4/8/2014: Second midterm practice B

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

Solution:
Differentiate the right hand side to check.

2) T F The fundamental theorem of calculus implies that $\int_0^1 f'(x)\ dx = f(1) - f(0)$.

Solution:
Yes this is a special case of the fundamental theorem.

3) T F The volume of truncated pyramid with a base square length 2 and top square length 3 is given by the integral $\int_2^3 x^2\ dx$.

Solution:
Yes the area of a slice is x^2.

4) T F The derivative of $\arctan(x)$ is $1/\cos^2(x)$.

Solution:
The derivative of $\tan(x)$ is $1/\cos(x)^2$.

5) T F The mean value theorem implies $\int_a^b f'(x)\ dx = f'(c)(b-a)$ for some c in the interval (a, b).

Solution:
This is a typical application of the mean value theorem.

6) T F If $F(x) = \int_0^x f(t)\ dt$ has an critical point at $x = 1$ then f has a root at $x = 1$.

Solution:
The first derivative of F is f.

7) T F The anti-derivative of the derivative of f is equal to $f + C$ where C is a constant.

Solution:
This is a consequence of the fundamental theorem.

8) T **F** If we blow up a balloon so that the volume V changes with constant rate, then the radius $r(t)$ changes with constant rate.

Solution:
This is a related rates problem. The balloon radius grows slower for large volumes.

9) T **F** The identity $\frac{d}{dx} \int_5^9 f(x)\, dx = f(9) - f(5)$ holds for all continuous functions f.

Solution:
We differentiate a constant.

10) **T** F Two surfaces of revolution which have the same cross section area $A(x)$ also have the same volume.

Solution:
This is Archimedes insight and true.

11) **T** F If $x^2 + y^2 = 2$ and $x(t), y(t)$ depend on time and $x' = 1$ at $x = 1$ then $y' = -1$ is possible.

Solution:
This is a simple example of related rates.

12) **T** F The identity $\int_2^9 7f(x)\, dx = 7\int_2^9 f(x)\, dx$ is true for all continuous functions f.

Solution:
Yes, we can take the 7 constant outside the integral.

13) T **F** The improper integral $\int_1^\infty 1/x\, dx$ in the sense that $\int_1^R 1/x\, dx$ converges for $R \to \infty$ to a finite value.

Solution:
This is what we mean with the existence. But the integral does not exist.

14) T F If $f_c(x)$ has a local minimum at $x = 2$ for $c < 1$ and no local minimum anywhere for $c > 1$, then $c = 1$ is a catastrophe.

Solution:
This is a definition.

15) T F An improper integral is an indefinite integral which does not converge.

Solution:
These two terms are easy to mix up. Improper means that we either have a discontinuity of f or integrate over an infinite interval. Indefinite means that we do not specify bounds.

16) T F If $f(-5) = 0$ and $f(5) = 10$ then $f' = 1$ somewhere on the interval $[-5, 5]$.

Solution:
Yes this is the mean value theorem.

17) T F The sum $\frac{1}{n}\sum_{k=0}^{n-1}\frac{k}{n} = \frac{1}{n}[\frac{0}{n} + \frac{1}{n} + \cdots + \frac{n-1}{n}]$ is a Riemann sum to the integral $\int_0^1 x\,dx$.

Solution:
Yes, this is the Riemann sum.

18) T F The anti-derivative of $\text{sinc}(x) = \sin(x)/x$ is equal to $\sin(\log(x)) + C$.

Solution:
Differentiate the right hand side to see that this is not true

19) T F The anti-derivative of $\log(x)$ is $1/x + C$.

20) T F We have $\int_0^x t f(t)\,dt = x\int_0^x f(t)\,dt$ for all functions f.

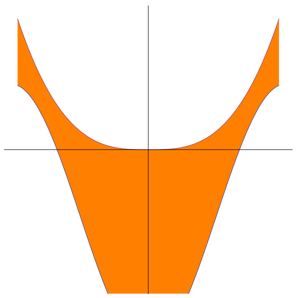
Solution:
It is already false for the constant function $f(t) = 1$.

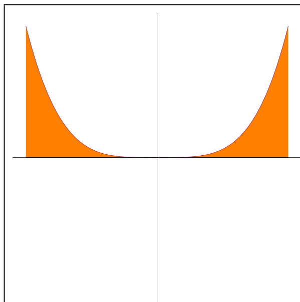
Problem 2) Matching problem (10 points) No justifications are needed.

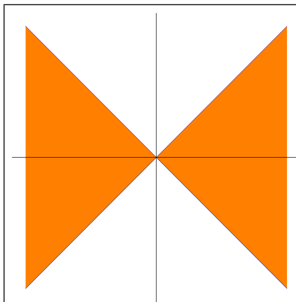
a) (6 points) Match the integrals with the pictures.

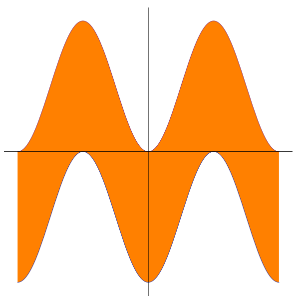
Integral	Enter 1-6
$\int_{-1}^{1}(1-x)^2\,dx$	
$\int_{-1}^{1}\lvert x\rvert\,dx$	
$\int_{-1}^{1}x^4\,dx$	

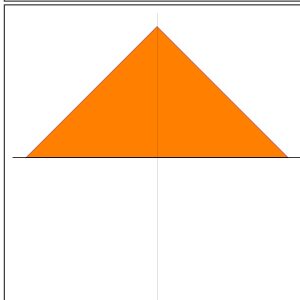
Integral	Enter 1-6
$\int_{-1}^{1}\lvert x\rvert^3-\cos(3x)\,dx$	
$\int_{-1}^{1}[\sin^2(\pi x)-\cos^2(\pi x)]\,dx$	
$\int_{-1}^{1}1-\lvert x\rvert\,dx$	

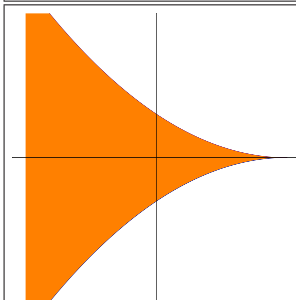

1)


2)


3)

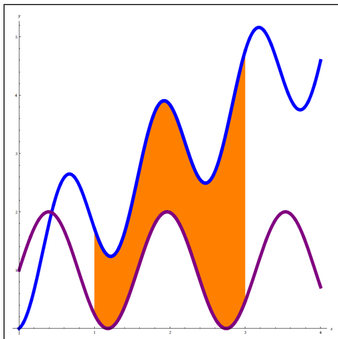

4)

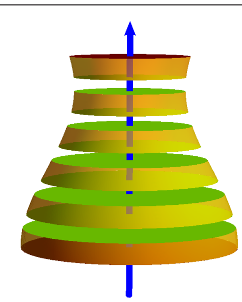

5)

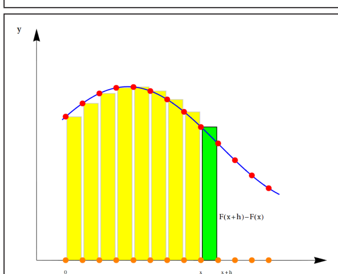

6)

Solution:
6,3,2,1,4,5.

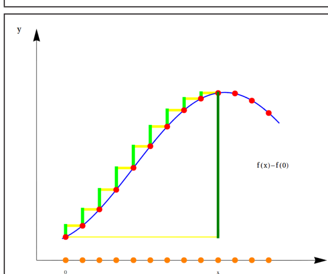
b) (4 points) Match the concepts: each of the 4 figures illustrates one of the formulas which are the centers of the **mind map** we have drawn for this exam:


1)


2)



3)



4)

Formula	Enter 1-4
$\int_a^b A(z)\, dz$	
$\int_a^b g(x) - f(x)\, dx$	

Formula	Enter 1-4
$\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$	
$\int_0^x f'(t)\, dt = f(x) - f(0)$	

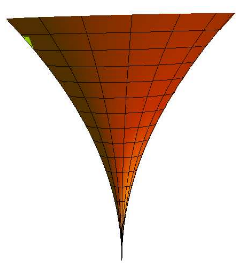
Solution:
2,1 and 3,4

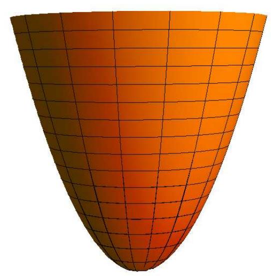
Problem 3) Matching problem (10 points) No justifications are needed.

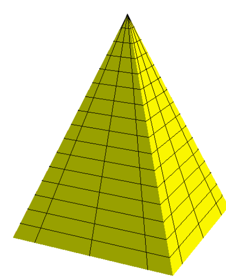
a) (6 points) Match the volumes of solids.

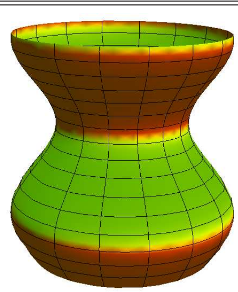
Integral	Enter 1-6
$\int_0^1 \pi z^4\, dz$	
$\int_0^1 \pi z\, dz$	
$\int_0^1 \pi(4 + \sin(4z))\, dz$	

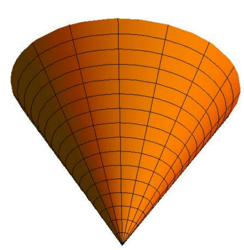
Integral	Enter 1-6
$\int_{-1}^1 \pi e^{-4z^2}\, dz$	
$\int_0^1 \pi z^2\, dz$	
$\int_0^1 (1 - z)^2\, dz$	

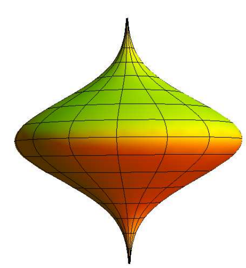

1)


2)


3)


4)


5)


6)

Solution:
1,2,4,6,5,3

b) (4 points) Fill in the missing word which links **applications** of integration.

The probability density function is the		of the cumulative distribution function.
The total cost is the		of the marginal cost.
The volume of a solid is the		of the cross section area function.
The velocity of a ball is the		of the acceleration of the ball.

Solution:
derivative, antiderivative, antiderivative, antiderivative

Problem 4) Area computation (10 points)

Find the area of the region enclosed the graphs of $y = x^4 - 12$ and $y = 8 - x^2$.

Solution:
The two curves intersect at $x = 2$ and $x = -2$.

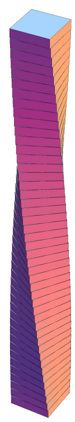
$$\int_{-2}^{2} 8 + x^2 - (x^4 - 12)\, dx = ... = \frac{928}{15}$$

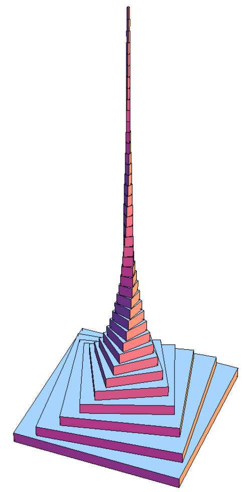
Problem 5) Volume computation (10 points)

The **infinity tower** in Dubai of height 330 meters has floors which can rotate. After much delay, it is expected to be completed this year. Inspired by the name "infinity", we build a new but twisted science center for which the side length of the square floor is

$$l(z) = \frac{1}{1+z} .$$

Find the volume of this new **Harvard needle building** which extends from 0 to ∞. We are the best!





Solution:
The cross section area is $A(z) = 1/(1+z)^2$. The integral is

$$\int_0^\infty \frac{1}{(1+z)^2} \, dz = -\frac{1}{1+z}|_0^\infty = 1$$

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. You should get a definite real number in each case.

a) (2 points) $\int_0^\infty e^{-x} \, dx$

b) (3 points) $\int_0^1 x^{1/5} + x^3 \, dx$.

c) (3 points) $\int_{-1}^1 \frac{1}{1+x^2} \, dx$

d) (2 points) $\int_0^{e-1} \frac{2}{1+x} \, dx$

Solution:
a) 1
b) 13/12
c) $\pi/2$
d) 2

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (2 points) $\int \frac{3}{\sqrt{1+3x}} + \cos(x)\, dx$

b) (3 points) $\int e^{x/5} - 7x^6 + \frac{4}{x^2+1}\, dx$

c) (2 points) $\int \frac{4}{e^{4x+5}} + 3\sin(x)\, dx$

d) (3 points) $\int \frac{1}{\sin^2(x)} + \frac{4}{x}\, dx$

Solution:
a) $2\sqrt{1+3x} + \sin(x) + C$.
b) $5e^{x/5} - x^7 + 4\arctan(x) + C$.
c) $-e^{-4x-5} - 3\cos(x) + C$.
d) $-\cot(x) + 4\log(x) + C$.

Problem 8) Implicit differentiation and related rates (10 points)

a) (5 points) The implicit equation

$$x^3 + y^4 = y + 1$$

defines a function $y = y(x)$ near $(x, y) = (-1, -1)$. Find the slope $y'(x)$ at $x = -1$.

b) (5 points) An ice cube of side length x melts and changes volume V with a rate $V' = -16$. What is the rate of change of the length x at $x = 4$?

Solution:
a) Differentiate the equation to get

$$3x^2x' + 4y^3y' = y'$$

and solve for y' using $x = -1, y = -1$ to get $\boxed{-3/5}$.
b) Differentiate $V = x^3$ to get $-16 = V' = 3x^2x'$ which gives for $x = 4$ the solution $x' = -16/(3 \cdot \cdot 4^2) = -1/3$.

Problem 9) Catastrophes (10 points)

Verify first for each of the following functions that $x = 0$ is a critical point. Then give a criterium for stability of $x = 0$. The answer will depend on c.

a) (3 points) $f(x) = x^5 + 2x^2 - cx^2$.

b) (3 points) $f(x) = x^4 + cx^2 - x^2$.

Determine now in both examples for which parameter c the catastrophe occurs

c) (2 points) in the case $f(x) = x^5 + 2x^2 - cx^2$.

d) (2 points) in the case $f(x) = x^4 + cx^2 - x^2$.

Solution:
a) $f'(x) = 5x^4 + 4x - 2xc$ has $x = 0$ as a root. Its stability is determined by $f''(0) = 4 - 2c$.
b) $f'(x) = 4x^3 + 2cx - 2x$ has $x = 0$ as a root. Its stability is determined by $f''(0) = 2c - 2$.
c) For $c < 2$ the point $x = 0$ is a local minimum. For $c > 2$ it is a local maximum. $c = 2$ is a catastrophe.
d) For $c < 1$ the point $x = 0$ is a local minimum. For $c > 1$ it is a local maximum. $c = 1$ is a catastrophe.