Lecture 10: 3/4/2004, DIRECTIONAL DERIVATIVE — Math21a

HOMEWORK. 11.5: 22,26,42,56

REVIEW. CHAIN RULE. The chain rule in multivariable calculus is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) .$$

where $\nabla f = (f_x, f_y)$ in two dimensions or $\nabla f = (f_x, f_y, f_z)$ in three dimensions.

DIRECTIONAL DERIVATIVE.
If f is a function of several variables and $\vec{v}$ is a vector, then $\nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$. One writes $\nabla_{\vec{v}} f$ or $D_{\vec{v}} f$.

$$D_v f(x, y, z) = \nabla f(x, y, z) \cdot v$$

It is usually assumed that $\vec{v}$ is a unit vector but we do not insist on that. Using the chain rule, one can write $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$.

EXAMPLE. PARTIAL DERIVATIVES ARE SPECIAL DIRECTIONAL DERIVATIVES.

If $\vec{v} = (1, 0, 0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$.
If $\vec{v} = (0, 1, 0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_y$.
If $\vec{v} = (0, 0, 1)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_z$.

The directional derivative is a generalization of the partial derivatives. Like the partial derivatives, it is a **scalar**.

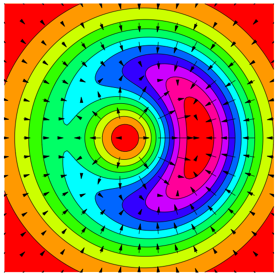
EXAMPLE. DIRECTIONAL DERIVATIVE ALONG A CURVE.

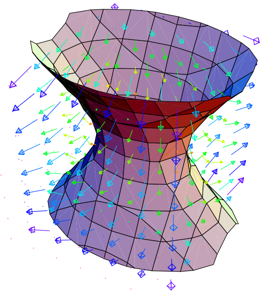
If f is the temperature in a room and $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$, then $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures on the point moving on a curve $\vec{r}(t)$ experiences: the chain rule told us that this is $d/dt f(\vec{r}(t))$.

GRADIENTS AND LEVEL CURVES/SURFACES.

Gradients are orthogonal to level curves and level surfaces.

Every vector $\vec{x} - \vec{x}_0$ in the tangent line satisfies $\nabla f \cdot (\vec{x} - \vec{x}_0) = 0$ and is so orthogonal to ∇f. The same argument holds for surfaces (see Friday).





STEEPEST DECENT. The directional derivative satisfies

$$|D_{\vec{v}} f| \leq |\nabla f||\vec{v}|$$

because $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}|| \cos(\phi)| \leq |\nabla f||\vec{v}|$. The direction $\vec{v} = \nabla f$ is the direction, where f **increases** most, the direction $-\nabla f$ is the direction where f **decreases most**. It is the direction of steepest decent.

IN WHICH DIRECTION DOES f INCREASE? If $\vec{v} = \nabla f$, then the directional derivative is $\nabla f \cdot \nabla f = |\nabla f|^2$. This means that f **increases**, if we move into the direction of the gradient!

EXAMPLE. You are on a trip in a air-shop at $(1, 2)$ and want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x, y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest?

Parameterize the direction by $\vec{v} = (\cos(\phi), \sin(\phi))$. The pressure gradient is $\nabla p(x, y) = (2x, 4y)$. The directional derivative in the ϕ-direction is $\nabla p(x, y) \cdot v = 2\cos(\phi) + 4\sin(\phi)$. This is maximal for $-2\sin(\phi) + 4\cos(\phi) = 0$ which means $\tan(\phi) = 1/2$.

ZERO DIRECTIONAL DERIVATIVE. The rate of change in all directions is zero if and only if $\nabla f(x, y) = 0$: if $\nabla f \neq \vec{0}$, we can choose $\vec{v} = \nabla f$ and get $D_{\nabla f} f = |\nabla f|^2$.

We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x, y), where $\nabla f(x, y) = (0, 0)$ are called **stationary points** or **critical points**. Knowing the critical points is important to understand the function f.

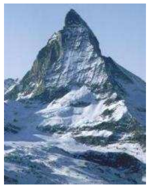
PROPERTIES DIRECTIONAL DERIVATIVE.

$$\begin{aligned} D_v(\lambda f) &= \lambda D_v(f) \\ D_v(f + g) &= D_v(f) + D_v(g) \\ D_v(fg) &= D_v(f)g + f D_v(g) \end{aligned}$$

PROPERTIES GRADIENT

$$\begin{aligned} \nabla(\lambda f) &= \lambda \nabla(f) \\ \nabla(f + g) &= \nabla(f) + \nabla(g) \\ \nabla(fg) &= \nabla(f)g + f\nabla(g) \end{aligned}$$

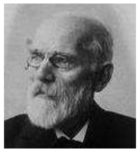
THE MATTERHORN is a popular climbing mountain in the Swiss alps. Its height is 4478 meters (14,869 feet). It is quite easy to climb with a guide. There are ropes and ladders at difficult places. Even so, about 3 people die each year from climbing accidents at the Matterhorn, this does not stop you from trying an ascent. In suitable units on the ground, the height $f(x, y)$ of the Matterhorn is approximated by $f(x, y) = 4000 - x^2 - y^2$. At height $f(-10, 10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the north-east direction $v = (1, -1)$. Calculate the rate of change in that direction. We have $\nabla f(x, y) = (-2x, -2y)$, so that $(20, -20) \cdot (1, -1) = 40$. This is a place, with a ladder, where you climb 40 meters up when advancing 1m forward.



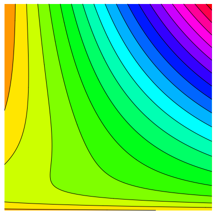
THE VAN DER WAALS (1837-1923) equation for real gases is

$$(p + a/V^2)(V - b) = RT(p, V) ,$$

where $R = 8.314 J/K$mol is a constant called the **Avogadro number**. This law relates the pressure p, the volume V and the temperature T of a gas. The constant a is related to the molecular interactions, the constant b to the finite rest volume of the molecules.



The **ideal gas** law $pV = nRT$ is obtained when a, b are set to 0. The level curves or **isotherms** $T(p, V) = const$ tell much about the properties of the gas. The so called **reduced van der Waals law** $T(p, V) = (p + 3/V^2)(3V - 1)/8$ is obtained by scaling p, T, V depending on the gas. Calculate the directional derivative of $T(p, V)$ at the point $(p, V) = (1, 1)$ into the direction $v = (1, 2)$. We have $T_p(p, V) = (3V - 1)/8$ and $T_V(p, V) = 3p/8 - (9/8)1/V^2 - 3/(4V^3)$. Therefore, $\nabla T(1, 1) = (1/4, 0)$ and $D_v T(1, 1) = 1/5$.



TANGENT LINE. Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b \rangle$ is perpendicular to the level curve $f(x, y) = c$ through (x_0, y_0), the equation for the tangent line is

$$ax + by = d,\ a = f_x(x_0, y_0),\ b = f_y(x_0, y_0),\ d = ax_0 + by_0$$

EXAMPLE. The isotherme in the previous example through $(1, 1)$ has there the tangent $(1/4)x + 0 \cdot y = (1/4)1 + 0 \cdot 1 = 1/4$ which is the horizontal line $x = 1$.