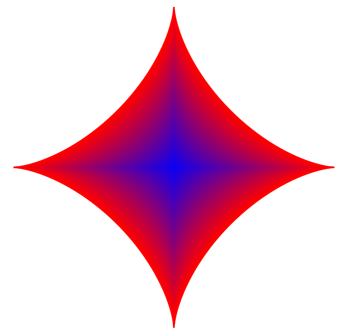
5/5/2004 INTEGRAL THEOREM PROBLEMS — Math21a, O. Knill

HOMEWORK. Problems 9,10,11 on the homeworksheet.

SUMMARY. This is a collection of problems on line integrals, Green's theorem, Stokes theorem and the divergence theorem. Some of them are more challenging.

LINE INTEGRALS GREEN THEOREM.

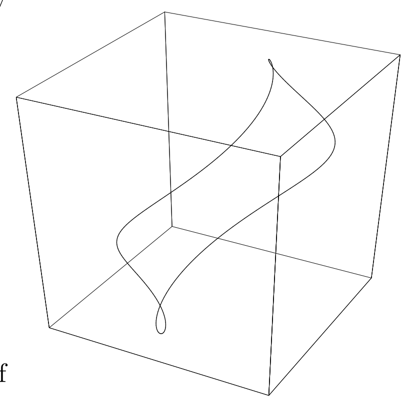
The curve $r(t) = (\cos^3(t), \sin^3(t))$ is called a **hypocycloid**. It bounds a region R in the plane.
a) Calculate the line integral of the vector field $F(x, y) = (x, y)$ along the curve.
b) Find the area of the hypocycloid.



a) Because $\text{curl}(F) = 0$ the result is zero by Green's theorem.
b) Use the vector field $F(x, y) = (0, x)$ which has $\text{curl}(F) = 1$. The line integral is $\int_0^{2\pi} F(r(t)) \cdot r'(t) \; dt = \int_0^{2\pi} \cos^3(t) 3\sin^2(t)\cos(t) \; dt$ $= \int_0^{2\pi} 3\cos^4(t)\sin^2(t) \; dt = 3\pi/8$. (To compute the integral, use that $8\cos^4(t)\sin^2(t) = \cos(2t)\sin^2(2t) + \sin^2(2t)$).

LENGTH OF CURVE AND LINE INTEGRALS.

Assume $C : t \mapsto r(t)$ is a closed path in space and $F(r(t))$ is the unit tangent vector to the curve (that is a vector parallel to the velocity vector which has length 1).
a) What is $\int_C F \, dr$?
b) Can F be a gradient field?



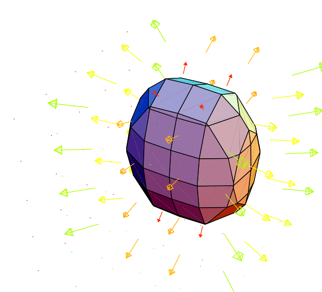
Answer:
a) $F(r(t)) = r'(t)/|r'(t)|$. By definition of the line integral,

$$\int_C F(r(t)) \cdot r'(t) \; dt = \int_a^b \frac{r'(t)}{|r'(t)|} \cdot r'(t) \; dt = \int_a^b |r'(t)| \; dt ,$$

which is the length of the curve.
b) No: If F were a gradient field, then by the fundamental theorem of line integrals, we would have that the line integral along a closed curve is zero. But because this is the length of the curve, this is not possible.

SURFACE AREA AND FLUX.

Assume $S : (u, v) \mapsto r(u, v)$ is a closed surface in space and $F(r(u, v))$ is the unit normal vector on S (which points in the direction of $r_u \times r_v$).
a) What is $\int\int_S F \cdot dS$?
b) Is it possible that F is the curl of an other vector field?
c) Is it possible that $\text{div}(F) = (0, 0, 0)$ everywhere inside the surface.



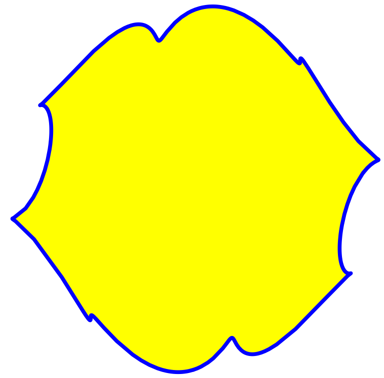
Answer:
a) $F(r(u, v)) = (r_u \times r_v)/|r_u \times r_v|$. By definition of the flux integral, $\int\int_S F \cdot dS = \int\int_R F(r(u, v)) \cdot r_u \times r_v = \int\int_R (r_u \times r_v/|r_u \times r_v|) \cdot r_u \times r_v =$ $\int\int_R |r_u \times r_v| \; dudv$ which is the area of the surface.
b) No, if F were the curl of an other field G, then the flux of F through the closed surface would be zero. But since it is the area, this is not possible.
c) From the divergence theorem follows that $\text{div}(F)$ is nonzero somewhere inside the surface.

GREENS THEOREM AND LAPLACIAN.

Assume R is a region in the plane and let n denote the unit normal vector to the boundary C of R. For any function $u(x,y)$, we use the notation $\partial f/\partial u = \text{grad}(u)\cdot\vec{n}$ which is the directional derivative of u into the direction $\vec{n}$ normal to C. We also use the notation $\Delta u = u_{xx} + u_{yy}$. Show that

$$\int_C \partial u/\partial n \cdot dr = \int\int_R \Delta u \, dA$$

Answer: Define $F(x,y) = (-B, A)$ if $\partial u/\partial n = (A, B)$. The left integral is the line integral of F along C. The right integral is the double integral over $\Delta u = \text{curl}(F)$.



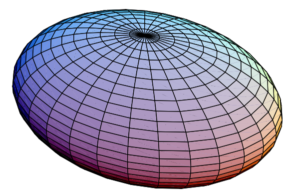
STOKES THEOREM OR DIVERGENCE THEOREM

Find $\int\int_S \text{curl}(F)\cdot dS$, where S is the ellipsoid $x^2 + y^2 + 2z^2 = 10$ and $F(x,y,z) = (\sin(xy), e^x, -yz)$.

Answer. The integral is zero because the boundary of S is empty. This fact can be seen using Stokes theorem. It can also been seen by divergence theorem

$$\int\int_S \text{curl}(F)\cdot dS = \int\int\int \text{divcurl}(F)\, dV\ .$$

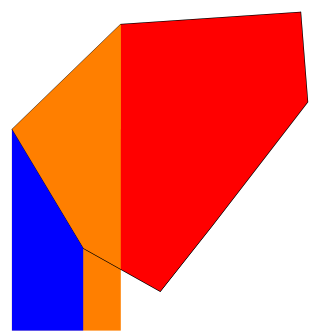
using $\text{div}(\text{curl})(F) = 0$.



AREA OF POLYGONS.

If $P_i = (x_i, y_i), i = 1, \dots, n$ are the edges of a polygon in the plane, then its area is $A = \sum_i (x_i - x_{i+1})(y_{i+1} + y_i)/2$.
The proof is an application of Green's theorem. The line integral of the vector field $F(x,y) = (-y, 0)$ through the side P_i, P_{i+1} is $(x_i - x_{i+1})(y_{i+1} + y_i)/2$, because $(x_{i+1} - x_i)$ is the projected area onto the x-axis and $(y_{i+1} + y_i)/2$ is the average value of the vector field on that side. Because $\text{curl}(F)(x,y) = 1$ for all (x,y), the result follows from Greens theorem.

The result can also be seen geometrically: $(x_i - x_{i+1})(y_{i+1} + y_i)/2$ is the signed area of the trapezoid $(x_i, 0), (x_{i+1}, 0), (x_{i+1}, y_{i+1})$. In the picture, we see two of them. The second one is taken negatively.

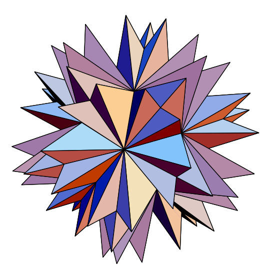


VOLUME OF POLYHEDRA.

Verify with the divergence theorem: If $P_i = (x_i, y_i, z_i)$ are the edges of a polyhedron in space and $F_j = \{P_{i_1}, ... P_{i_{k_j}}\}$ are the faces, then $V = \sum_j A_j \overline{z}_j$ where A_j is the area of the xy-projection (*) of the polygon F_j and $\overline{z}_j = (z_{i_1} + ... + z_{k_j})/k_j$ is the average z value of the face F_j.

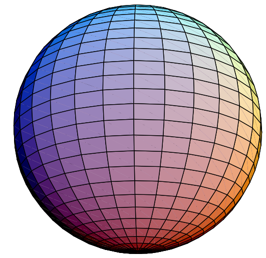
Solution. The vector field $F(x,y,z) = z$ has divergence 1. The flux through a face F is $|F_j|(z_{i_1} + ... + z_{k_j})/k_j$. Gauss theorem assures that the volume is the sum of the fluxes $A_j \overline{z}_j$ through the faces.

(*) The projection of a polygon is the "shadow" when projecting from space along the z-axes onto the xy-plane. A triangle $(1,0,1), (1,1,0), (0,1,2)$ for example would be projected to the triangle $(1,0), (1,1), (0,1)$.



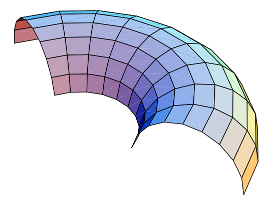
STOKES AND GAUSS TOGETHER.

Can you derive $\text{div}(\text{curl}(F)) = 0$ using Gauss and Stokes theorem? Consider a sphere S of radius r around a point (x, y, z). It bounds a ball G. Consider a vector field F. The flux of $\text{curl}(F)$ through S is zero because of Stokes theorem. Gauss theorem tells that the integral of $f = \text{div}(\text{curl}(F))$ over G is zero. Because S was arbitrary, f must vanish everywhere.



FUNDAMENTAL THEOREM AND STOKES.

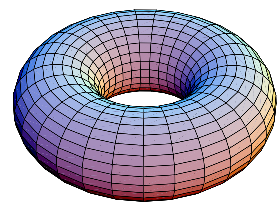
Can you derive the identity $\text{curl}(\text{grad}(F)) = 0$ from integral theorems?



To see that the vector field $G = \text{curl}(\text{grad}(F)) = 0$ is identically zero, it is enough to show that the flux of G trough any surface S is zero. By Stokes theorem, the flux trough S is $\int_C \text{grad}(F) \cdot dr$. By the fundamental theorem of line integrals, this is zero.

VOLUME COMPUTATION WITH GAUSS.

Calculate the volume of the torus $T(a, b)$ enclosed by the surface $r(u, v) = ((a + b\cos(v))\cos(u), (a + b\cos(v))\sin(u), b\sin(v))$ using Gauss theorem and the vector field $F(x, y, z) = (x, y, 0)/2$.

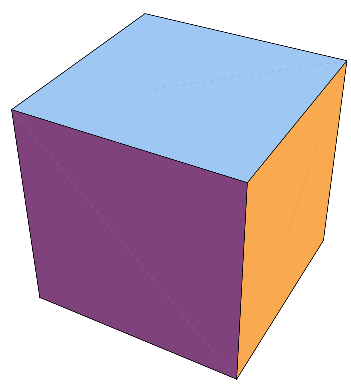


The vector field F has divergence 1. The parameterization of the torus gives

$$r_u \times r_v = b(a + b\cos(v))(\cos(u)\cos(v), \cos(v)\sin(u), \sin(v)) \; .$$

The flux of this vectorfield through the boundary of the torus is $\int_0^{2\pi} \int_0^{2\pi} b(a + b\cos(v))^2 \cos(v) \; dudv = 2\pi^2 ab^2$.

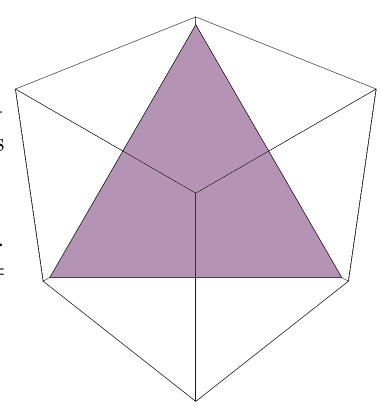
GAUSS OR STOKES?



You know that the flux of the vector field $G = \text{curl}(F)(x, y, z)$ through 5 faces of a cube D is equal to 1 each. What is the flux of the same vector field G through the 6'th face?

Solution: the problem is best solved with the divergence theorem: because the flux of G through the entire surface is zero, the flux through the 6'th face must cancel the sum of the fluxes 5 through the other 5 surfaces. The result is -5.

WORK COMPUTATION USING STOKES.

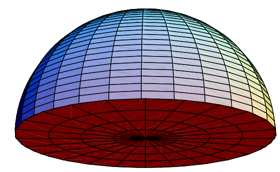


Calculate the work of the vector field $F(x, y, z) = (x - y + z, y - z + x, z - x + y)$, along the path C which connects the points $(1, 0, 0) \to (0, 1, 0) \to (0, 0, 1) \to (1, 0, 0)$ in that order.

Answer. The line integral over each part is each 1. The total is 3. $\text{curl}(F) = (2, 2, 2)$ and $S : (u, v) \mapsto r(u, v) = (u, v, 1 - u - v)$ $r_u \times r_v = (1, 1, 1)$ $\int_S \text{curl}(F) \cdot r_u \times r_v \; dudv = 6$ area of$S = 3$,

STOKES OR GAUSS?

Compute the flux of the vector field $F(x,y,z) = (x - x\sin(\sin(z)), 2y, 3z + \sin(\sin(z)))$ through the upper hemisphere $S = \{ (x,y,z) | \ x^2+y^2+z^2 = 1, z \geq 0\}$.
Answer. We use Gauss: $\mathrm{div}(F) = 6$ and $\int\int\int_B \mathrm{div}(F)\ dV = 6\mathrm{Vol}(B) = 4\pi$. We can not easily compute the flux through the hemisphere. However, we can see that the flux through the floor of the region is zero because the normal component P of of the vector field $F = (M,N,P)$ is zero on $z = 0$. So: the result is $4\pi - 0 = 4\pi$.

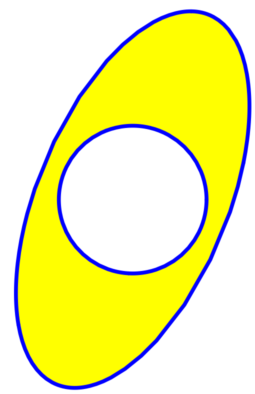


GREENS THEOREM.

Calculate the work of the vector field $F(x,y) = \frac{1}{x^2+y^2}(-y,x)$ along the boundary of the ellipse $r(t) = (3\cos(t)+\sin(t), 5\sin(t)+\cos(t))$.

Solution. Take an other curve $C : x^2+y^2 \leq 4$ and apply Green's theorem to the region R bounded by the ellipse and the circle. Because $\mathrm{curl}(F)$ is zero in D, the line integral along the ellipse is the same as the line integral along the circle: $t \mapsto r(t) = (2\cos(t), 2\sin(t))$ with velocity $r'(t) = (-2\sin(t), 2\cos(t))$:

$$\int F\cdot dr = \int_0^{2\pi} \frac{(-2\cos(t), 2\sin(t))}{4} \cdot (-2\sin(t), 2\cos(t) = \int_0^{2\pi} 1\ dt = 2\pi\ .$$



TRUE/FALSE QUESTIONS ON INTEGRAL THEOREMS.

(TF) The flux of the curl of a vector field through the unit sphere is zero.

(TF) The line integral of the curl of a vector field along a closed curve is zero.

(TF) The line integral $\int_C F \cdot dr$ is independent of how a curve $C : t \mapsto r(t)$ is parametrized.

(TF) The maximal speed of a curve is independent on how the curve is parametrized.

(TF) The flux integral $\int_S F \cdot dS$ through a surface is independent on how the surface S is parametrized.

(TF) The area $\int_S dS$ of a surface is independent on how the surface S is parametrized.

(TF) The maximal value of $r_u \times r_v$ on a surface S is independent on how the surface is parametrized.

(TF) There exists a vector field in space which has zero divergence, zero curl but is not a constant field.

(TF) There exists a vector field in space which has zero gradient but is a constant vector field $F(x,y,z) = (a,b,c)$.

(TF) There exists a function in space which has zero Laplacian $f_{xx} + f_{yy} + f_{zz} = 0$ but which is not constant.

(TF) $\mathrm{div}(\mathrm{grad}(F)) = 0$ and $\mathrm{div}(\mathrm{curl}(F)) = 0$ and $\mathrm{curl}(\mathrm{grad}(F)) = 0$.

(TF) The line integral of a gradient field along any part of a level curve $F = const$ is zero.

(TF) If $\mathrm{div}(F) = 0$, then the line integral along any closed curve is zero.

(TF) If $\mathrm{curl}(F) = 0$, then the line integral along any closed curve is zero.

(TF) If $\mathrm{div}(F) = 0$ then the flux integral along any sphere in space is zero.

(TF) If $\mathrm{curl}(F) = 0$ then the flux integral along any sphere in space is zero.