GENERAL TIPS.

- Review the online quizzes.
- Make list of facts on a sheet of paper.
- Fresh up short-term memory before test.
- Review homework. Find error patterns.
- Ask questions:

> "Ask a question and you're a fool for three minutes; do not ask a question and you're a fool for the rest of your life." - Chinese Proverb

- During the exam: read the questions carefully. Wrong understanding could lead you to solve an other problem:

> There was a college student trying to earn some pocket money by going from house to house offering to do odd jobs. He explained this to a man who answered one door. "How much will you charge to paint my porch?" asked the man. "Forty dollars." "Fine" said the man, and gave the student the paint and brushes. Three hours later the paint-splattered lad knocked on the door again. "All done!", he says, and collects his money. "By the way," the student says, "That's not a Porsche, it's a Ferrari."

MIDTERM TOPICS.

- Properties of dot, cross and triple product
- Orthogonality, parallel, vector projection
- Parametrized Lines and Planes
- Given line and plane, find intersection
- Given plane and plane, find intersection
- Given line and point, find plane
- Given two points, find line
- Given three points, find plane
- Distances: point-line, line-line,point-plane
- Distinguish and analyse curves
- Determine curves from acceleration
- Know Keplers laws, polar form of ellipse
- Recognize functions $f(x,y)$ of two variables.
- Tangent lines, tangent curves
- Distance between two lines
- Distance between two planes
- Angle between two vectors
- Angle between two planes
- Area of parallelogram, triangle in space
- Volume of parallelepiped
- Distinguish contour maps, graphs
- Compute velocity, acceleration, speed
- Integrate from velocity to get position
- Find length of curves
- Level curves, level surfaces
- Directional derivative
- Chain rule
- Implicit differentiation
- Tangent planes

VECTORS.

Two points $P = (1,2,3), Q = (3,4,6)$ define a vector $\vec{v} = \vec{PQ} = \langle 2,2,3 \rangle$. If $\vec{v} = \lambda \vec{w}$, then the vectors are **parallel** if $\vec{v} \cdot \vec{w} = 0$, then the vectors are called **orthogonal**. For example, $(1,2,3)$ is parallel to $(-2,-4,-6)$ and orthogonal to $(3,-2,1)$. The addition, subtraction and scalar multiplication of vectors is done componentwise. For example: $(3,2,1) - 2((1,1,1) + (-1,-1,0)) = (3,2,-1)$.

A nonzero vector $\vec{v}$ and a point $P = (x_0, y_0, z_0)$ define a line $\vec{r}(t) = P + t\vec{v}$. Two nonzero, non-parallel vectors $\vec{v}, \vec{w}$ and a point P define a plane $P + t\vec{v} + s\vec{s}$. The vector $\vec{n} = \vec{v} \times \vec{w} = (a,b,c)$ is orthogonal to the plane. Points on the line satisfy the symmetric equation $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. Points on the plane satisfy an equation $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$. Using the dot product for projection and the vector product to get orthogonal vectors, one can solve many geometric problems in 3D.

DOT PRODUCT (is scalar)

$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$	commutative
$\lvert\vec{v} \cdot \vec{w}\rvert = \lvert\vec{v}\rvert\lvert\vec{w}\rvert \cos(\alpha)$	angle
$(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$	linearity
$(\vec{u} + \vec{v}) \cdot \vec{w} == \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$	distributivity
$\{1,2,3\}.\{3,4,5\}$	in Mathematica
$\frac{d}{dt}(\vec{v} \cdot \vec{w}) = (\frac{d}{dt}\vec{v}) \cdot \vec{w} + (\vec{v} \cdot \frac{d}{dt}\vec{w})$	product rule

CROSS PRODUCT (is vector)

$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$	anti-commutative
$\lvert\vec{v} \times \vec{w}\rvert = \lvert\vec{v}\rvert\lvert\vec{w}\rvert \sin(\alpha)$	angle
$(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})$	linearity
$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$	distributivity
Cross[{1,2,3},{3,4,5}]	Mathematica
$\frac{d}{dt}(\vec{v} \times \vec{w}) = (\frac{d}{dt}\vec{v}) \times \vec{w} + \vec{v} \times (\frac{d}{dt}\vec{w})$	product rule

Vector projection:

$$\text{proj}_{\vec{v}}(\vec{w}) = \frac{(\vec{v}\cdot\vec{w})\vec{v}}{|\vec{v}|^2}.$$

Is a vector parallel to $\vec{w}$.

Scalar projection:

$$\text{comp}_{\vec{v}}(\vec{w}) = |\text{proj}_{\vec{v}}(\vec{w})| = \frac{|\vec{v}\cdot\vec{w}|}{|\vec{v}|}$$

the length of the projected vector.

Applications:

- Distance $P+t\vec{v}, Q+s\vec{w}$ is scalar projection of $\vec{PQ}$ onto $\vec{v}\times\vec{w}$.
- Distance $P, Q+t\vec{v}+s\vec{w}$ is scalar projection of $\vec{PQ}$ onto $\vec{n} = \vec{v}\times\vec{w}$.

SURFACES $\{f(x,y,z) = c\}$.

Examples are **graphs**, where $f(x,y,z) = z - g(x,y) = 0$ or planes, where $f(x,y,z) = ax + by + cz = c$. Surfaces can be analyzed by looking at intersections with planes parallel to the coordinate planes. For graphs, the traces $f(x,y) = c$ are **contour lines**. Most important fact:

The gradient $\nabla f(x_0,y_0,z_0)$ is normal to the surface $f(x,y,z) = c$ containing (x_0,y_0,z_0).

SURFACES EXAMPLES

- sphere $x^2 + y^2 + z^2 = 1$
- cylinder $x^2 + y^2 = 1$
- ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$
- cone $x^2 + y^2 - z^2 = 0$
- plane $ax + by + cz = d$
- one sheeted hyperboloid $x^2 + y^2 - z^2 = 1$
- two sheeted hyperboloid $x^2 + y^2 - z^2 = -1$
- paraboloid $x^2 + y^2 - z = 0$
- hyperbolic paraboloid $x^2 - y^2 - z = 0$
- graph of function $g(x,y) - z = 0$

can be identified using **traces**, the intersections with planes.

CURVES.

$\vec{r}(t) = (x(t), y(t), z(t))$, $t \in [a,b]$ defines a curve. By differentiation, we obtain the **velocity** $\vec{r}\,'(t)$ and **acceleration** $\vec{r}\,''(t)$. If we integrate the speed $|\vec{r}\,'(t)|$ over the interval $[a,b]$, we obtain the **length** of the curve.

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Example: $\vec{r}(t) = (1, 3t^2, t^3)$, $\vec{r}'(t) = (0, 6t, 3t^2)$, so that $|\vec{r}'(t)| = 3t(4+t^2)$. The length of the curve between 0 and 1 is $\int_0^1 3t(4+t^2)\, dt = 6t^2 + 3\frac{t^4}{4}|_0^1 = 6 \cdot \frac{3}{4}$.

DIRECTIONAL DERIVATIVE.

For any vector $\vec{v}$ and a function $f(x,y)$, define $D_{\vec{v}}f(x,y) = \nabla f(x,y)\cdot\vec{v}$. Unlike in many textbooks:

We do not divide by $|\vec{v}|$ to compute the directional derivative.

CHAIN RULE.

We have seen that $d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$. This works also, if $\vec{r}(t,s)$ is a function of two variables:
$f_t(x(t,s), y(t,s)) = \nabla f(\vec{r}(t,s)) \cdot \vec{r}_t(t,s)$.
$f_s(x(t,s), y(t,s)) = \nabla f(\vec{r}(t,s)) \cdot \vec{r}_s(t,s)$.

Other variables: $w(u,v)$ function of u,v, where u,v are functions of x and eventually of y:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$