HOMEWORK: 10.6: 8,14,20,22

PLANE CURVE

$\vec{r}(t) = (x(t), y(t))$ **position**
$\vec{r}\,'(t) = (x'(t), y'(t))$ **velocity**
$|\vec{r}\,'(t)| = (x'(t), y'(t))$ **speed**
$\vec{r}\,''(t) = (x''(t), y''(t))$ **acceleration** $\vec{r}\,'''(t) = (x'''(t), y'''(t))$ **jerk**

SPACE CURVE

$\vec{r}(t) = (x(t), y(t), z(t))$ **position**
$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$ **velocity**
$|\vec{r}\,'(t)| = (x'(t), y'(t))$ **speed**
$\vec{r}\,''(t) = (x''(t), y''(t), z''(t))$ **acceleration** $\vec{r}\,'''(t) = (x'''(t), y'''(t), z'''(t))$ **jerk**

ARC LENGTH. If $t \in [a,b] \mapsto \vec{r}(t)$ with velocity $\vec{v}(t) = \vec{r}\,'(t)$ and speed $|\vec{v}(t)|$, then $\int_a^b |\vec{v}(t)|\,dt$ is called the **arc length of the curve**. For space curves this is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

PARAMETER INDEPENDENCE. The arc length is independent of the parameterization of the curve.

REASON. Changing the parameter is a change of variables (substitution) in the integration.

EXAMPLE. The circle parameterized by $\vec{r}(t) = (\cos(t^2), \sin(t^2))$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r}\,'(t) = 2t(-\sin(t), \cos(t))$ and speed $2t$. The arc length is $\int_0^{\sqrt{2\pi}} 2t\,dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

REMARK. Often, there is no closed formula for the arc length of a curve. For example, the **Lissajoux figure** $\vec{r}(t) = (\cos(3t), \sin(5t))$ has the length $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\,dt$. This integral must be evaluated numerically. If you do the Mathematica Lab, you will see how to do that with the computer.

THE MATERIAL BELOW IS NOT PART OF THIS COURSE.

CURVATURE.

$\vec{T}(t) = \vec{r}\,'(t)|/|\vec{r}\,'(t)|$ **unit tangent vector**
$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)|$ **curvature**

CURVATURE FORMULA
($(a,b) \times (c,d) = ad - bc$ in 2D)

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

EXAMPLE. CIRCLE

$\vec{r}(t) = (r\cos(t), r\sin(t))$.
$\vec{r}\,'(t) = (-r\sin(t), r\cos(t))$.
$|\vec{r}\,'(t)| = r$.
$\vec{T}(t) = (-\sin(t), \cos(t))$.
$\vec{r}\,''(t) = (-r\cos(t), -r\sin(t))$.
$\vec{T}'(t) = (-\cos(t), -\sin(t))$.
$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = 1/r$.

EXAMPLE. HELIX

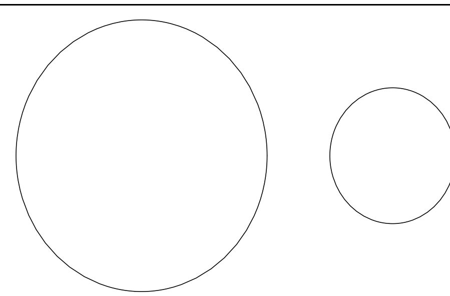
$\vec{r}(t) = (\cos(t), \sin(t), t)$.
$\vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$.
$|\vec{r}\,'(t)| = (-\sin(t), \cos(t), 1) = \sqrt{2}$.
$\vec{T}(t) = (-\sin(t), \cos(t), 1)/\sqrt{2}$.
$\vec{r}\,''(t) = (-\cos(t), -\sin(t), 0)$.
$\vec{T}'(t) = (-\cos(t), \sin(t), 0)/\sqrt{2}$.
$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = 1/2$.

INTERPRETATION.

If $s(t) = \int_0^t |\vec{r}\,'(t)|\,dt$, then $s'(t) = ds/dt = |\vec{r}\,'(t)|$. Because $\vec{T}'(t) = d\vec{T}/dt = d\vec{T}/ds \cdot ds/dt$, we have $|d\vec{T}/ds| = |\vec{T}'(t)|/|\vec{r}\,'(t)| = \kappa(t)$.

"The curvature is the length of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1."

A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.



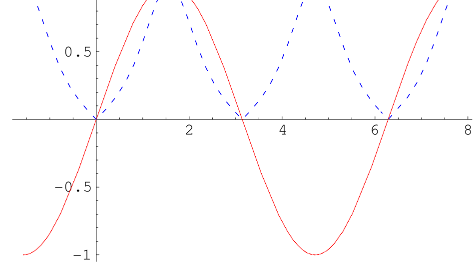
Small curvature $\kappa = 1/r = 1/2$

Large curvature $\kappa = 1/r = 2$

The curve $\vec{r}(t) = (t, f(t))$, which is the graph of a function $f$ has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1+f'(t)^2}^2$ and after some simplification

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1+f'(t)^2}^3$$

EXAMPLE. $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1+\cos^2(t)}^3$.



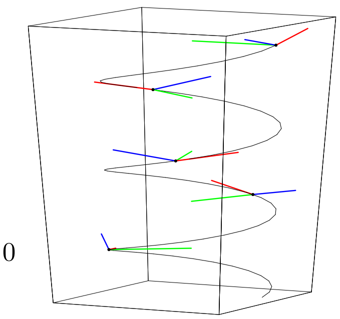

TANGENT/NORMAL/BINORMAL.

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ **tangent vector**

$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$ **unit normal vector**

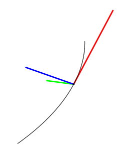
$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ **binormal vector**

Because $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get after differentiation $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$.



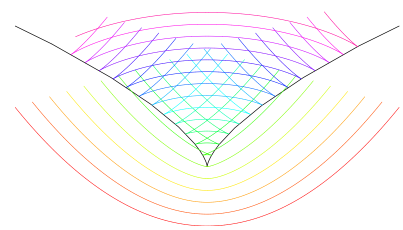
The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

Note. In order that $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ exist, we need that $\vec{r}'(t)$ is not zero.



WHERE IS CURVATURE NEEDED?

OPTICS. If a curve $\vec{r}(t)$ represents a wavefront and $\vec{n}(t)$ is a unit vector normal to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call that curve the **evolute** of the original curve.



HISTORY.

**Aristotle:** (350 BC) distinguishes between straight lines, circles and "mixed behavior"
**Oresme:** (14'th century): measure of twist called "curvitas"
**Kepler:** (15'th century): circle of curvature.
**Huygens:** (16'th century): evolutes and involutes in connection with optics.
**Newton:** (17'th century): circle has constant curvature inversely proportional to radius. (using infinitesimals)
**Simpson:** (17'th century): string construction of evolutes, description using fluxions.
**Euler:** (17'th century): first formulas of curvature using second derivatives.
**Gauss:** (18'th century): modern description, higher dimensional versions.

COMPUTING CURVATURE WITH MATHEMATICA

```
x[t_]:=Cos[3 t];
y[t_]:=Sin[5 t];
r[t_]:={x[t],y[t]};
dr[t_]:=D[r[s],s] /. s->t;
L[{a_,b_}]:=Sqrt[a^2+b^2];
T[t_]:=dr[t]/L[dr[t]];
dT[t_]:=D[T[s],s] /. s->t;
kappa[t_]:=L[dT[t]]/L[dr[t]]
```