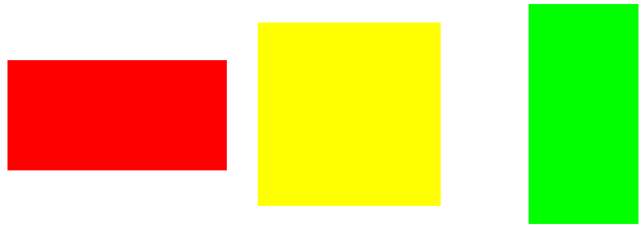
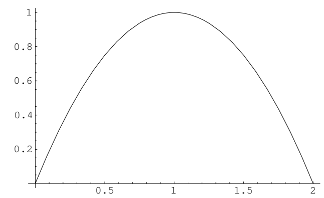
Lecture 16: 4/17/2004, MORE ON EXTREMA — Math21a, O. Knill

HOMEWORK: Section 11.8: 10,20,24,42

MAXIMAL AREA OF RECTANGLE. We want to extremize the area of a rectangle for which the length of the boundary is fixed 4. If the sides are x and y, then we want to extremize $f(x,y) = xy$ under the constraint $g(x,y) = 2x + 2y = 4$. The Lagrange equations $y = 2\lambda, x = 2\lambda$ show that $x = y$ and so $x = y = 1$.



The last problem could also been solved by substituting $y = 2 - x$ into the area formula $A = xy = x(2-x)$ leading to a one-dimensional extremal problem: maximize $f(x) = x(2-x)$ on the interval $[0,2]$. To do so, we have to find the extrema inside the interval and then consider also the boundary points $x = 0, x = 2$. Again, we get $x = 1$.



VOLUME OF CUBE. Extremize the volume $f(x,y,z) = xyz$ of a box with fixed surface area $xy + yz + xz = 3$. To solve $yz = \lambda(y+z), xz = \lambda(x+z), xy = \lambda(x+y), xy + yz + xz = 1$, take quotients: $z/x = (y+z)/(y+x), z/y = (z+x)/(y+x)$ which gives $z(y+x) = x(y+z), z(y+x) = y(z+x)$ so that either $xz = yz$ or $z = 0$. Similarly, we get $y = z$ or $y = 0$. The solution is $x = y = z = 1$.

AN OTHER SOLUTION. For a solution without Lagrange multipliers, we would plug in $z = (1-xy)/(y+x)$ and try to find the maximum of $f(x,y) = xy(1-xy)/(y+x)$ on the domain $D = \{x > 0, y > 0, xy \leq 1\}$.

We first would have to find critical points inside the region D:

$$f_x(x,y) = -y(1-2xy)/(x+y) - xy(1-xy)/(x+y)^2 = 0$$

$$f_y(x,y) = -x(1-2xy)/(x+y) - xy(1-xy)/(x+y)^2 = 0$$

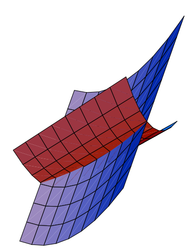
The difference of these two equations gives $(x-y)(1-2xy) = 0$ so that either $x = y$ or $xy = 1/2$. The second case can not give us $f_x = f_y = 0$. The first condition $x = y$ gives $x = y = 1$ which is not inside the region. However, on the boundary $g(x,y) = xy = 1$, the Lagrange equations $\nabla f = \lambda \nabla g$ have a solution with $(x,y) = (1,1)$.

The example illustrates the power of Lagrange multipliers. The substitution method is more complicated.

TWO CONSTRAINTS. (informal) The calculation with Lagrange multipliers can be generalized: if the goal is to optimize a function $f(x,y,z)$ under the constraints $g(x,y,z) = c$, $h(x,y,z) = d$, take the Lagrange equations

$$\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$$

which are 5 equations for the 5 unknowns (x,y,z,λ,μ). Geometrically the gradient of f is in the plane spanned by the gradients of g and h. (This is the plane orthogonal to the curve $\{g = c, h = d\}$.)

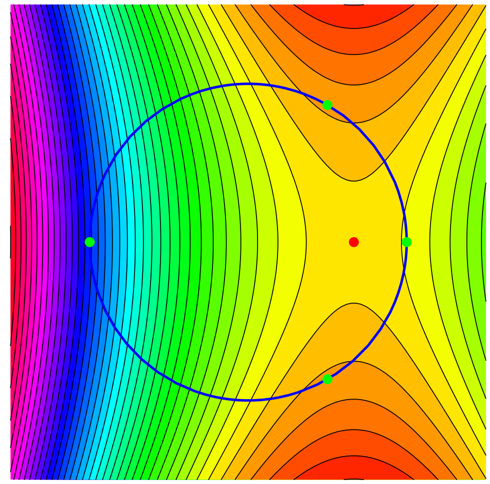


GENERAL PROBLEM. Given a region G whose boundary is given by $g(x,y) = c$. The task to maximize or minimize $f(x,y)$ on G has the following steps:
I) Find extrema inside the region: compute critical points $\nabla f = (0,0)$ and classify them using the second derivative test.
II) Find extrema on the boundary using Lagrange: $\nabla f = \lambda \nabla g, g = c$.
III) Compare the values of the functions obtained in I) and II) to find the maximum or minimum.

EXAMPLE. Extremize $f(x,y) = 3x^2-4x-y^2$ on the disc $x^2+y^2 \leq 1$.
I) Inside the disc. There is only one critical point $(2/3,0)$. The discriminant D is -6 so that $(2/3,0)$ is a saddle point.
II) On the boundary solve $6x-4 = 2\lambda x, -2y = 2\lambda y$. There are four solutions: $(1/2,-\sqrt{3}/2), (1/2,+\sqrt{3}/2), (1,0), (-1,0)$.
III) A list of all candidates:

(x,y)	f(x,y)
$(2/3,0)$	-4/3
$(1/2,-\sqrt{3}/2)$	-2
$(1/2,\sqrt{3}/2)$	-2
$(1,0)$	-1
$(-1,0)$	7

reveals that $(-1,0)$ is the maximum and $(1/2,-\pm\sqrt{3}/2)$ are minima.



IN MATHEMATICA.
Here is how a machine solves the above problem. After defining the functions f and g, the machine solves first the equations leading to critical points, and then the Lagrange equations (we put $L = \lambda$).

$f[x\_, y\_] := 3x^2 - 4x - y^2$
$g[x\_, y\_] := x^2 + y^2 - 1$
Solve$[\{D[f[x,y],x] == 0, D[f[x,y],y] == 0\}, \{x,y\}]$
Solve$[\{D[f[x,y],x] == L * D[g[x,y],x], D[f[x,y],y] == L * D[g[x,y],y], g[x,y] == 0\}, \{x,y,L\}]$

TRICKY LAGRANGE PROBLEM. Let p and q be positive constants such that $\frac{1}{p}+\frac{1}{q} = 1$. Use the method of Lagrange Multipliers to prove that for any $x>0$, $y>0$, the following inequality is true:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} .$$

SOLUTION.
We have to find the maximum of $f(x,y) = xy$ $(x>0,\ y>0)$ under the constraint $\frac{x^p}{p}+\frac{y^q}{q} = c$.)
The Lagrange equations $x = \lambda x^{p-1}, y = \lambda y^{q-1}$ gives $y/x = x^{p-1}/y^{q-1}$ so that $y^q = x^p$.
Plugging this into $x^p/q + y^q = c$ gives $x^p(1/p+1/q) = c$ or $x = c^{1/p}$ and so $y = c^{1/q}$. The maximal value of $f(x,y) = xy$ is $c^{1/p}c^{1/q} = c^1 = c$. Therefore, everywhere

$$xy = f(x,y) \leq c = x^p/p + y^q/q\ .$$

SNELLS LAW of refraction is the problem to determine the fastest path between two points, if the path crosses a border of two media and the media have different indices of refraction. The law can be derived from Lagrange:

PROBLEM. A light ray travels from $A = (-1,1)$ to the point $B = (1,-1)$ crossing a boundary between two media (air and water). In the air $(y>0)$ the speed of the ray is $v_1 = 1$ (in units of speed of light). In the second medium $(y<0)$ the speed of light is $v_2 = 0.9$. The light ray travels on a straight line from A to a point $P = (x,0)$ on the boundary and on a straight line from P to B. Verify Snell's law of refraction $\sin(\theta_1)/\sin(\theta_2) = v_1/v_2$, where θ_1 is the angle the ray makes in air with the y axis and where θ_2 is the angle, the ray makes in water with the y axis.

SOLUTION. Minimize $f(x,y) = \sqrt{(-1-x)^2+y^2}/v_1 + \sqrt{(1-x)^2+y^2}/v_2 = l_1/v_1 + l_2/v_2$ under the constraint $G(x,y) = y = 0$. The Lagrange equations show that $f_x(x,y) = 0$. This is already Snells law because $f_x = v_1 2(x+1)/(2l_1) + v_2 2(1-x)/(2l_2) = 0$ means $v_1/v_2 = \sin(\theta_1)/\sin(\theta_2)$. If v_1 is larger, then θ_1 is larger.