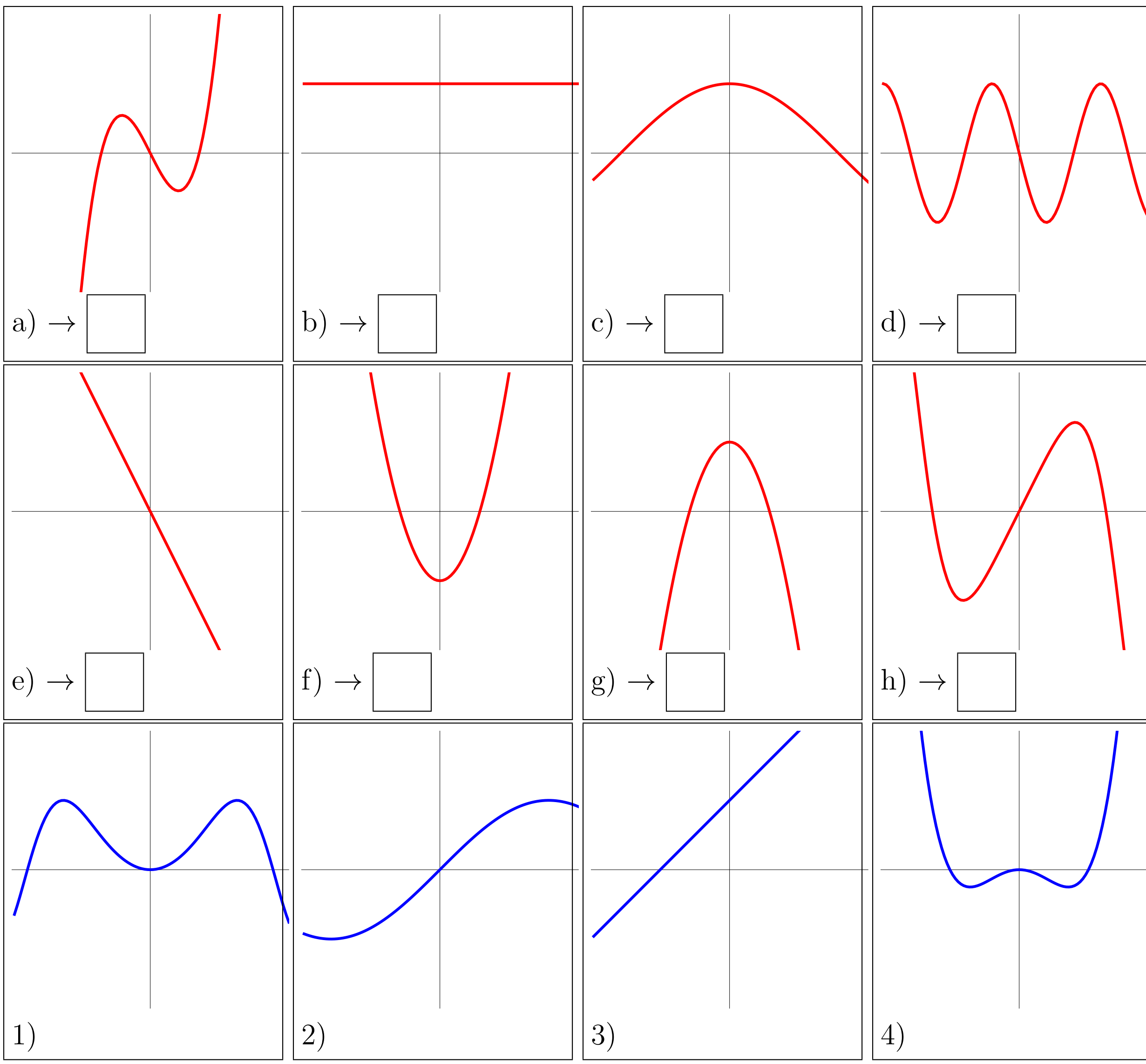
b) (2 points) Which theorem assures that on the closed interval $[0, 59]$ of 59 days, there is a global maximal temperature?

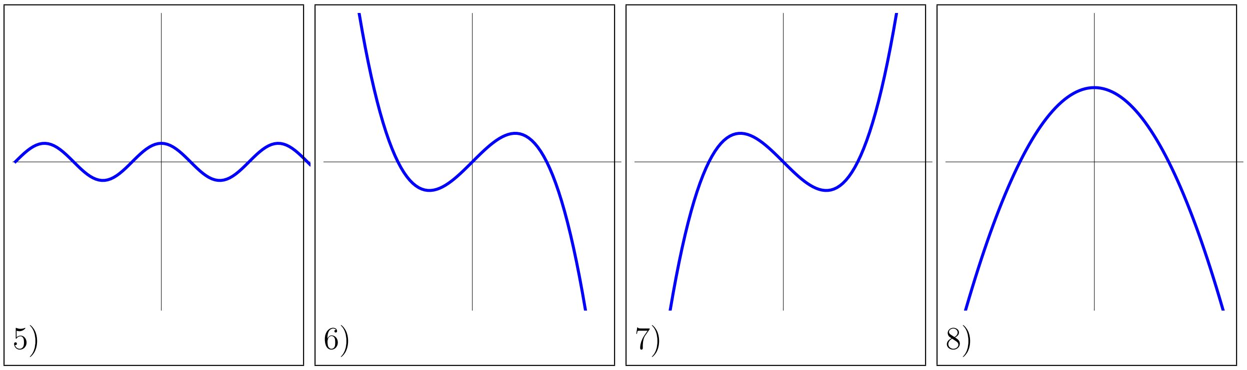
c) (3 points) Argue by citing a theorem why there is a time at which the temperature at Bedford was exactly 25 degree Fahrenheit.

Solution:
a) global min, global max, local min, local max, local min
b) Bolzano's theorem
c) Intermediate value theorem

Problem 3) Matching problem (10 points) No justifications are needed.

In the first pictures, we see the first derivatives f'. Match them with the functions f in 1-8. Note that the functions above are the derivative and the functions below are the functions.





Solution:
4325
8761

Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x \to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{1-2x^3}{1-x}$, at $x_0 = 1$.

b) (2 points) $f(x) = \sin(\sin(5x))/\sin(7x)$, at $x_0 = 0$.

c) (2 points) $f(x) = \frac{\exp(-3x)-1}{\exp(2x)-1}$, at $x_0 = 0$.

d) (2 points) $f(x) = \frac{2x}{\log(x)}$, at $x_0 = 0$.

e) (2 points) $f(x) = \frac{(x-1)^{10}}{(x+1)^{10}}$, at $x_0 = -1$.

Solution:
a) No Limit.
b) Use Hopital: $5/7$.
c) Use Hopital: $-3/2$
d) Use Hopital: 0
e) No limit.

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sqrt{\log(x+1)}$.

b) (3 points) $f(x) = 7\sin(x^3) + \frac{\log(5x)}{x}$.

c) (3 points) $f(x) = \log(\sqrt{x}) + \arctan(x^3)$.

d) (2 points) $f(x) = e^{5\sqrt{x}} + \tan(x)$.

Solution:
a) $1/[2(1+x)\sqrt{\log(1+x)}]$
b) $21x^2\cos(x^3) + [1-\log(5x)]/x^2$. c) $1/(2x) + 3x^2/(1+x^6)$
d) $\frac{5}{2\sqrt{x}}e^{5\sqrt{x}} + 1/\cos^2(x)$

Problem 6) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions:

a) (2 points) $f(x) = \frac{\exp(3x)-\exp(-3x)}{\exp(5x)-\exp(-5x)}$.

b) (3 points) $f(x) = \frac{\cos(3x)-1}{\sin^2(x)}$.

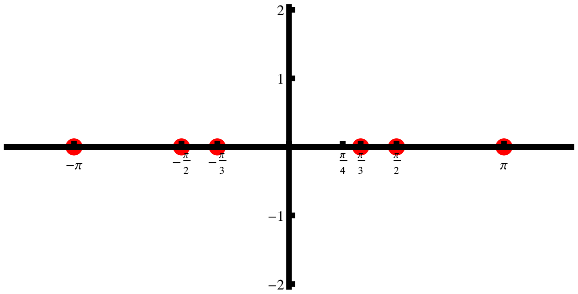
c) (3 points) $f(x) = [\arctan(x) - \arctan(0)]/x$.

d) (2 points) $f(x) = \frac{\log(7x)}{\log(11x)}$.

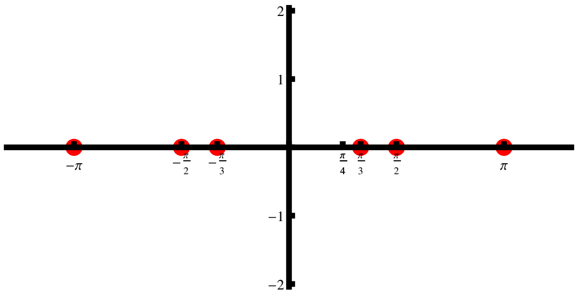
Solution:
a) Use Hopital: $\frac{3\exp(3x)+3\exp(-3x)}{5\exp(5x)+5\exp(-5x)}$. Now take the limits: $3/5$.
b) Use Hopital once $-3\sin(3x)/(2\sin(x)\cos(x))$ and then a second time $9\cos(3x)/2\cos^2(x) - 2\sin^2(x) \to -9/2$
c) This is just the derivative of arctan at 0 which is 1. Hopital gives the same.
d) Use Hopital using $d/dx \log(ax) = 1/x$ for all constants a and get 1.

Problem 7) Trig functions (10 points)

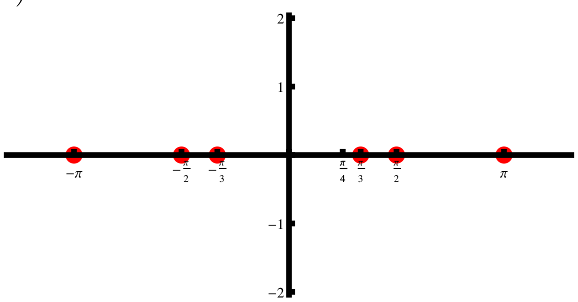
a) Draw the sin function and mark the values of $\sin(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



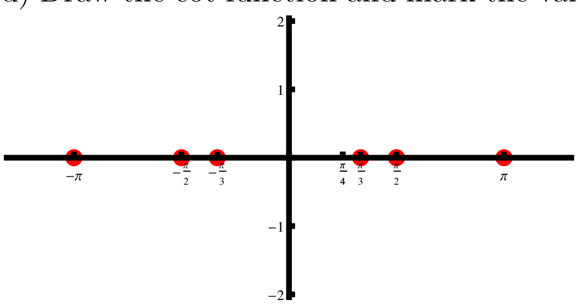
b) Draw the cos function and mark the values of $\cos(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



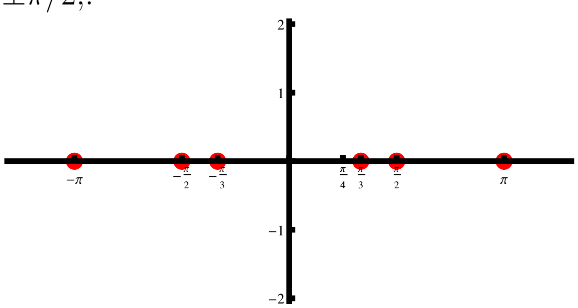
c) Draw the tan function and mark the values of $\tan(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



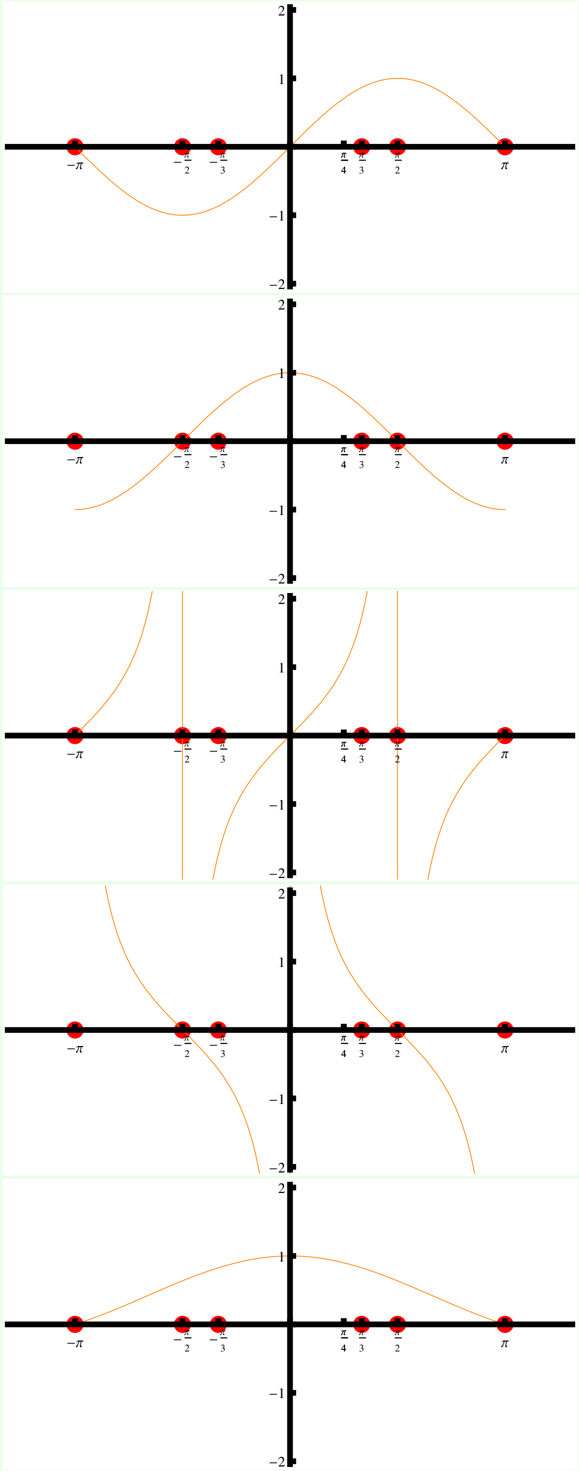
d) Draw the cot function and mark the values of $\cot(x)$ at $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$.



e) Draw the sinc function $f(x) = \sin(x)/x$ and mark the points $x = 0, x = \pm\pi/4, x = \pm\pi/3, x = \pm\pi/2$,.



Solution:



Problem 8) Extrema (10 points)

Last week, Oliver got a new batch of strong **Neodym magnets**. They are ring shaped. Assume the inner radius is x, the outer radius y is 1 and the height is $h = x$, we want to maximize the surface area $A = 2\pi(y - x)h + 2\pi(y^2 - x^2)$. This amount of maximizing

$$f(x) = 2\pi(1 - x)x + 2\pi(1 - x^2)$$

a) (2 points) Using that $f(x)$ is a surface area, on what interval $[a, b]$ needs f to be considered?

b) (3 points) Find the local maxima of f inside the interval.

c) (3 points) Use the second derivative test to verify it is a maximum.

d) (2 points) Find the global maximum on the interval.

Solution:
a) The surface area is nonnegative on $[0, 1]$.
b) $f'(x) = 2\pi(1 - 4x)$. It is zero at $x = 1/4$. This is the only critical point.
c) $f''(x) = -8\pi$ is negative. By the second derivative test, the critical point is a local maximum.
d) To find the global maximum, compare $f(0) = 2\pi = 6.28..$, $f(1/4) = 2\pi/16 + 2\pi(3/4) = 7.07..$ and $f(1) = 0$. Since the graph of f is quadratic and is everywhere concave down, one could see also that 1/4 is a global maximum also without evaluating the end points.

Problem 9) Trig and Exponential functions (10 points)

Simplify the following terms. log denotes the natural log and $\log_{10}$ the log to the base 10. Each result in a)-c) is an integer or a fraction

a) (2 points) $\exp(\log(2)) + e^{3\log(2)}$

b) (2 points) $\log(1/e) + \exp(\log(2)) + \log(\exp(3))$.

c) (2 points) $\log_{10}(1/100) + \log_{10}(10000)$

d) (4 points) Produce the formula for $\arccos'(x)$ by taking the derivative of the identity

$$\cos(\arccos(x)) = x \; .$$

Your answer should be simplified as we did when deriving the derivatives of arcsin, arctan in class or when you derived the derivative of arccot and arcsinh, arccosh in the homework.

Solution:
a) $\boxed{10}$
b) -1+2+3= $\boxed{4}$
c) -2+4 = $\boxed{2}$
d) Differentiate $\cos(\arccos(x)) = x$ and simplify.