

REMINDER: INTEGRATION POLAR COORDINATES.

$$\int \int_R f(r, \theta) \boxed{r} d\theta dr .$$

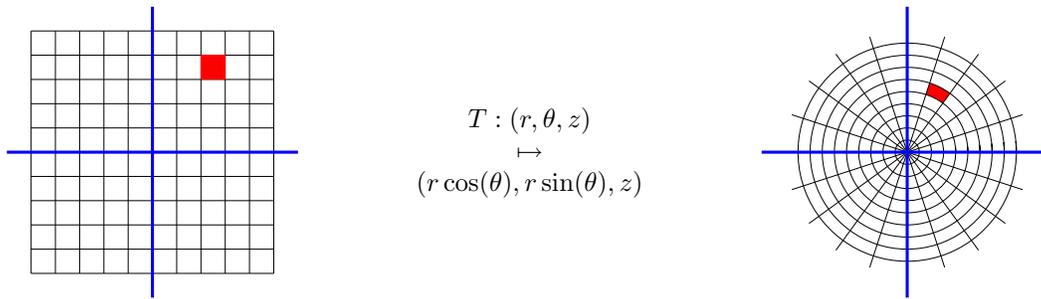
EXAMPLE 1. Area of a disk of radius  $R$

$$\int_0^R \int_0^{2\pi} r d\theta dr = 2\pi \frac{r^2}{2} \Big|_0^R = R^2\pi .$$

WHERE DOES THE FACTOR "r" COME FROM?

1. EXPLANATION. A small rectangle with dimensions  $d\theta dr$  in the  $(r, \theta)$  plane is mapped to a sector segment in the  $(x, y)$  plane. It has approximately the area  $r d\theta dr$ . It is small for small  $r$ .

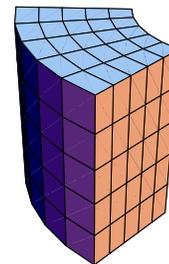
2. EXPLANATION. The map  $(r, \theta) \mapsto (r \cos(\theta), r \sin(\theta))$  which changes from Cartesian to polar coordinates has the **Jacobian** is  $T' = \begin{bmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{bmatrix} = \begin{bmatrix} f_r & f_\theta \\ g_r & g_\theta \end{bmatrix}$  with determinant  $f_r g_\theta - f_\theta g_r = r$ . This is a special case of a more general formula which we do not cover in this class.



CYLINDRICAL COORDINATES. Use polar coordinates in the  $x-y$  plane and leave the  $z$  coordinate. Take  $T(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z)$ . The integration factor  $r$  is the same as in polar coordinates.

$$\int \int \int_{T(R)} f(x, y, z) dx dy dz = \int \int \int_R g(r, \theta, z) \boxed{r} dr d\theta dz .$$

For example, if  $f(x, y, z) = (x^2 + y^2) + xz$ , then  $g(r, \theta, z) = r^2 + r \cos(\theta)z$ .



EXAMPLE. Calculate the volume bounded by the parabolic  $z = 1 - (x^2 + y^2)$  and the  $x-y$  plane. In cylindrical coordinates, the paraboloid is given by the relation  $z = 1 - r^2$ :

$$\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} r dz d\theta dr = \int_0^1 \int_0^{2\pi} (r - r^3) d\theta dr = 2\pi(r^2/2 - r^4/4) \Big|_0^1 = \pi .$$

USE A GOOD PICTURE! A good conceptual picture not only helps to solve double and triple integral problems. Sometimes, it is even virtually impossible to solve the problem without having a good picture.

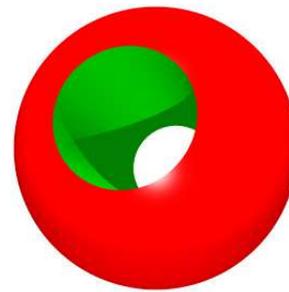
PROBLEM. Find the volume of the solid obtained by taking a sphere  $x^2 + y^2 + z^2 = 1$  into which a hole  $x^2 + y^2 \leq 1/2$  has been drilled.

SOLUTION.

$$2\pi \int_{1/2}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr = 2\pi \int_{1/2}^1 2r\sqrt{1-r^2} dr$$

which is

$$-2\pi \frac{2}{3} (1-r^2)^{3/2} \Big|_{1/2}^1 = \frac{4\pi}{3} \frac{\sqrt{27}}{8} = \pi\sqrt{3}/2$$

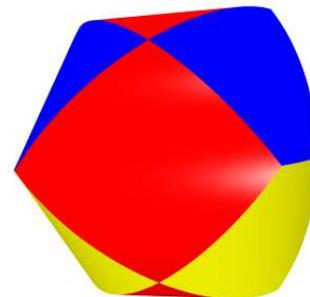
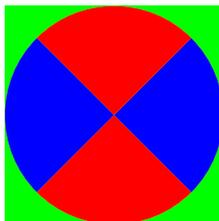


PROBLEM. Find the volume of the intersection of the three cylinders  $x^2 + y^2 \leq 1$ ,  $x^2 + z^2 \leq 1$  and  $y^2 + z^2 \leq 1$ .

SOLUTION.

$$8 \int_{-\pi/4}^{\pi/4} \int_0^1 \sqrt{1-r^2 \sin^2(\theta)} r dr d\theta$$

which is  $-16/3 + 8\sqrt{2}$ .



PROBLEM. Find the volume of the intersection of the two solid cylinders  $x^2 + y^2 \leq 1$  and  $x^2 + z^2 \leq 1$ .

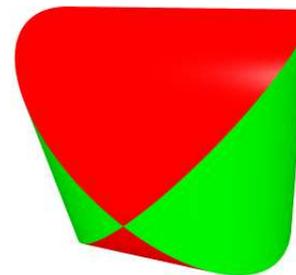
SOLUTION. We compute the volume in one of the 8 octants and multiply by 8 in the end.

$$8 \int_0^1 \int_0^{\pi/2} \sqrt{1-r^2 \cos^2(\theta)} r d\theta dr = 16/3.$$

Here is how one would evaluate this integral with Mathematica:

```
8 Integrate[Sqrt[1 - r^2 Cos[theta]^2] r, {r, 0, 1}, {theta, 0, Pi/2}]
```

Note the order in which the integration range is entered the computer algebra system!



PROBLEM. Find  $\int \int \int_R z^2 dV$ , where  $R$  is the solid obtained by intersecting  $\{1 \leq x^2 + y^2 + z^2 \leq 4\}$  with the double cone  $\{z^2 \geq x^2 + y^2\}$ .

SOLUTION. We split the integral up into a "cone part"  $z \in [-\sqrt{2}, \sqrt{2}]$ , and the cup part  $|z| > \sqrt{2}$  and evaluate each separately. The double cone has the volume  $2\pi \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^z r dr dz = 2\pi\sqrt{2}/3$ . One cup has the volume  $2\pi \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-z^2}} r dr dz = \pi \int_{\sqrt{2}}^2 (4-z^2) dz$ . The total volume is  $\pi(2 - \sqrt{2})16/3$ .

