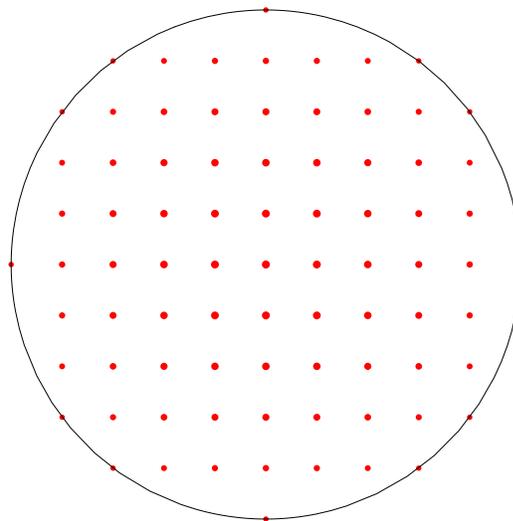
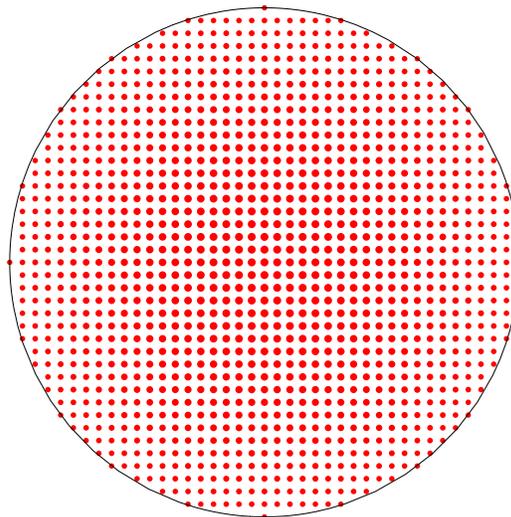
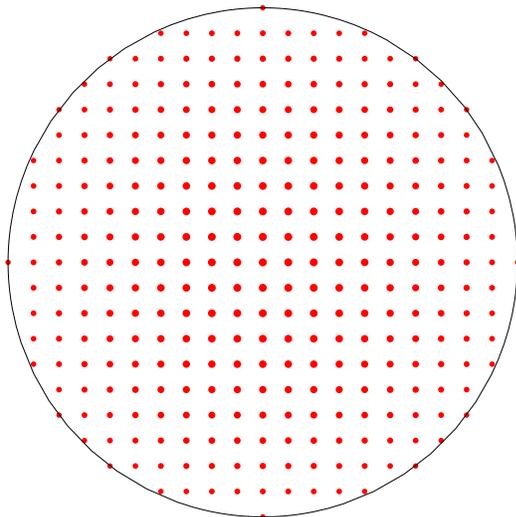


We have defined the double integral of a function  $f$  over a region  $R$  as the limit  $\frac{1}{n^2} \sum_{x_{ij} \in R} f(x_{ij})$ , where  $x_{ij} = (i/n, j/n)$ . Let us look at the case of a disc of radius  $r$  and  $n = 5$  and the case  $f(x, y) = 1$ , where we compute area. What is the approximation area, which we get by counting the number of lattice points inside  $R$ ?



The **Gauss circle problem** asks to estimate the number of lattice points  $g(r) = \pi r^2 + E(r)$  enclosed by the circle  $r$ .

Here are some numbers  $g(10) = 317, g(100) = 31417, g(1000) = 3141549, g(10000) = 314159053$ .



Estimating the error  $g(n)/n^2 - \pi$  is a famous open problem called the **"Gauss circle problem"**. If one writes  $E(n) = g(n) - \pi n^2$ , one believes that for every  $\theta > 1/2$  there is a constant  $C$  such that  $E(n) \leq Cn^\theta$ . Gauss knew already that this is true for  $\theta = 1$ . This has been improved over the 20'th century to  $\theta = 46/73$ .

Let's experiment with  $\theta = 1/2$ :  $g(1000) - 1000^2\pi = -43.65$  is in absolute value less than  $C = 2$  times  $\sqrt{1000} = 31.6$ ,  $g(10000) - 10000^2\pi = -212.359$  is in absolute value less than  $C = 3$  times  $\sqrt{10000} = 100$ . Letting Mathematica count for a bit longer:  $g(100'000) - 100'000^2\pi = -1078.9$  which is in absolute value smaller than  $C = 3$  times  $\sqrt{100000} = 316.2$ .

