# 4/11/2013: Second midterm practice E

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total: | | 110 |

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F** The anti derivative of $\log(3x)$ is $x\log(3x) - 3x + C$.

**Solution:**
Differentiate the right hand side to check. The result is $log(3x) - 2$. Its almost right but not quite.

2) T **F** The fundamental theorem of calculus assures that $\int_a^b f'(x)\, dx = f(a) - f(b)$.

**Solution:**
The order is reversed on the right hand side.

3) **T** F If $\int_0^x f(t)\, dt$ is monotonically increasing in $x$ for $0 \leq x \leq 1$, then $f(x) \geq 0$ on $0 \leq x \leq 1$.

**Solution:**
The derivative of the integral is $f(x)$ and because the integral is monotonically increasing, this derivative is positive or zero.

4) **T** F The volume of a cone of base radius 1 and height 1 is given by the integral $\int_0^1 \pi x^2\, dx$.

**Solution:**
Yes, the radius is $x$ and the cross section area is $\pi x^2$.

5) **T** F The mean value theorem assures that if you run from Cambridge to Boston with a $10 Miles/hour$ average, then there was moment along the run, where the velocity was exactly $10 Miles/hour$.

**Solution:**
This is a typical application of the mean value theorem.

6) T **F** For any continuous function $f$, the integral $\int_a^b f(x)\, dx$ is the area under a curve and therefore always positive or zero.

**Solution:**
It can be a signed area and negative. The statement is false

7) T F The sum $(1/n^2 + (2/n)^2 + ... + ((n-1)/n))^2)/n$ is a Riemann sum approximation to $\int_0^1 x^2\, dx$.

**Solution:**
Yes, this is it.

8) T F If a differentiable function $f$ has a critical point at 1, then the function $F(x) = \int_0^x f(t)\, dt$ has an inflection point at 1.

**Solution:**
This means $F$ has a second derivative which is zero and that is an inflection point.

9) T F The derivative of the anti derivative of a differentiable function $f$ is always the same as a particular anti derivative of the derivative of $f$.

**Solution:**
This is a consequence of the fundamental theorem.

10) T F If $f$ is constant 1 then $\int_a^b f(x)\, dx$ is the length of the interval $[a, b]$.

**Solution:**
Indeed the integral is $b - a$ then. Since we wrote $[a, b]$ this indicates that $a \leq b$.

11) T F Subtract the volume of a cone $C$ from a cylinder $Z$ with the same circular base of radius 1 and height 1 and you get the volume of a half sphere of radius 1.

**Solution:**
Again, this is Archimedes insight.

12) T F If $x + y = 10$ is constant and $x' = 3$ then $y' = -3$.

**Solution:**
This is a simple example of related rates. —

13) T F The identity $\int_a^b -f(x)\ dx = -\int_a^b f(x)\ dx$ is always true for continuous functions $f$.

**Solution:**
Yes, we can take the $-$ sign outside the integral.

14) T **F** If $f(x)$ approaches zero for $x \to \infty$, then $\int_1^\infty f(x)\, dx$ is finite.

**Solution:**
A counter example is $f(x) = 1/x$.

15) **T** F If $f_c(x)$ has a minimum $x_c$ which is present for $c < 0$ and disappears for $c > 0$, then $c = 0$ is a catastrophe.

**Solution:**
This is a definition.

16) T **F** An indefinite integral is an improper integral which converges.

**Solution:**
These two terms are easy to mix up. Improper means that we either have a discontinuity of $f$ or integrate over an infinite interval. Indefinite means that we do not specify bounds.

17) **T** F Rolle's theorem assures that if $f(a) = f(b) = 0$ then $f'$ has a root inside the interval $(a, b)$.

**Solution:**
Yes this is pretty much Rolle's theorem.

18) **T** F The sum $\frac{1}{n}\sum_{k=0}^{n-1} \sin(k/n)$ is a Riemann sum to the integral $\int_0^1 \sin(x)\, dx$.

**Solution:**
Yes, this is the Rieman sum.

19) T **F** The derivative of $\tan(x)$ is $1/(1 + x^2)$.

**Solution:**
The derivative of $\arctan(x)$ is $1/(1 + x^2)$, not tan itself.

20) T F There are continuous functions for which the anti derivative can not be expressed using known elementary functions.

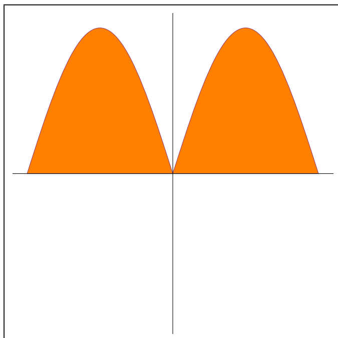
**Solution:**
We have seen various examples, like $\exp(-x^2)$.

Problem 2) Matching problem (10 points) No justifications are needed.

a) (6 points) Match the following integrals with the graphs and areas. Each integral matches exactly one area.

| Integral | Enter 1-6 |
|---|---|
| $\int_{-1}^{1} \lvert x\rvert \, dx$ | |
| $\int_{-1}^{1} \lvert\sin(\pi x)\rvert \, dx$ | |
| $\int_{-1}^{1} \lvert\sin(\pi x) - (-\sin(\pi x))\rvert \, dx$ | |

| Integral | Enter 1-6 |
|---|---|
| $\int_{-1}^{1} 2 - 2x^2 \, dx$ | |
| $\int_{-1}^{1} 2x^2 \, dx$ | |
| $\int_{-1}^{1} \lvert x\rvert - x^2 \, dx$ | |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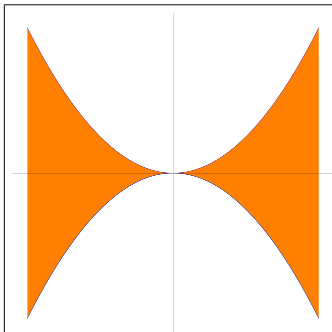
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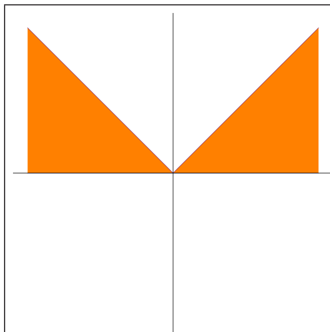
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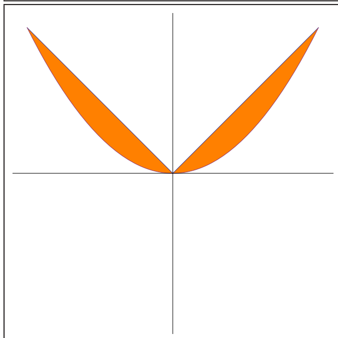
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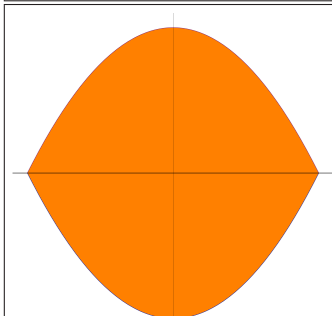
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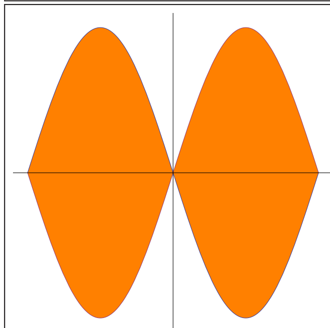
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b) (4 points) Which of the following statements are true because of the mean value theorem? (All functions are differentiable).

| Result | Check |
|---|---|
| If $f(0) = -1$ and $f(1) = 1$ then there is a root $p$ of $f$ in $(0, 1)$ | |
| If $f(0) = -1$ and $f(1) = 1$ then there is a critical point $p$ of $f$ in $(0, 1)$ | |
| If $f(0) = -1$ and $f(1) = 1$ then there is point where $f(p) = 2$ in $(0, 1)$ | |
| If $f(0) = -1$ and $f(1) = 1$ then there is point where $f'(p) = 2$ in $(0, 1)$ | |

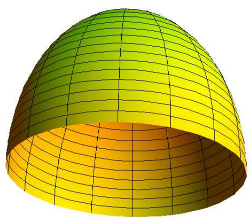
**Solution:**
a) 3,1,6,5,2,4
b) No, no, no, Yes

Problem 3) Matching problem (10 points) No justifications are needed.

a) (6 points) The following integrals match the volumes of solids. Each integral matches exactly one solid.

| Integral | Enter 1-6 |
|---|---|
| $\int_0^1 \pi x^2 \, dx$ | |
| $\int_0^1 \pi \, dx$ | |
| $\int_0^1 \pi(1-x^2) \, dx$ | |

| Integral | Enter 1-6 |
|---|---|
| $\int_0^1 \pi \sin^2(\pi x) \, dx$ | |
| $\int_0^1 \pi(1+x)^2 \, dx$ | |
| $\int_0^1 \pi \cos^2(\pi x) \, dx$ | |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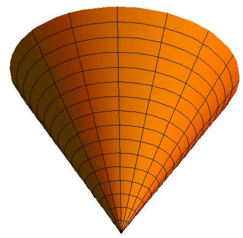
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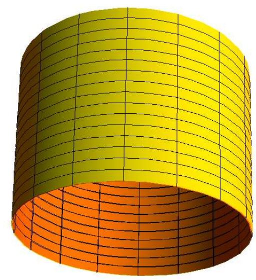
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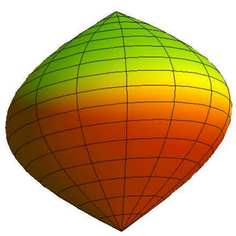
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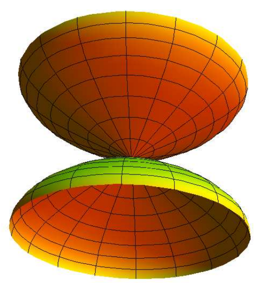
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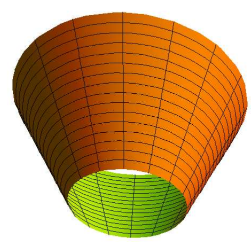

6)

b) (4 points) Which of the following results are part of the fundamental theorem of calculus? (any number of results can apply).

| Result | Check |
|---|---|
| $\int_a^b f'(x) \, dx = f(b) - f(a)$ | |
| $\frac{d}{dx} \int_0^x f(t) \, dt = f(x)$ | |
| $\int_0^x f'(t) \, dt = f(x) - f(0)$ | |
| $\int_0^x f(x) \, dx = f'(x)$ | |

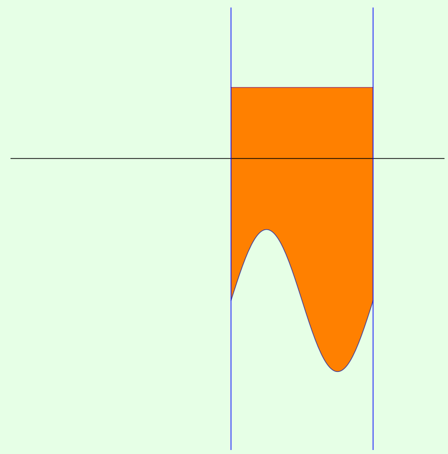
**Solution:**
2,3,1 and 4,6,5. The first three statements follow from the fundamental theorem. The last one is nonsense.

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the four curves $x = 2, x = 4, y = 1, y = -2 + \sin(\pi x)$.

**Solution:**
Make a good picture. The integral is

$$\int_2^4 1 - (-2 + \sin(\pi x)\ dx = \int_2^4 3 - \sin(\pi x)/\pi\ dx = 3x + \cos(\pi x)/\pi|_2^4 = 6\ .$$

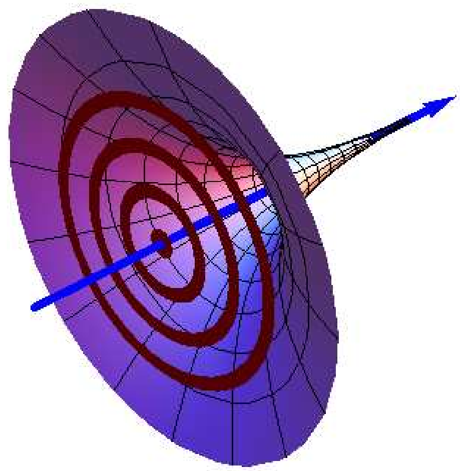


Problem 5) Volume computation (10 points)

In this problem we deal with an **extraordinary exponential trumpet**.

a) (7 points) Find the volume of the rotationally symmetric solid for which the radius at position $x$ is $f(x) = \exp(-x)$ and $0 \leq x \leq R$.

b) (3 points) Does the volume limit $R \to \infty$ stay finite? If yes, what is the limit?



**Solution:**
The radius is $\exp(-x)$ so that the cross section area is $\pi \exp(-2x)$. We can integrate this

$$\pi \int_0^R \exp(-2x)\, dx = -\pi \exp(-2x)/2|_0^R = \pi/2 - \pi \exp(-2R)/2 \; .$$

This converges to $\pi/2$ for $R \to \infty$.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals

a) (5 points) $\int_1^2 \sqrt{x} + x^2 - \frac{1}{\sqrt{x}} + \frac{1}{x}\, dx$.

b) (5 points) $\int_1^3 \sqrt{1+x} + \frac{4}{1+x^2}\, dx$

**Solution:**
a) $x^{3/2}\frac{2}{3} + x^3/3 - 2\sqrt{2} + \log(x)|_1^2 = (11/2) - \sqrt{2}(2/3) + \log(2)$.
b) $(2/3)(1+x)^{3/2} + 4\arctan(x)|_1^3 = (2/3)4^{3/2} + 4\arctan(3) - (2/3)2^{3/2} - \pi$

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

a) (5 points) $\int \frac{1}{\sqrt{1+x}} + \exp(4x) + 2x^5 \; dx$

b) (5 points) $\int \cos(3x) + \sin^2(x) + \frac{1}{\cos^2(x)} \; dx$

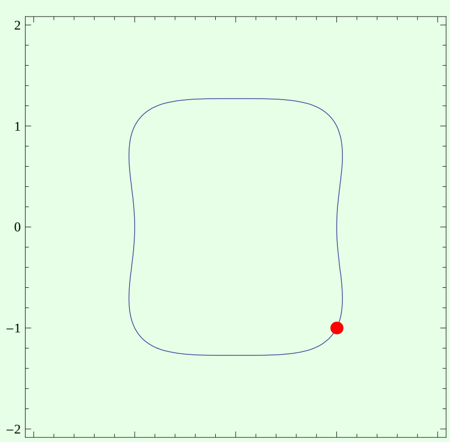
**Solution:**
a) $2\sqrt{(1+x)} + \exp(4x)/4 + 2x^6/6 + C$.
b) For the middle term, use $1 - 2\sin^2(x) = \cos(2x)$ to get the antiderivative of $[1 - \cos(2x)]/2 + C$ which is $x/2 - \sin(2x)/4 + C$. The result is $\sin(3x)/3 + x/2 - \sin(2x)/4 + \tan(x) + C$.

Problem 8) Implicit and related rates (10 points)

a) (5 points) The implicit equation $x^4 + y^4 = y^2 + 1$ defines a function $y = y(x)$ near $(x, y) = (1, -1)$. Find the slope $y'(x)$ at this point.

b) (5 points) Take the same equation as before and assume now that $x(t) = t^2$ depends on an external parameter $t$. This produces a function $y(t)$ near $y = -1$. Find $y'(t)$ at $t = 1$.

**Solution:**
a) $4x^3 + 4y^3y' = 2yy'$. Solve for $y'$ to get $y' = 4x^3/(2y - 4y^3)$. At the point $(x, y) = (1, -1)$, this is $4/2 = 2$.
b) $4x^3x' + 4y^3y' = -2y'$. Solve for $y'$ to get $y' = 4x^3x'/(4y^3 + 2)$. At the point $(x, y) = (1, -1)$, using $x' = 2$, this is 4.



Problem 9) Catastrophes (10 points)

Consider the family of functions $f(x) = x^3 + cx$ on the real line.

a) (5 points) Find all critical points of $f$, depending on $c$.

b) (2 points) Using the second derivative test, determine which are minima and which are maxima.

b) (3 points) For which value of $c$ does a catastrophe occur?

**Solution:**
a) The critical points are $3x^2 + c = 0$ which means $x = \sqrt{-c/3}$. There are no critical points for $c > 0$ and two different critical points for $c < 0$.
b) The second derivative is $6x$ which is negative for $x < 0$ and positive for $x > 0$. The point $\sqrt{-c/3}$ is a local minimum for $c < 0$. The point $-\sqrt{-c/3}$ is a local maximum for $c < 0$.
c) The catastrophe appears at the parameter $c = 0$ because a local minimum, present for $c < 0$ disappears for $c \geq 0$.

Problem 10) Applications (10 points)

a) Check that the function

$$f(x) = \begin{cases} 6x(1-x) & 0 \leq x \leq 1 \\ 0 \; otherwise & \end{cases}$$

is a probability density function.
b) The integral

$$\int_0^1 x f(x) \, dx$$

is called the expectation. Find the expectation.

**Solution:**
a) Check that $\int_0^1 6x(1-x) \, dx = 1$ and that $6x(1-x)$ is never zero on $[0, 1]$.
b) $\int_0^1 x6x(1-x) \, dx = 1/2$.