Multivariable Calculus

Math 21a

Harvard University

Spring 2004

Oliver Knill

These are some class notes distributed in a multivariable calculus course tought in Spring 2004. This was a physics flavored section.

Some of the pages were developed as complements to the text and lectures in the years 2000-2004. While some of the pages are proofread pretty well over the years, others were written just the night before class.

The last lecture was "calculus beyond calculus". Glued with it after that are some notes from "last hours" from previous semesters.

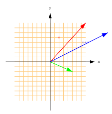
Oliver Knill. 5/7/2004

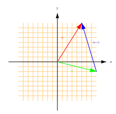
Lecture 1: VECTORS DOT PRODUCT **O. Knill, Math21a**

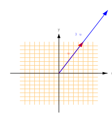
HOMEWORK: Section 10.1: 42,60: Section 10.2: 4,16

VECTORS. Two points $P_1 = (x_1, y_1)$, $Q = P_2 = (x_2, y_2)$ in the plane determine a **vector** $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. It points from P_1 to P_2 and we can write $P_1 + \vec{v} = P_2$.
COORDINATES. Points P in space are in one to one correspondence to vectors pointing from 0 to P. The numbers $\vec{v}_i$ in a vector $\vec{v} = (v_1, v_2)$ are also called **components** or of the vector.
REMARKS: vectors can be drawn **everywhere** in the plane. If a vector starts at 0, then the vector $\vec{v} = \langle v_1, v_2 \rangle$ points to the point $\langle v_1, v_2 \rangle$. That's is why one can identify points $P = (a, b)$ with vectors $\vec{v} = \langle a, b \rangle$. Two vectors which can be translated into each other are considered **equal**. In three dimensions, vectors have three components. In some Encyclopedias like Encylopedia Britannica define vectors as objects which have "both magnitude and direction". This is unprecise and strictly speaking incorrect because the zero vector is also a vector but has no direction.

ADDITION SUBTRACTION, SCALAR MULTIPLICATION.







$\vec{u} + \vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$
$= \langle u_1 + v_1, u_2 + v_2 \rangle$

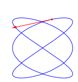
$\vec{u} - \vec{v} = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle$
$= \langle u_1 - v_1, u_2 - v_2 \rangle$

$\lambda \vec{u} = \lambda \langle u_1, u_2 \rangle$
$= \langle \lambda u_1, \lambda u_2 \rangle$

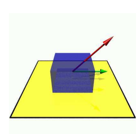
BASIS VECTORS. The vectors $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ are called **standard basis vectors** in the plane. In space, one has the basis vectors $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$.
Every vector $\vec{v} = (v_1, v_2)$ in the plane can be written as a sum of standard basis vectors: $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$. Every vector $\vec{v} = (v_1, v_2, v_3)$ in space can be written as $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$.

WHERE DO VECTORS OCCUR? Here are some examples:

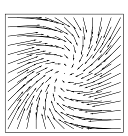
Velocity: if $(f(t), g(t))$ is a point in the plane which depends on time t, then $\vec{v} = \langle f'(t), g'(t) \rangle$ is the **velocity vector** at the point $(f(t), g(t))$.



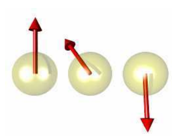
Forces: Some problems in statics involve the determination of a forces acting on objects. Forces are represented as vectors



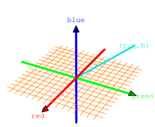
Fields: fields like electromagnetic or gravitational fields or velocity fields in fluids are described with vectors.



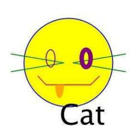
Qbits: in quantum computation, one does not work with bits, but with **qbits**, which are vectors.



Color can be written as a vector $\vec{v} = (r, g, b)$, where r is red, g is green and b is blue. An other coordinate system is $\vec{v} = (c, m, y) = (1 - r, 1 - g, 1 - b)$, where c is cyan, m is magenta and y is yellow.



SVG. Scalable Vector Graphics is an emerging standard for the web for describing two-dimensional graphics in XML.



VECTOR OPERATIONS: The addition and scalar multiplication of vectors satisfy "obvious" properties. There is no need to memorize them. We write $*$ here for multiplication with a scalar but usually, the multiplication sign is left out.

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$	commutativity
$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$	additive associativity
$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{0}$	null vector
$r * (s * \vec{v}) = (r * s) * \vec{v}$	scalar associativity
$(r + s)\vec{v} = \vec{v}(r + s)$	distributivity in scalar
$r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$	distributivity in vector
$1 * \vec{v} = \vec{v}$	the one element

LENGTH. The length $|\vec{v}|$ of $\vec{v}$ is the distance from the beginning to the end of the vector.

EXAMPLES. 1) If $\vec{v} = (3, 4)$, then $|\vec{v}| = \sqrt{25} = 5$. 2) $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$, $|\vec{0}| = 0$.

UNIT VECTOR. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is a unit vector.
EXAMPLE: If $\vec{v} = (3, 4)$, then $\vec{v} = (2/5, 3/5)$ is a unit vector, $\vec{i}, \vec{j}, \vec{k}$ are unit vectors.

PARALLEL VECTORS. Two vectors $\vec{v}$ and $\vec{w}$ are called **parallel**, if $\vec{v} = r\vec{w}$ with some constant r.

DOT PRODUCT. The **dot product** of two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is defined as

$$\boxed{\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3}$$

Remark: in science, other notations are used: $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$ (mathematics) $< \vec{v} | \vec{w} >$ (quantum mechanics) $v_i w^i$ (Einstein notation) $g_{ij} v^i w^j$ (general relativity). The dot product is also called **scalar product**, or **inner product**.

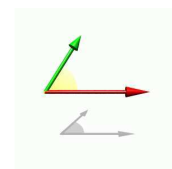
LENGTH. Using the dot product one can express the length of $\vec{v}$ as $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$.

CHALLENGE. Express the dot product in terms of the length alone.

SOLUTION: $(\vec{v} + \vec{w}, \vec{v} + \vec{w}) = (\vec{v}, \vec{v}) + (\vec{w}, \vec{w}) + 2(\vec{v}, \vec{w})$ can be solved for $(\vec{v}, \vec{w})$.

ANGLE. Because $|\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}, \vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v}, \vec{w})$ is by the **cos-theorem** equal to $|\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha)$, where α is the angle between the vectors $\vec{v}$ and $\vec{w}$, we get the important formula

$$\boxed{\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)}$$

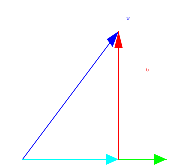


CAUCHY-SCHWARZ INEQUALITY: $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$ follows from that formula because $|\cos(\alpha)| \leq 1$.
TRIANGLE INEQUALITY: $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ follows from $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$.

FINDING ANGLES BETWEEN VECTORS. Find the angle between the vectors $(1, 4, 3)$ and $(-1, 2, 3)$.
ANSWER: $\cos(\alpha) = 16/(\sqrt{26}\sqrt{14}) \sim 0.839$. So that $\alpha = \arccos(0.839..) \sim 33°$.

ORTHOGONAL VECTORS. Two vectors are called **orthogonal** if $v \cdot w = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. EXAMPLE: $\vec{v} = (2, 3)$ is orthogonal to $\vec{w} = (-3, 2)$.

PROJECTION. The vector $\boxed{\vec{a} = \mathrm{proj}_{\vec{w}}(\vec{v}) = \vec{w}(\vec{v} \cdot \vec{w}/|\vec{w}|^2)}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$.
The **scalar projection** is defined as $\boxed{\mathrm{comp}_{\vec{w}(\vec{v})} = (\vec{v} \cdot \vec{w})/|\vec{w}|}$. (Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$.) The vector $\vec{b} = \vec{v} - \vec{a}$ is called the **component** of $\vec{v}$ orthogonal to the $\vec{w}$-direction.

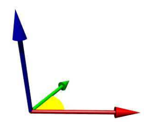


EXAMPLE. $\vec{v} = (0, -1, 1)$, $\vec{w} = (1, -1, 0)$, $\mathrm{proj}_{\vec{w}}(\vec{v}) = (1/2, -1/2, 0)$, $\mathrm{comp}_{\vec{w}}(\vec{v}) = 1/\sqrt{2}$.

Lecture 2: CROSS PRODUCT — **Math21a, O. Knill**

HOMEWORK: section 10.2: 28,42, section 10.3: 12,20

CROSS PRODUCT. The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ is defined as the vector $\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$.



To compute it: multiply diagonally at the crosses.

$$\begin{array}{ccccccccc} v_1 & & v_2 & & v_3 & & v_1 & & v_2 \\ & & & X & & X & & X & \\ w_1 & & w_2 & & w_3 & & w_1 & & w_2 \end{array}$$

DIRECTION OF $\vec{v} \times \vec{w}$: $\boxed{\vec{v} \times \vec{w} \text{ is orthogonal to } \vec{v} \text{ and orthogonal to } \vec{w}.}$

Proof. Check that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$.

LENGTH: $\boxed{|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)}$

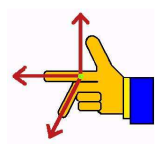
Proof. The identity $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ can be proven by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}|\cos(\alpha)$.

AREA. The length $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

Proof. Because $|\vec{w}|\sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$, the area is $|\vec{v}||\vec{w}|\sin(\alpha)$ which is by the above formula equal to $|\vec{v} \times \vec{w}|$.
EXAMPLE. If $\vec{v} = (a, 0, 0)$ and $\vec{w} = (b\cos(\alpha), b\sin(\alpha), 0)$, then $\vec{v} \times \vec{w} = (0, 0, ab\sin(\alpha))$ which has length $|ab\sin(\alpha)|$.

ZERO CROSS PRODUCT. We see that $\vec{v} \times \vec{w}$ is zero if $\vec{v}$ and $\vec{w}$ are **parallel**.

ORIENTATION. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. The right hand rule is: put the first vector $\vec{v}$ on the thumb, the second vector $\vec{w}$ on the pointing finger and the third vector $\vec{v} \times \vec{w}$ on the third middle finger.

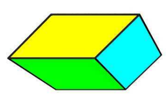


EXAMPLE. $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ forms a right handed coordinate system.

DOT PRODUCT (is a scalar)		CROSS PRODUCT (is a vector)	
$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$	commutative	$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$	anti-commutative
$\lvert\vec{v} \cdot \vec{w}\rvert = \lvert\vec{v}\rvert\lvert\vec{w}\rvert\cos(\alpha)$	angle	$\lvert\vec{v} \times \vec{w}\rvert = \lvert\vec{v}\rvert\lvert\vec{w}\rvert\sin(\alpha)$	angle
$(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$	linearity	$(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})$	linearity
$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$	distributivity	$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$	distributivity
$\{1,2,3\}.\{3,4,5\}$	in Mathematica	Cross[$\{1,2,3\},\{3,4,5\}$]	in Mathematica
$\frac{d}{dt}(\vec{v} \cdot \vec{w}) = \dot{v} \cdot \vec{w} + \vec{v} \cdot \dot{w}$	product rule	$\frac{d}{dt}(\vec{v} \times \vec{w}) = \dot{\vec{v}} \times \vec{w} + \vec{v} \times \dot{\vec{w}}$	product rule

TRIPLE SCALAR PRODUCT. The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$.

PARALLELEPIPED. $[\vec{u}, \vec{v}, \vec{w}]$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ because $h = \vec{u} \cdot \vec{n}/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram which has area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = \vec{u} \cdot \vec{n}|\vec{n}|/|\vec{n}| = |\vec{u} \cdot (\vec{v} \times \vec{w})|$.

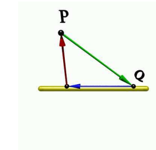


EXAMPLE. Find the volume of the parallel epiped which has the one corner $O = (1, 1, 0)$ and three corners $P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$ connected to it.
ANSWER: The parallelepiped is spanned by $\vec{u} = (1, 2, 1)$, $\vec{v} = (3, 2, 1)$, and $\vec{w} = (0, 3, 2)$. We get $\vec{v} \times \vec{w} = (1, -6, 9)$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = -2$. The volume is 2.

DISTANCE POINT-LINE (3D). If P is a point in space and L is the line which contains the vector $\vec{u}$, then

$$d(P, L) = |\vec{PQ} \times \vec{u}|/|\vec{u}|$$

is the distance between P and the line L.



PLANE THROUGH 3 POINTS P, Q, R:
The vector $\vec{n} = \vec{PQ} \times \vec{PR}$ is orthogonal to the plane. We will see next week that $\vec{n} = (a, b, c)$ defines the plane $ax + by + cz = d$, with $d = ax_0 + by_0 + cz_0$ which passes through the points $P = (x_0, y_0, z_0), Q, R$.

The cross product appears in many different applications:

ANGULAR MOMENTUM. If a mass point of mass m moves along a curve $\vec{r}(t)$, then the vector $\vec{L}(t) = m\vec{r}(t) \times \vec{r}\,'(t)$ is called the **angular momentum** of the point. It is coordinate system dependent.

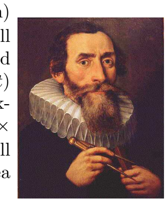
ANGULAR MOMENTUM CONSERVATION.

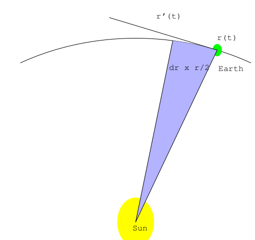
$$\frac{d}{dt}\vec{L}(t) = m\vec{r}'(t) \times \vec{r}'(t) + m\vec{r}(t) \times \vec{r}''(t) = \vec{r}(t) \times \vec{F}(t)$$

In a central field, where $\vec{F}(t)$ is parallel to $\vec{r}(t)$, we get $d/dtL(t) = 0$ which means $L(t)$ is constant.

TORQUE. The quantity $\vec{r}(t) \times \vec{F}(t)$ is also called the **torque**. The time derivative of the **momentum** $p = m\vec{r}'$ is the **force** F. the time derivative of the **angular momentum** $\vec{L} = \vec{r}(t) \times \vec{p}(t) = m\vec{r}(t) \times \vec{r}'(t)$ is the **torque**.

KEPLER'S AREA LAW. (Proof by Newton) The fact that $\vec{L}(t)$ is constant means first of all that $\vec{r}(t)$ stays in a plane spanned by $\vec{r}(0)$ and $\vec{r}'(0)$. The experimental fact that the vector $\vec{r}(t)$ sweeps over **equal areas in equal times** expresses angular momentum conservation: $|\vec{r}(t) \times \vec{r}'(t)dt/2| = |\vec{L}dt/m/2|$ is the area of a small triangle. The vector $\vec{r}(t)$ sweeps over an area $\int_0^T |\vec{L}|dt/(2m) = |\vec{L}|T/(2m)$ in time $[0, T]$.

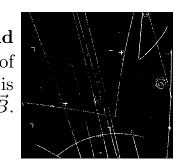




MORE PLACES IN PHYSICS WHERE THE CROSS PRODUCT OCCURS:

The **top**, the motion of a rigid body is describe by the angular momentum L and the angular velocity vector Ω in the body. Then $\dot{L} = L \times \Omega + M$, where M is an external **torque** obtained by external forces.

Electromagnetism: (informal) a particle moving along $\vec{r}(t)$ in a **magnetic field** $\vec{B}$ for example experiences the force $\vec{F}(t) = q\vec{r}\,'(t) \times \vec{B}$, where q is the charge of the particle. In a constant magnetic field, the particles move on circles: if m is the mass of the particle, then $m\vec{r}\,''(t) = q\vec{r}\,'(t) \times \vec{B}$ implies $m\vec{r}\,'(t) = q\vec{r}\,(t) \times \vec{B}$. Now $d/dt|\vec{r}|^2 = 2\vec{r} \cdot \vec{r}\,' = \vec{r} \cdot q\vec{r}\,(t) \times \vec{B} = 0$ so that $|\vec{r}|$ is constant.



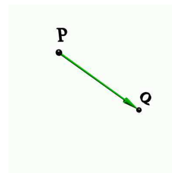
Hurricanes are powerful storms with wind velocities of 74 miles per hour or more. On the northern hemisphere, hurricanes turn counterclockwise, on the southern hemisphere clockwise. This is a feature of all low pressure systems and can be explained by the Coriolis force. In a rotating coordinate system a particle of mass m moving along $\vec{r}(t)$ experience the following forces: $m\vec{\omega}' \times \vec{r}$ (inertia of rotation), $2m\vec{\omega} \times \vec{r}'$ (Coriolis force) and $m\omega \times (\vec{\omega} \times \vec{r})$ (Centrifugal force). The Coriolis force is also responsible for the circulation in Jupiter's Red Spot.

Lecture 3. DISTANCES — Math21a, O. Knill

DISTANCE POINT-POINT (3D). If P and Q are two points, then

$$d(P,Q) = |\vec{PQ}|$$

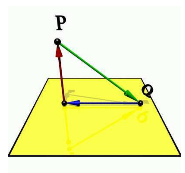
is the distance between P and Q.



DISTANCE POINT-PLANE (3D). If P is a point in space and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P,\Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$$

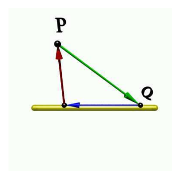
is the distance between P and the plane.



DISTANCE POINT-LINE (3D). If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P,L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$$

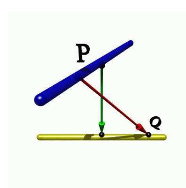
is the distance between P and the line L.



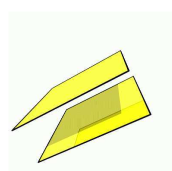
DISTANCE LINE-LINE (3D). L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$$

is the distance between the two lines L and M.



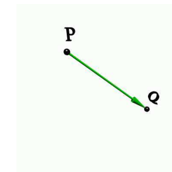
DISTANCE PLANE-PLANE (3D). If $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$ are two parallel planes, then their distance is $(e-d)/|\vec{n}|$. Nonparallel planes have distance 0.



EXAMPLES

DISTANCE POINT-POINT (3D). $P = (-5,2,4)$ and $Q = (-2,2,0)$ are two points, then

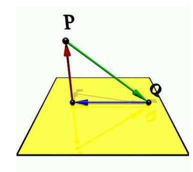
$$d(P,Q) = |\vec{PQ}| = \sqrt{(-5+2)^2 + (2-2)^2 + (0-4)^2} = 5$$



DISTANCE POINT-PLANE (3D). $P = (7,1,4)$ is a point and $\Sigma : 2x+4y+5z = 9$ is a plane which contains the point $Q = (0,1,1)$. Then

$$d(P,\Sigma) = |(7,0,3) \cdot (2,4,5)|/|\sqrt{45}| = 29/\sqrt{45}$$

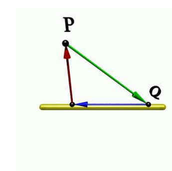
is the distance between P and Σ.



DISTANCE POINT-LINE (3D). $P = (2,3,1)$ is a point in space and L is the line $\vec{r}(t) = (1,1,2) + t(5,0,1)$. Then

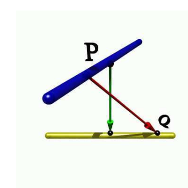
$$d(P,L) = |(1,2,-1) \times (5,0,1)|/\sqrt{26} = |(2,-6,-10)|/\sqrt{26} = \sqrt{140}/\sqrt{26}$$

is the distance between P and L.

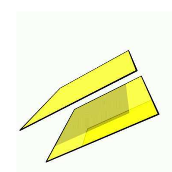


DISTANCE LINE-LINE (3D). L is the line $\vec{r}(t) = (2,1,4) + t(-1,1,0)$ and M is the line $\vec{s}(t) = (-1,0,2) + t(5,1,2)$. The cross product of $(-1,1,0)$ and $(5,1,2)$ is $(2,2,-6)$. The distance between these two lines is

$$d(L,M) = |(3,1,2) \cdot (2,2,-6)|/\sqrt{44} = 4/\sqrt{44} \ .$$



DISTANCE PLANE-PLANE (3D). $5x+4y+3z = 8$ and $5x+4y+3z = 1$ are two parallel planes. Their distance is $7/\sqrt{50}$.



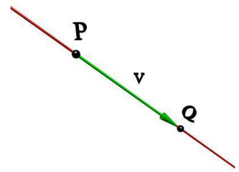
LINES and PLANES **Math 21a, O. Knill**

HOMEWORK: Section 10.3: 18,42, section 10.3: 30,32.

PROBLEM SESSIONS: Tuesdays 3-4pm SC 411

LINES. A point P and a vector $\vec{v}$ define a line L. It is the set of points

$$\boxed{L = \{P + t\vec{v}, \text{ where } t \text{ is a real number }\}}$$



The line contains the point P and points into the direction $\vec{v}$.

EXAMPLE. $L = \{(x, y, z) = (1, 1, 2) + t(2, 4, 6)\ \}$.

This description is called the **parametric equation** for the line.

EQUATIONS OF LINE. We can write $(x, y, z) = (1, 1, 2) + t(2, 4, 6)$ so that $x = 1 + 2t, y = 1 + 4t, z = 2 + 6t$. If we solve the first equation for t and plug it into the other equations, we get $y = 1 + (2x - 2), z = 2 + 3(2x - 2)$. We can therefore describe the line also as

$$L = \{(x, y, z) \mid y = 2x - 1, z = 6x - 4\ \}$$

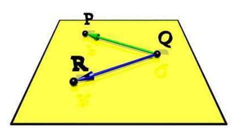
SYMMETRIC EQUATION. The line $\vec{r} = P + t\vec{v}$ with $P = (x_0, y_0, z_0)$ and $\vec{v} = (a, b, c)$ satisfies the **symmetric equations** $\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$ (every expression is equal to t).

PROBLEM. Find the equations for the line through the points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$.

SOLUTION. The parametric equations are $(x, y, z) = (0, 1, 1) + t(2, 2, 3)$ or $x = 2t, y = 1 + 2t, z = 1 + 3t$. Solving each equation for t gives the symmetric equations $x/2 = (y - 1)/2 = (z - 1)/3$.

PLANES. A point P and two vectors $\vec{v}, \vec{w}$ define a plane Σ. It is the set of points

$$\boxed{\Sigma = \{P + t\vec{v} + s\vec{w}, \text{ where } t, s \text{ are real numbers }\}}$$



The line contains the point P.

EXAMPLE. $\Sigma = \{(x, y, z) \mid (1, 1, 2) + t(2, 4, 6) + s(1, 0, -1)\ \}$.

This is called the **parametric description** of a plane.

EQUATION OF PLANE. Given a plane as a parametric equation $P = Q + t\vec{v} + s\vec{w}$. The vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{n}$ and $\vec{w}$. Because also the vector $PQ = Q - P$ is perpendicular to $\vec{n}$ we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = (a, b, c)$, this means $ax + by + cz = ax_0 by_0 + cz_0 = d$. The plane is therefore described by a single equation $\boxed{ax + by + cz = d}$.

PROBLEM. Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.

SOLUTION. The plane contains the two vectors $\vec{v} = (1, 2, 0)$ and $\vec{w} = (2, 2, 2)$. We have $\vec{n} = (4, -2, -2)$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in one point: $\boxed{4x - 2y - 2z = -4}$.

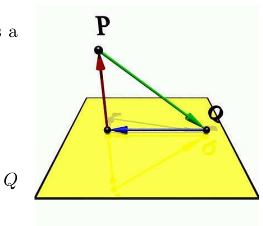
LINES AND PLANES IN MATHEMATICA.
Plotting a line: $\boxed{\mathrm{ParametricPlot3D}[\{1,1,1\} + \mathrm{t}\{3,4,5\}, \{\mathrm{t}, -2, 2\}]}$
Plotting a plane: $\boxed{\mathrm{ParametricPlot3D}[\{1,1,1\} + \mathrm{t}\{3,4,5\} + \mathrm{s}\{1,2,3\}, \{\mathrm{t}, -2, 2\}, \{\mathrm{s}, -2, 2\}]}$
Finding the equation of a plane
$\boxed{\mathrm{P} = \{-1,-1,1\}; \mathrm{Q} = \{0,1,1\}; \mathrm{R} = \{1,1,3\}; \mathrm{n} = \mathrm{Cross}[\mathrm{Q} - \mathrm{P}, \mathrm{R} - \mathrm{P}]; \mathrm{n}.\{\mathrm{x},\mathrm{y},\mathrm{z}\} - \mathrm{n.P}}$

DISTANCE POINT-PLANE (3D). If P is a point in space and $\vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P, L) = |(P - Q) \cdot \vec{n}|/|\vec{n}|$$

is the distance between P and the plane.

You recognize that this is just the scalar projection of the vector $\vec{QP} = P - Q$ onto the vector $\vec{n}$.

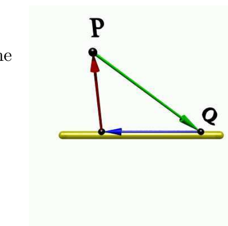


DISTANCE POINT-LINE (3D). If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = |(P - Q) \times \vec{u}|/|\vec{u}|$$

is the distance between P and the line L.

This formula is verified by writing $(P - Q) \times \vec{u} = |P - Q||\vec{u}| \sin(\theta)$.

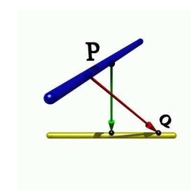


DISTANCE LINE-LINE (3D). L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L, M) = |(P - Q) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$$

is the distance between the two lines L and M.

This formula is just the scalar projection of $\vec{QP} = P - Q$ onto the vector $\vec{n} = \vec{u} \times \vec{v}$ normal to both $\vec{u}$ and $\vec{v}$.



PLANE THROUGH 3 POINTS P, Q, R: The vector $(a, b, c) = \vec{n} = (Q - P) \times (R - P)$ is normal to the plane. Therefore, the equation is $ax + by + cz = d$. The constant is $d = ax_0 + by_0 + cz_0$ because the point $P = (x_0, y_0, z_0)$ is on the plane.

PLANE THROUGH POINT P AND LINE $\vec{r}(t) = Q + t\vec{u}$. The vector $(a, b, c) = \vec{n} = \vec{u} \times (Q \vec{-} P)$ is normal to the plane. Therefore the plane is given by $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$ and $P = (x_0, y_0, z_0)$.

LINE ORTHOGONAL TO PLANE ax+by+cz=d THROUGH POINT P. The vector $\vec{n} = (a, b, c)$ is normal to the plane. The line is $\vec{r}(t) = P + \vec{n}t$.

ANGLE BETWEEN PLANES. The angle between the two planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is $\arccos(\vec{n}_1 \cdot \vec{n}_2/(|\vec{n}_1||\vec{n}_2|)$, where $\vec{n}_i = (a_i, b_i, c_i)$. Alternatively, it is $\arcsin(|\vec{n}_1 \times \vec{n}_2|/(|\vec{n}_1||\vec{n}_2|))$.

INTERSECTION BETWEEN TWO PLANES. Find the line which is the intersection of two non-parallel planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$. Find first a point P which is in the intersection. Then $\vec{r}(t) = P + t(\vec{n}_1 \times \vec{n}_2)$ is the line, we were looking for.

LINES IN THE PLANE.

LINE $P = Q + t\vec{v}$. Eliminating t gives a single equation. For example, $(x, y) = (1, 2) + t(3, 4)$ is equivalent to $x = 1 + 3t, y = 2 + 4t$ and $4x - 3y = -2$. The general equation for a line in the plane is $\boxed{ax + by = d}$.

Lecture 4 2/18/2004, CURVES — Math21a

HOMEWORK FOR WEDNESDAY: Section 10.5: 4,10,26,42 (can assume 4.1)

PARAMETRIC PLANE CURVES. If $x(t)$, $y(t)$ are functions of one variable, defined on the **parameter interval** $I = [a, b]$, then $\vec{r}(t) = \langle f(t), g(t)\rangle$ is a **parametric curve** in the plane. The functions $x(t), y(t)$ are called **coordinate functions**.

PARAMETRIC SPACE CURVES. If $x(t), y(t), z(t)$ are functions, then $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a **space curve**. Always think of the **parameter** t as **time**. For every fixed t, we have a point $(x(t), y(t), z(t))$ in space. As t varies, we move along the curve.

EXAMPLE 1. If $x(t) = t$, $y(t) = t^2 + 1$, we can write $y(x) = x^2 + 1$ and the curve is a **graph**.

EXAMPLE 2. If $x(t) = \cos(t), y(t) = \sin(t)$, then $\vec{r}(t)$ follows a **circle**.

EXAMPLE 3. If $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$, then $\vec{r}(t)$ describes a **spiral**.

EXAMPLE 4. If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in example 3 but we traverse it **faster**. The **parameterization** changed.

EXAMPLE 5. If $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$, then we have the same curve as in example 3 but we traverse it in the **opposite direction**.

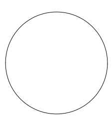
EXAMPLE 6. If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = \langle a+t(u-a), b+t(v-b), c+t(w-c)\rangle$ defined on $t \in [0, 1]$ is a **line segment** connecting P with Q.

ELIMINATION: Sometimes it is possible to eliminate the parameter t and write the curve using equations (one equation in the plane or two equations in space).

EXAMPLE: (circle) If $x(t) = \cos(t), y(t) = \sin(t)$, then $x(t)^2 + y(t)^2 = 1$.

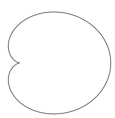
EXAMPLE: (spiral) If $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$, then $x = \cos(z), y = \sin(z)$. The spiral is the intersection of two graphs $x = \cos(z)$ and $y = \sin(z)$.

CIRCLE



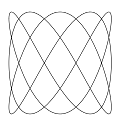
$(\cos(t), 3\sin(t))$

HEART



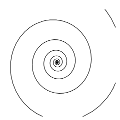
$(1 + \cos(t))(\cos(t), \sin(t))$

LISSAJOUS



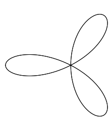
$(\cos(3t), \sin(5t))$

SPIRAL



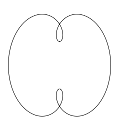
$e^{t/10}(\cos(t), \sin(t))$

TRIFOLIUM



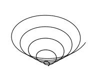
$-\cos(3t)(\cos(t), \sin(t))$

EPICYCLE



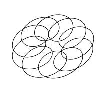
$(\cos(t) + \cos(3t)/2, \sin(t) + \sin(3t)/2)$

SPRING (3D)



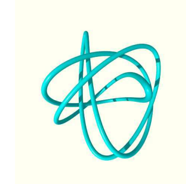
$(\cos(t), \sin(t), t)$

TORAL KNOT (3D)



$(\cos(t) + \cos(9t)/2, \sin(t) + \cos(9t)/2, \sin(9t)/2)$

WHERE DO CURVES APPEAR? Objects like particles, celestial bodies, or quantities change in time. Their motion is described by curves. Examples are the motion of a star moving in a galaxy, or data changing in time like (DJIA(t),NASDAQ(t),SP500(t))



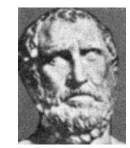
Strings or knots are closed curves in space.
Molecules like RNA or proteins can be modeled as curves.
Computer graphics: surfaces are represented by mesh of curves.
Typography: fonts represented by Bezier curves.
Space time A curve in space-time describes the motion of particles.
Topology Examples: space filling curves, boundaries of surfaces or knots.

DERIVATIVES. If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a curve, then $\boxed{\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle \dot{x}, \dot{y}, \dot{z}\rangle}$ is called the **velocity**. Its length $|\vec{r}\,'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**.

$\boxed{\text{The velocity vector } \vec{r}(t) \text{ is tangent to the curve at } \vec{r}(t).}$

EXAMPLE. If $\vec{r}(t) = \langle \cos(3t), \sin(2t), 2\sin(t)\rangle$, then $\vec{r}\,'(t) = \langle -3\sin(3t), 2\cos(2t), 2\cos(t)\rangle$, $\vec{r}\,''(t) = \langle -9\cos(3t), -4\sin(2t), -2\sin(t)\rangle$ and $\vec{r}\,'''(t) = \langle 27\sin(3t), 8\cos(2t), -2\cos(t)\rangle$.

WHAT IS MOTION? **The paradoxon of Zeno of Elea**: "When looking at a body at a specific time, the body is fixed. Being fixed at each instant, there is no motion". While one might wonder today a bit about Zeno's thoughts, there were philosophers like Kant, Hume or Hegel, who thought seriously about Zeno's challenges. Physicists continue to ponder about the question: "what is time and space?" Today, the derivative or rate of change is defined as a **limit** $(\vec{r}(t + dt) - \vec{r}(t))/dt$, where dt approaches zero. If the limit exists, the velocity is defined.



EXAMPLES OF VELOCITIES.

Person walking:	1.5 m/s
Signals in nerves:	40 m/s
Plane:	70-900 m/s
Sound in air:	Mach1=340 m/s
Speed of bullet:	1200-1500 m/s
Earth around the sun:	30'000 m/s
Sun around galaxy center:	200'000 m/s
Light in vacuum:	300'000'000 m/s

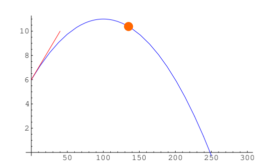
EXAMPLES OF ACCELERATIONS.

Train:	0.1-0.3 m/s^2
Car:	3-8 m/s^2
Space shuttle:	$\leq 3G = 30m/s^2$
Combat plane (F16) (blackout):	9G=90 m/s^2
Ejection from F16:	14G=140 m/s^2.
Free fall:	1G = 9.81 m/s^2
Electron in vacuum tube:	10^{15} m/s^2

DIFFERENTIATION RULES.
The rules in one dimensions $(f + g)' = f' + g'$ $(cf)' = cf'$, $(fg)' = f'g + fg'$ (Leibniz), $(f(g))' = f'(g)g'$ (chain rule) generalize for vector-valued functions: $(\vec{v} + \vec{w})' = \vec{v}\,' + \vec{w}\,'$, $(c\vec{v})' = c\vec{v}\,'$, $(\vec{v} \cdot \vec{w})' = \vec{v}\,' \cdot \vec{w} + \vec{v} \cdot \vec{w}\,'$ $(\vec{v} \times \vec{w})' = \vec{v}\,' \times \vec{w} + \vec{v} \times \vec{w}\,'$ (Leibniz), $(\vec{v}(f(t)))' = \vec{v}\,'(f(t))f'(t)$ (chain rule). The Leibniz rule for the triple dot product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is $d/dt[\vec{u}, \vec{v}, \vec{w}] = [\vec{u}\,', \vec{v}, \vec{w}] + [\vec{u}, \vec{v}\,', \vec{w}] + [\vec{u}, \vec{v}, \vec{w}\,']$ (see homework).

INTEGRATION. If $\vec{r}\,'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}\,'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}\,''(t)$ as well as initial velocity and position $\vec{r}\,'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}\,'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.



EXAMPLE. Shooting a ball. If $\vec{r}\,''(t) = \langle 0, 0, -10\rangle$, $\vec{r}\,'(0) = \langle 0, 1000, 2\rangle, \vec{r}(0) = \langle 0, 0, h\rangle$, then $\vec{r}(t) = \langle 0, 1000t, h + 2t - 10t^2/2\rangle$.

KNOTS **O. Knill, Math 21a**

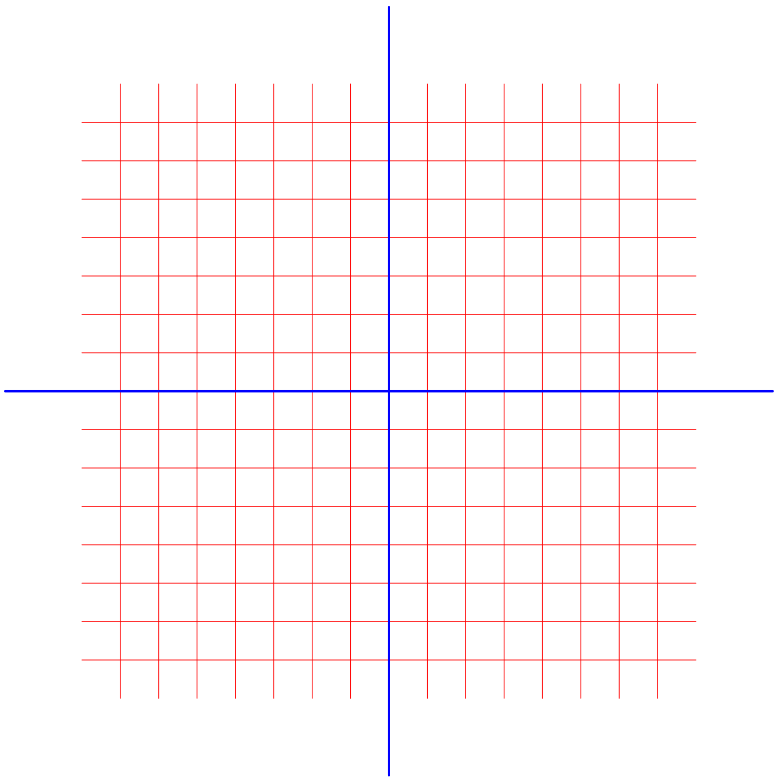
The **threefoil knot** is the space curve

$$\vec{r}(t) = ((2+\cos(3t/2))\cos(t), (2+\cos(3t/2))\sin(t), \sin(3t/2))$$

1) Find an interval $[a,b]$ on which $\vec{r}(t)$ parameterizes a closed curve in space. A closed curve in space is called a **knot**. These objects are not only interesting in mathematics or physics. For example, DNA of bacteria and some proteins form knots (see the back of the page).

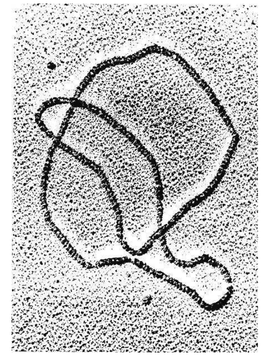
2) Verify that the projection $\vec{r}(t) = \langle x(t), y(t)\rangle$ of this curve onto the xy-plane is in polar coordinates by $\vec{r}(t) = 2+\cos(3t/2), \theta(t) = t$.

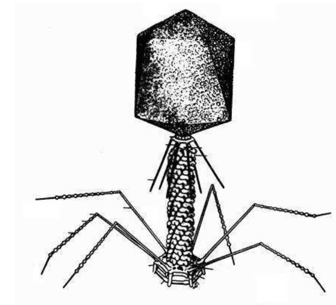
3) Sketch this curve.

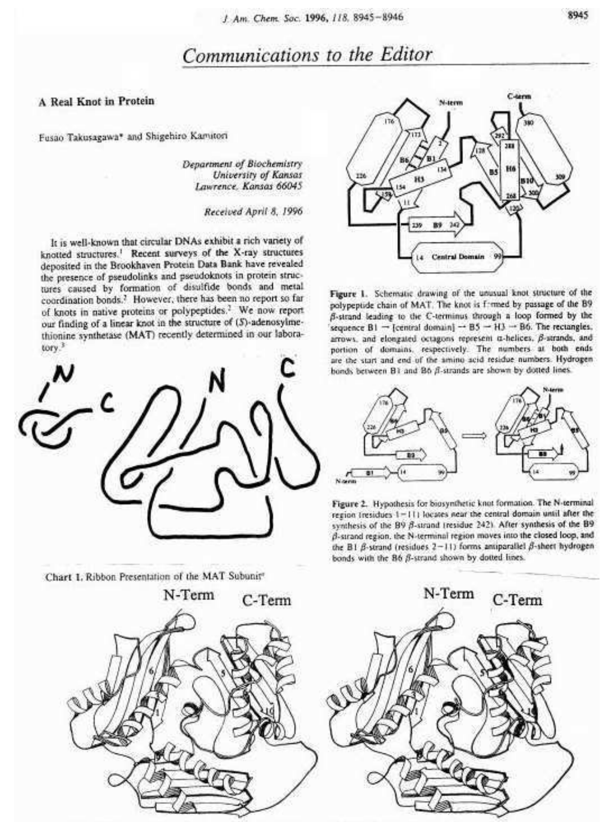


Indicate at the crossings, which part of the curve is above the other.

4) Calculate the velocity vector $\vec{r}'(t)$ of $\vec{r}(t)$ at $t = 0$. If this vector is $\vec{v} = \langle a,b,c\rangle$, then (a,b) is its projection on the xy plane. Draw this vector $\langle a,b\rangle$ in the above xy-projection of the knot.







J. Am. Chem. Soc. **1996,** *118,* 8945–8946 8945

Communications to the Editor

A Real Knot in Protein

Fusao Takusagawa* and Shigehiro Kamitori

Department of Biochemistry
University of Kansas
Lawrence, Kansas 66045

Received April 8, 1996

It is well-known that circular DNAs exhibit a rich variety of knotted structures.[1] Recent surveys of the X-ray structures deposited in the Brookhaven Protein Data Bank have revealed the presence of pseudolinks and pseudoknots in protein structures caused by formation of disulfide bonds and metal coordination bonds.[2] However, there has been no report so far of knots in native proteins or polypeptides.[2] We now report our finding of a linear knot in the structure of (*S*)-adenosylmethionine synthetase (MAT) recently determined in our laboratory.[3]

Figure 1. Schematic drawing of the unusual knot structure of the polypeptide chain of MAT. The knot is formed by passage of the B9 β-strand leading to the C-terminus through a loop formed by the sequence B1 → [central domain] → B5 → H3 → B6. The rectangles, arrows, and elongated octagons represent α-helices, β-strands, and portion of domains, respectively. The numbers at both ends are the start and end of the amino acid residue numbers. Hydrogen bonds between B1 and B6 β-strands are shown by dotted lines.

Figure 2. Hypothesis for biosynthetic knot formation. The N-terminal region (residues 1−11) locates near the central domain until after the synthesis of the B9 β-strand (residue 242). After synthesis of the B9 β-strand region, the N-terminal region moves into the closed loop, and the B1 β-strand (residues 2−11) forms antiparallel β-sheet hydrogen bonds with the B6 β-strand shown by dotted lines.

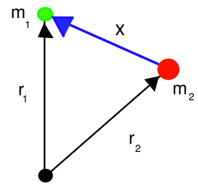
Chart 1. Ribbon Presentation of the MAT Subunit[a]

N-Term C-Term N-Term C-Term

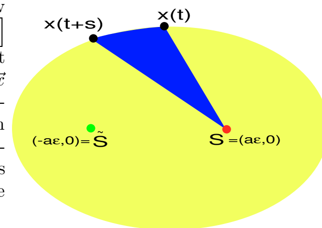
For more information on DNA knots, see the MSRI talk by DeWitt Sumners:
http://www.msri.org/publications/ln/msri/2000/molbio/sumners/1/

KEPLER'S LAWS — Oliver Knill

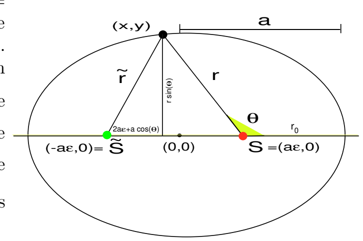
FROM THE 2 BODY TO THE KEPLER PROBLEM. Two bodies with position vectors $\vec{r}_1, \vec{r}_2$ and mass m_1, m_2 attract each other with the force $\vec{F} = -Gm_1m_2\frac{\vec{x}}{r^3}$, where $\vec{x} = \vec{r}_1 - \vec{r}_2$ is the difference vector between the bodies. The length $|\vec{F}|$ of the force is proportional to $1/r^2$. The vector $\vec{x} = \vec{r}_1 - \vec{r}_2$ of length r satisfies $\vec{x}'' = \vec{r}_1'' - \vec{r}_2'' = m_2G(\vec{r}_2 - \vec{r}_1)/|\vec{r}_2 - \vec{r}_1|^3 - m_1G(\vec{r}_1 - \vec{r}_2)/|\vec{r}_2 - \vec{r}_1|^3 = -(m_1 + m_2)G\vec{x}/r^3$. The new problem describes a mass-point with position $\vec{x}$ and mass $m = m_1 + m_2$ which is attracted to a fixed force centered at the origin. This problem $\boxed{\vec{x}'' = -mG\vec{x}/r^3}$ is called the **Kepler problem**. The **angular momentum** $\vec{L} = m\vec{x} \times \vec{x}'$ and the **energy** $E = m|\vec{x}'|^2/2 + mG/|\vec{x}|$ do not change in time.



THE 2. KEPLER LAW. The "area law" or Keplers second law is: $\boxed{\text{"The radius vector } \vec{x} \text{ passes the equal area in equal time."}}$
Proof. Because $\vec{x}''$ is parallel to $\vec{x}$, and $L = m\vec{x}' \times \vec{x}$, we get $\vec{L}' = 0$ (use the product law). If follows that the vector $\vec{x}$ stays in the plane spanned by $\vec{x}$ and $\vec{x}'$. We can now use coordinates $\vec{x} = (r\cos(\theta), r\sin(\theta), 0)$ to describe the point. With $\vec{x}' = (r'\cos(\theta), r'\sin(\theta), 0) + (-r\sin(\theta), r\cos(\theta), 0)\theta'$. The conserved quantity $|L| = m|\vec{x}' \times \vec{x}| = mr^2\theta'$ can be interpreted as $2mf'$, where $f(t)$ is the **area** swept over by the vector x in the interval $[0, t]$. We will use the formula $\boxed{|L| = r^2\theta'}$ later.



ELLIPSES. An **ellipse** with **focal points** $\tilde{S} = (-a\epsilon, 0), S = (a\epsilon, 0)$ with **eccentricity** ϵ is defined as the set of points in the plane whose distances $\tilde{r}$ and r to $\tilde{S}$ and S satisfy $\tilde{r} + r = 2a$. From $(2a - r)^2 = \tilde{r}^2 = r^2\sin^2(\theta) + (2a\epsilon + r\cos(\theta))^2$, we obtain $\boxed{r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta)}}$ the **polar form** of the ellipse. One can replace a with the constant $r_0 = (1 - \epsilon)a$, the length of $\vec{r}$ at $\theta = 0$. The polar form becomes then $\boxed{r = \frac{r_0(1+\epsilon)}{1+\epsilon\cos(\theta)}}$. The **semiaxes** of the ellipse have length a and $b = a\sqrt{1+\epsilon^2}$. The **area** is $\boxed{A = \pi ab}$ as we will see later in the course.



THE 1. KEPLER LAW. $\boxed{\text{The radius vector } \vec{x}(t) \text{ describes an ellipse.}}$
Proof. (We show that a particle on an ellipse satisfying the 2. law has the correct acceleration of the Kepler law). From the 2. law, we know $\theta' = L/(mr^2)$. The polar form allows us to find the time derivatives

$$r' = \frac{a(1-\epsilon^2)\sin(\theta)}{(1+\epsilon\cos(\theta))^2}\frac{L}{mr^2} = \frac{L\epsilon\sin(\theta)}{ma(1-\epsilon^2)}$$

of $r(t)$. (In the last step, we replaced r^2 in the denominator with the polar formula.) The second derivative is $\boxed{r'' = \frac{L^2\epsilon\cos(\theta)}{(m^2a(1-\epsilon^2)r^2)}}$ (1).
With the unit vector $\vec{n} = \vec{x}/r$, one has $\vec{x}'' = (\vec{x}'' \cdot \vec{n})\vec{n}$. From $\vec{x} = \vec{n}r$, we get $\vec{x}' = \vec{n}'r + \vec{n}r'$, $\vec{x}'' = \vec{n}''r + 2\vec{n}'r' + \vec{n}r''$. Therefore $\boxed{\vec{x}'' \cdot \vec{n} = \vec{n}'' \cdot \vec{n}r + 2\vec{n}' \cdot \vec{n}r' + r''}$ (2).
From $\vec{n} = (\cos(\theta), \sin(\theta), 0)$, we have $\vec{n}' = \theta'\vec{n}^\perp$ where $\vec{n}^\perp = (-\sin(\theta), \cos(\theta), 0)$. This gives $\vec{n}' \cdot \vec{n}' = (\theta')^2$. Using $\vec{n} \cdot \vec{n} = 1$, we have $\boxed{\vec{n} \cdot \vec{n}' = 0}$ (3). From $\vec{n}'' \cdot \vec{n} + \vec{n}' \cdot \vec{n}' = 0$ and $\vec{n}' \cdot \vec{n}' = (\theta')^2 = L^2/(mr^2)^2$ we obtain $\boxed{\vec{n}'' \cdot \vec{n}r = -\frac{L^2r}{(mr^2)^2} = -\frac{L^2}{m^2}\frac{(1+\epsilon\cos(\theta))}{r^2a(1-\epsilon^2)}}$ (4). Plugging (1),(3),(4) into (2) gives

$$\vec{x}'' \cdot \vec{n} = r'' + \vec{n}'' \cdot \vec{n}r = \frac{L^2\epsilon\cos(\theta)}{m^2a(1-\epsilon^2)r^2} - \frac{L^2}{m^2}\frac{(1+\epsilon\cos(\theta))}{r^2a(1-\epsilon^2)} = \frac{-L^2}{m^2a(1-\epsilon^2)r^2} .$$

Therefore,

$$\vec{x}'' = (\vec{x}'' \cdot \vec{n})\vec{n} = (\vec{x}'' \cdot \vec{n})\frac{\vec{x}}{r} = \frac{-L^2}{m^2a(1-\epsilon^2)}\frac{\vec{x}}{r^3} = -mG\frac{\vec{x}}{r^3} .$$

THE 3. KEPLER LAW. Let T be the **period** of the orbit. It is the time the body needs to go around the ellipse once. (If S is the sun and $r(t)$ the orbit of the earth, then T is one year.) The third Kepler law states that $\boxed{T^2/a^3 = 4\pi^2/(Gm) \text{ is a constant}}$.
Proof. If $f(t)$ is the area swept by the radial vector from time 0 to time t, then $f'(t) = L/(2m)$ implies that the area of the ellipse $A = \pi a^2\sqrt{1-\epsilon^2}$ is equal to $LT/(2m)$. From $T = 2m\pi a^2\sqrt{1-\epsilon^2}/L$, we obtain $T^2/a^3 = 4m^2\pi^2a(1-\epsilon^2)/L^2 = 4\pi^2/(Gm)$.

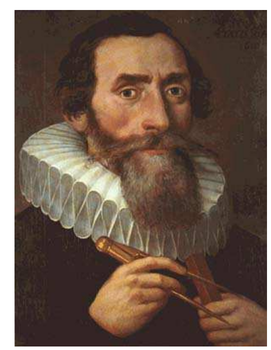
MEASURING THE GRAVITATIONAL CONSTANT. Note that the 3. Kepler law allows us to compute the gravitational constant G from the period, the total mass (essentially the mass of the sun) and the geometry of the ellipse.
The case $\epsilon > 1$ corresponds to a negative G, where particles repell each other. The third Kepler law does then no more apply and ellipses become a hyperbola. The second law is unchanged.

REMARKS.
If the force is changed to $\vec{x}'' = -mG\vec{x}/r^\alpha$, (note that $\alpha = 3$ is the Kepler case), then the second Kepler law still applies, the other two laws not. The formula $\dot{\theta} = L/(mr^2)$ still applies. Also the derivation of the formula for $\vec{x}'' \cdot n = r'' - L^2/(m^2r^3)$ is valid. The left hand side is $-mG/r^{\alpha-1}$, which leads to the ordinary differential equation $r'' = -mG/r^{\alpha-1} + L^2/(m^2r^3)$ for $r(t)$. Knowing $r(t)$ gives then $\theta(t)$ from $\dot{\theta} = L/(mr^2)$. The global behavior depends on the constants G, L. If $\alpha = 4$, which corresponds to the two body problem in 4 dimensions, then $r'' = C/r^3$, where $C = -mG + L^2/m^2$ is a constant. If $C > 0$, the bodies seperate to infinity. If $C < 0$, then $r(t) \to 0$. Only if the angular momentum is such that $C = 0$, there is a bounded motion. **Stable planetary systems would not exist in four dimensions**. A theorem of Bertant states that only for $\alpha = 3$ (the Kepler case) and $\alpha = -1$ (the harmonic oscillator), all bounded orbits are periodic.

COLLISIONS.
If the two podies collide, we get a **collision singularity**. Collisions can occur in the 2-body problem, if the total angular momentum of the two bodies is zero. Analysing collision singularities involving more than two bodies helps to understand what happens when particles are close to such collision configurations. It is known that initial conditions leading to collisions are rare in the n-body problem. Noncollision singularities in which particles escape to infinity in finite time exist already for the 5-body problem.
Our galaxy and M31, the Andromeda galaxy, form a relatively isolated system known as the **local group**. The center of mass of M31 approaches the center of mass our galaxy with a velocity of 119 km/s. In about 10^{10} years, these galaxies are likely to collide. Such a collision will have dramatic consequences for both systems. Nevertheless, even a direct encounter would probably not lead to any collision of any two stars.

SOME HISTORY.

- Nicolas Kopernicus (1473-1543) had a heliocentric system.
- Galileo Galilei (1564-1642) discovers Jupiter moons.
- Johannes Kepler (1571-1630) builds on the observations of Tycho Brahe. Finds the first and second Kepler Law in 1609, the third in 1619.
- Before Newton, the dynamics of celestial objects was described empirically, first by circular, later by epicycle approximations which effectively where Fourier approximations of the actual elliptic motion.
- Issac Newton (1643-1727) led the foundations of mathematical description and developed calculus simultaneously with Leibnitz. Newton had already solved the Kepler problem geometrically.
- Johan Bernoulli proved in 1710 that solutions to the 2 body problem move on conic sections.
- Laplace, Lagrange, Hamilton and Poisson belong to the ancestors of celestial mechanics, the study of the n-body problem.
- With Poincaré (1854-1912) at the end of the 19'th century, and Birkhoff (at Harvard!), the subject of the three body problem was studied with new geometric and topological methods.



Lecture 5: ARC LENGTH — Math21a

HOMEWORK: 10.6: 8,14,20,22

PLANE CURVE

$\vec r(t) = (x(t), y(t))$ **position**
$\vec r\,'(t) = (x'(t), y'(t))$ **velocity**
$|\vec r\,'(t)| = (x'(t), y'(t))$ **speed**
$\vec r\,''(t) = (x''(t), y''(t))$ **acceleration** $\vec r\,'''(t) = (x'''(t), y'''(t))$ **jerk**

SPACE CURVE

$\vec r(t) = (x(t), y(t), z(t))$ **position**
$\vec r\,'(t) = (x'(t), y'(t), z'(t))$ **velocity**
$|\vec r\,'(t)| = (x'(t), y'(t))$ **speed**
$\vec r\,''(t) = (x''(t), y''(t), z''(t))$ **acceleration** $\vec r\,'''(t) = (x'''(t), y'''(t), z'''(t))$ **jerk**

ARC LENGTH. If $t \in [a,b] \mapsto \vec r(t)$ with velocity $\vec v(t) = \vec r\,'(t)$ and speed $|\vec v(t)|$, then $\int_a^b |\vec v(t)|\,dt$ is called the **arc length of the curve**. For space curves this is

$$\boxed{L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt}$$

PARAMETER INDEPENDENCE. $\boxed{\text{The arc length is independent of the parameterization of the curve.}}$

REASON. Changing the parameter is a change of variables (substitution) in the integration.

EXAMPLE. The circle parameterized by $\vec r(t) = (\cos(t^2), \sin(t^2))$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec r\,'(t) = 2t(-\sin(t), \cos(t))$ and speed $2t$. The arc length is $\int_0^{\sqrt{2\pi}} 2t\,dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

REMARK. Often, there is no closed formula for the arc length of a curve. For example, the **Lissajoux figure** $\vec r(t) = (\cos(3t), \sin(5t))$ has the length $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\,dt$. This integral must be evaluated numerically. If you do the Mathematica Lab, you will see how to do that with the computer.

THE MATERIAL BELOW IS NOT PART OF THIS COURSE.

CURVATURE.

$\vec T(t) = \vec r\,'(t)/|\vec r\,'(t)|$ **unit tangent vector**
$\kappa(t) = |\vec T'(t)|/|\vec r\,'(t)|$ **curvature**

CURVATURE FORMULA
($(a,b) \times (c,d) = ad - bc$ in 2D)

$$\boxed{\kappa(t) = \frac{|\vec r\,'(t) \times \vec r\,''(t)|}{|\vec r\,'(t)|^3}}$$

EXAMPLE. CIRCLE

$\vec r(t) = (r\cos(t), r\sin(t))$.
$\vec r\,'(t) = (-r\sin(t), r\cos(t))$.
$|\vec r\,'(t)| = r$.
$\vec T(t) = (-\sin(t), \cos(t))$.
$\vec r\,''(t) = (-r\cos(t), -r\sin(t))$.
$\vec T'(t) = (-\cos(t), -\sin(t))$.
$\kappa(t) = |\vec T'(t)|/|\vec r\,'(t)| = 1/r$.

EXAMPLE. HELIX

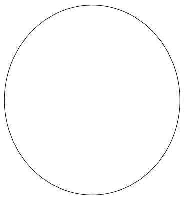
$\vec r(t) = (\cos(t), \sin(t), t)$.
$\vec r\,'(t) = (-\sin(t), \cos(t), 1)$.
$|\vec r\,'(t)| = (-\sin(t), \cos(t), 1) = \sqrt{2}$.
$\vec T(t) = (-\sin(t), \cos(t), 1)/\sqrt{2}$.
$\vec r\,''(t) = (-\cos(t), -\sin(t), 0)$.
$\vec T'(t) = (-\cos(t), \sin(t), 0)/\sqrt{2}$.
$\kappa(t) = |\vec T'(t)|/|\vec r\,'(t)| = 1/2$.

INTERPRETATION.

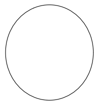
If $s(t) = \int_0^t |\vec r\,'(t)|\,dt$, then $s'(t) = ds/dt = |\vec r\,'(t)|$. Because $\vec T'(t) = d\vec T/dt = d\vec T/ds \cdot ds/dt$, we have $|d\vec T/ds| = |\vec T'(t)|/|\vec r\,'(t)| = \kappa(t)$.

> "The curvature is the length of the acceleration vector if $\vec r(t)$ traces the curve with constant speed 1."

A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.



Small curvature $\kappa = 1/r = 1/2$



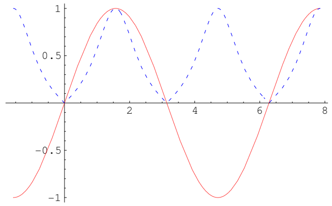
Large curvature $\kappa = 1/r = 2$

CURVATURE OF A GRAPH.

The curve $\vec r(t) = (t, f(t))$, which is the graph of a function f has the velocity $\vec r\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec T(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$ and after some simplification

$$\boxed{\kappa(t) = |\vec T'(t)|/|\vec r\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3}$$

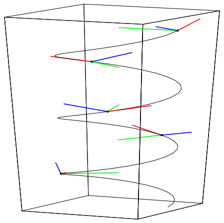
EXAMPLE. $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.



TANGENT/NORMAL/BINORMAL.

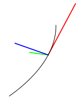
$\vec T(t) = \frac{\vec r\,'(t)}{|\vec r\,'(t)|}$ **tangent vector**
$\vec N(t) = \frac{\vec T'(t)}{|\vec T'(t)|}$ **unit normal vector**
$\vec B(t) = \vec T(t) \times \vec N(t)$ **binormal vector**

Because $\vec T(t) \cdot \vec T(t) = 1$, we get after differentiation $\vec T'(t) \cdot \vec T(t) = 0$ and $\vec N(t)$ is perpendicular to $\vec T(t)$.



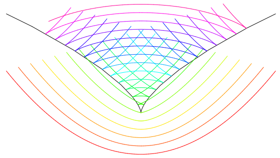
$\boxed{\text{The three vectors } (\vec T(t), \vec N(t), \vec B(t)) \text{ are unit vectors orthogonal to each other.}}$

Note. In order that $(\vec T(t), \vec N(t), \vec B(t))$ exist, we need that $\vec r\,'(t)$ is not zero.



WHERE IS CURVATURE NEEDED?

OPTICS. If a curve $\vec r(t)$ represents a wavefront and $\vec n(t)$ is a unit vector normal to the curve at $\vec r(t)$, then $\vec s(t) = \vec r(t) + \vec n(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call that curve the **evolute** of the original curve.

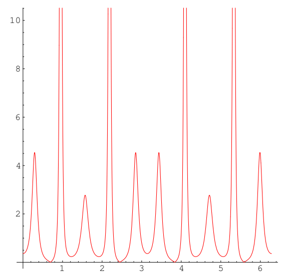


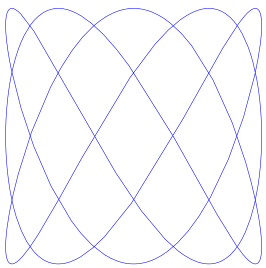
HISTORY.

Aristotle: (350 BC) distinguishes between straight lines, circles and "mixed behavior"
Oresme: (14'th century): measure of twist called "curvitas"
Kepler: (15'th century): circle of curvature.
Huygens: (16'th century): evolutes and involutes in connection with optics.
Newton: (17'th century): circle has constant curvature inversely proportional to radius. (using infinitesimals)
Simpson: (17'th century): string construction of evolutes, description using fluxions.
Euler: (17'th century): first formulas of curvature using second derivatives.
Gauss: (18'th century): modern description, higher dimensional versions.

COMPUTING CURVATURE WITH MATHEMATICA

```
x[t_]:=Cos[3 t];
y[t_]:=Sin[5 t];
r[t_]:={x[t],y[t]};
dr[t_]:=D[r[s],s] /. s->t;
L[{a_,b_}]:=Sqrt[a^2+b^2];
T[t_]:=dr[t]/L[dr[t]];
dT[t_]:=D[T[s],s] /. s->t;
kappa[t_]:=L[dT[t]]/L[dr[t]]
```





Lecture 7: 2/25/2004, FUNCTIONS AND LEVEL CURVES Math21a, O. Knill

Section 11.1: 4(abc) 6(abc),14,16,18,42, Section 11.3: 6 18

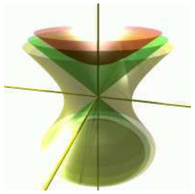
FUNCTIONS, DOMAIN AND RANGE. We deal with functions $f(x,y)$ of two variables defined on a **domain** D. The domain is usually the entire plane like for $f(x,y) = x^2 + \sin(xy)$. But there are cases like in $f(x,y) = 1/\sqrt{1-(x^2+y^2)}$, where the domain is a subset of the plane. The **range** is the set of possible values of f.

LEVEL CURVES
2D: If $f(x,y)$ is a function of two variables, then $f(x,y) = c = const$ is a **curve** or a collection of curves in the plane. It is called **contour curve** or **level curve**. For example, $f(x,y) = 4x^2 + 3y^2 = 1$ is an ellipse. Level curves allow to visualize functions of two variables $f(x,y)$.

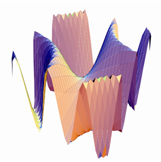
LEVEL SURFACES. We will later see also 3D analogues: if $f(x,y,z)$ is a function of three variables and c is a constant then $f(x,y,z) = c$ is a surface in space. It is called a **contour surface** or a **level surface**. For example if $f(x,y,z) = 4x^2 + 3y^2 + z^2$ then the contour surfaces are ellipsoids.

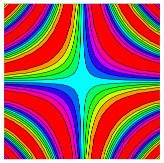
EXAMPLE. Let $f(x,y) = x^2 - y^2$. The set $x^2 - y^2 = 0$ is the union of the sets $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with with their tips at $(-1,0)$ and $(1,0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their tips at $(0,\pm 1)$.

EXAMPLE. Let $f(x,y,z) = x^2 + y^2 - z^2$. $f(x,y,z) = 0, f(x,y,z) = 1, f(x,y,z) = -1$. The set $x^2+y^2-z^2 = 0$ is a **cone** rotational symmetric around the z-axis. The set $x^2+y^2-z^2 = 1$ is a **one-sheeted hyperboloid**, the set $x^2+y^2-z^2 = -1$ is a **two-sheeted hyperboloid**. (To see that it is two-sheeted note that the intersection with $z = c$ is empty for $-1 \leq z \leq 1$.)



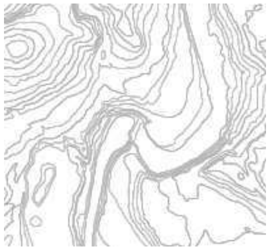
CONTOUR MAP. Drawing several contour curves $\{f(x,y) = c\}$ or several contour surfaces $\{f(x,y,z) = c\}$ produces a **contour map**.



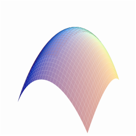


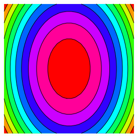
The example shows the graph of the function $f(x,y) = \sin(xy)$. We draw the contour map of f: The curve $\sin(xy) = c$ is $xy = C$, where $C = \arcsin(c)$ is a constant. The curves $y = C/x$ are hyperbolas except for $C = 0$, where $y = 0$ is a line. Also the line $x = 0$ is a contour curve. The contour map is a family of hyperbolas and the coordinate axis.

TOPOGRAPHY. Topographical maps often show the curves of equal height. With the contour curves as information, it is usually already possible to get a good picture of the situation.



EXAMPLE. $f(x,y) = 1 - 2x^2 - y^2$. The contour curves $f(x,y) = 1 - 2x^2 + y^2 = c$ are the ellipses $2x^2 + y^2 = 1 - c$ for $c < 1$.



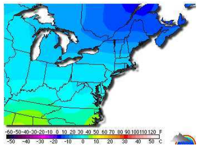


SPECIAL LINES. Level curves are encountered every day:

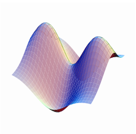
Isobars:	pressure
Isoclines:	direction

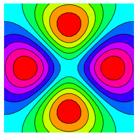
Isothermes:	temperature
Isoheight:	height

For example, the isobars to the right show the lines of constant temperature in the north east of the US.



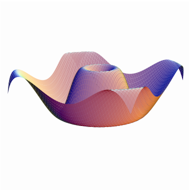
A SADDLE. $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$. We can here no more find explicit formulas for the contour curves $(x^2 - y^2)e^{-x^2-y^2} = c$. Lets try our best:

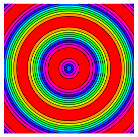




- $f(x,y) = 0$ means $x^2 - y^2 = 0$ so that $x = y, x = -y$ are contour curves.
- On $y = ax$ the function is $g(x) = (1-a^2)x^2 e^{-(1+a^2)x^2}$.
- Because $f(x,y) = f(-x,y) = f(x,-y)$, the function is symmetric with respect to reflections at the x and y axis.

A SOMBRERO. The surface $z = f(x,y) = \sin(\sqrt{x^2+y^2})$ has circles as contour lines.





ABOUT CONTINUITY. In reality, one sometimes has to deal with functions which are not smooth or not continuous: For example, when plotting the temperature of water in relation to pressure and volume, one experiences **phase transitions**, an other example are water waves breaking in the ocean. Mathematicians have also tried to explain "catastrophic" events mathematically with a theory called "catastrophe theory". Discontinuous things are useful (for example in switches), or not so useful (for example, if something breaks).

DEFINITION. A function $f(x,y)$ is **continuous** at (a,b) if $f(a,b)$ is finite and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. The later means that that along any curve $\vec{r}(t)$ with $r(0) = (a,b)$, we have $\lim_{t\to 0} f(\vec{r}(t)) = f(a,b)$. Continuity for functions of more variables is defined in the same way.

EXAMPLE. $f(x,y) = (xy)/(x^2+y^2)$. Because $\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} x^2/(2x^2) = 1/2$ and $\lim_{(x,0)\to(0,0)} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous.

EXAMPLE. $f(x,y) = (x^2 y)/(x^2+y^2)$. In polar coordinates this is $f(r,\theta) = r^3\cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly if $r \to 0$. The function is continuous.

Lecture 8: 2/26/2004, PARTIAL DERIVATIVES — Math21a, O. Knill

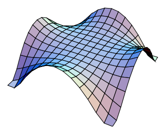
HOMEWORK. 11.3: 6 18 (if not done yet), 40,42,66,70

PARTIAL DERIVATIVE. If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y,z)$ is defined as the derivative of the function $g(x) = f(x,y,z)$, where y is fixed. The partial derivative with respect to y is defined similarly.

NOTATION. One also writes $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$ etc. For iterated derivatives the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$.

REMARK. The partial derivatives $\partial_x f, \partial_y f$ measure the rate of change of the function in the x or y directions. For functions of more variables, the partial derivatives are defined in a similar way.

EXAMPLE. $f(x,y) = x^4 - 6x^2y^2 + y^4$. We have $f_x(x,y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x,y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$. We see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation**.



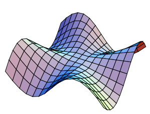
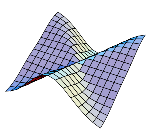
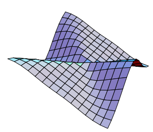
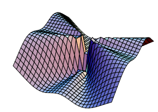
CLAIROT THEOREM. If f_{xy} and f_{yx} are both continuous, then $\boxed{f_{xy} = f_{yx}}$. Proof. Compare the two sides:

$dx f_x(x,y) \sim f(x+dx,y) - f(x,y)$ $dydx f_{xy}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y+dy) - (f(x+dx,y) - f(x,y))$	$dy f_y(x,y) \sim f(x,y+dy) - f(x,y)$. $dxdy f_{yx}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y) - (f(x,y+dy) - f(x,y))$

CONTINUITY IS NECESSARY. Example: $f(x,y) = (x^3y - xy^3)/(x^2+y^2)$ contradicts Clairot:

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,	$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

$\boxed{f(x,y)}$ $\boxed{f_x(x,y)}$ $\boxed{f_y(x,y)}$ $\boxed{f_{xy}(x,y)}$

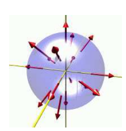





GRADIENT. If $f(x,y,z)$ is a function of three variables, then

$$\nabla f(x,y,z) = \left(\frac{\partial}{\partial x} f(x,y,z), \frac{\partial}{\partial y} f(x,y,z), \frac{\partial}{\partial z} f(x,y,z)\right)$$

is called the **gradient** of f. The symbol ∇ is called **Nabla**. It is named after an Egyptian harp, the Hebrew word "nevel"=harp seems to have the same aramaic origin). We will talk about the gradient in detail next week.

NORMAL. As we will see later, the gradient $\nabla f(x,y)$ is orthogonal to the level curve $f(x,y) = c$ and the gradient $\nabla f(x,y,z)$ is normal to the level surface $f(x,y,z)$. For example, the gradient of $f(x,y,z) = x^2 + y^2 - z^2$ at a point (x,y,z) is $(2x, 2y, -2z)$.



WHY ARE PARTIAL DERIVATIVES IMPORTANT?

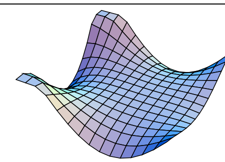
- Geometry. For example, the gradient $\nabla f(x,y,z)$ is a vector normal to a surface at the point (x,y,z). Tangent spaces.
- Approximations, linarizations.
- Partial differential equations: laws describing nature.
- Optimization problems, as we will see later.
- Solution to some integration problems using generalizations of fundamental theorem of calculus.
- In general helpful to understand and analyze functions of several variables.

PARTIAL DIFFERENTIAL EQUATIONS. An equation which involves partial derivatives of an unknown function is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

1) $f_{xx}(x,y) = f_{yy}(x,y)$ is an example of a partial differential equation 1) $f_x(x,y) = f_{xx}(x,y)$ is an example of an ordinary differential equation. The variable y can be considered as a parameter.

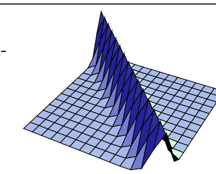
LAPLACE EQUATION

$\boxed{f_{xx} + f_{yy} = 0}$.
$f(x,t) = x^2 - y^2$.



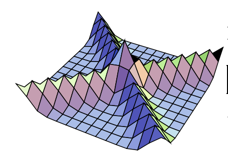
ADVECTION EQUATION

$\boxed{f_t = f_x}$.
$f(x,t) = g(x-t)$.



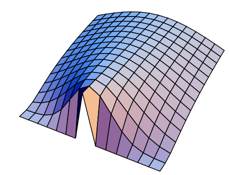
WAVE EQUATION

$\boxed{f_{tt} = f_{xx}}$. $f(t,x) = \sin(x-t) + \sin(x+t)$



HEAT EQUATION

$\boxed{f_t = f_{xx}}$
$f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$



"A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." - Paul A. M. Dirac.

Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpretated the later as an **antiparticle**: the existence of antiparticles was later confirmed.

Lecture 9: 3/1/2004, CHAIN RULE **Math21a, O. Knill**

HOMEWORK. Section 11.4: 8,28,36,38,46. 

REMEMBER? If f and g are functions of one variable t, then $d/dt f(g(t)) = f'(g(t))g'(t)$. For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$.

GRADIENT. Define $\nabla f(x,y) = (f_x(x,y), f_y(x,y))$. It is called the gradient of f.

THE CHAIN RULE. If $\vec{r}(t)$ is curve in space and f is a function of three variables, we get a function of one variables $t \mapsto f(\vec{r}(t))$.
The chain rule looks like the 1D chain rule, but where the derivative f' is replaced with the gradient $\nabla f = (f_x, f_y, f_z)$. $\boxed{d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$

WRITING IT OUT. Writing the dot product out gives

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) \; .$$

EXAMPLE. Let $z = \sin(x + 2y)$, where x and y are functions of t: $x = e^t, y = \cos(t)$. What is $\frac{dz}{dt}$?

Here, $z = f(x,y) = \sin(x+2y)$, $z_x = \cos(x+2y)$, and $z_y = 2\cos(x+2y)$ and $\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = -\sin(t)$ and $\frac{dz}{dt} = \cos(x+2y)e^x - 2\cos(x+2y)\sin(t)$.

EXAMPLE. If f is the temperature distribution in a room and $\vec{r}(t)$ is the path of the spider Shelob, then $f(\vec{r}(t))$ is the temperature, Shelob experiences at time t. The rate of change depends on the velocity $\vec{r}\,'(t)$ of the spider as well as the temperature gradient ∇f and the angle between gradient and velocity. For example, if the spider moves perpendicular to the gradient, its velocity is tangent to a level curve and the rate of change is zero.

EXAMPLE. A nicer spider called "Nabla" moves along a circle $\vec{r}(t) = (\cos(t), \sin(t))$ on a table with temperature distribution $T(x,y) = x^2 - y^3$. Find the rate of change of the temperature, "Nabla" experiences.

SOLUTION. $\nabla T(x,y) = (2x, -3y^2)$, $\vec{r}\,'(t) = (-\sin(t), \cos(t))$ $d/dt T(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}\,'(t) = (2\cos(t), -3\sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.

APPLICATION ENERGY CONSERVATION. If $H(x,y)$ is the energy of a system, the system moves satisfies the equations $\boxed{x'(t) = H_y, y'(t) = -H_x}$. For example, if $H(x,y) = y^2/2 + V(x)$ is a sum of kinetic and potential energy, then $x'(t) = y, y'(t) = V'(x)$ is equivalent to $x''(t) = -V'(x)$. In the case of the Kepler problem, we had $V(x) = Gm/|x|$. $\boxed{\text{The energy } H \text{ is conserved}}$. Proof. The chain rule shows that $d/dt H(x(t), y(t)) = H_x(x,y)x'(t) + H_y(x,y)y'(t) = H_x(x,y)H_y(x,y) - H_y(x,y)H_x(x,y) = 0$.

APPLICATION: IMPLICIT DIFFERENTIATION.
From $f(x,y) = 0$ one can express y as a function of x. From $d/df(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$ we obtain $\boxed{y' = -f_x/f_y}$.

EXAMPLE. $f(x,y) = x^4 + x\sin(xy) = 0$ defines $y = g(x)$. If $f(x, g(x)) = 0$, then $g_x(x) = -f_x/f_y = -(4x^3 + \sin(xy) + xy\cos(xy))/(x^2\cos(xy))$.

APPLICATION: DIFFERENTIATION RULES (OSCAR NIGHT). One ring of the chain rules them all!
$f(x,y) = x + y, x = u(t), y = v(t), d/dt(x+y) = f_x u' + f_y v' = u' + v'$.
$f(x,y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x,y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.

DIETERICI EQUATION. In thermodynamics the temperature T, the pressure p and the volume V of a gase are related. One refinement of the ideal gas law $pV = RT$ is the **Dieterici equation** $f(p,V,T) = p(V-b)e^{a/RVT} - RT = 0$. The constant b depends on the volume of the molecules and a depends on the interaction of the molecules. (A different variation of the ideal gas law is van der Waals law). Problem: compute V_T.

If $V = V(T,p)$, the chain rule says $f_V V_T + f_T = 0$, so that $\boxed{V_T = -f_T/f_V} = -(-ap(V-b)e^{a/RVT}/(RVT^2) - R)/(pVe^{a/RVT} - p(V-b)e^{a/RVT}/(RV^2T))$. (This could be simplified to $(R + a/TV)/(RT/(V-b) - a/V^2)$).

DERIVATIVE. If $f: \mathbf{R}^n \to \mathbf{R}^m$ is a map, its **derivative** f' is the $m \times n$ matrix $[f']_{ij} = \frac{\partial}{\partial x_j} f_i$.

MORE DERIVATIVES. (The last three derivatives will only appear later)

$f: \mathbf{R} \to \mathbf{R}^3$ curve f' velocity vector.	$f: \mathbf{R}^3 \to \mathbf{R}$ scalar function f' gradient vector.	$f: \mathbf{R}^2 \to \mathbf{R}^3$ surface f' tangent matrix	$f: \mathbf{R}^2 \to \mathbf{R}^2$ coordinate change f' Jacobean matrix.	$f: \mathbf{R}^3 \to \mathbf{R}^3$ gradient field f' Hessian matrix.

THE GENERAL CHAIN RULE. (for people who have seen some linear algebra). First of all, the $d/dt f(\vec{r}(t)) = \nabla f(r(t)) \cdot \vec{r}\,'(t)$ is true in any dimension. If f is vector valued, the same equation holds for each component $d/dt f_i(\vec{r}(t)) = \nabla f_i(r(t)) \cdot \vec{r}\,'(t)$. One can further assume that $\vec{r}$ depends on different variables. Then this formula holds for each variable x_i. Here is the general chain rule for the curious: If $f: \mathbf{R}^n \to \mathbf{R}^m$ and $g: \mathbf{R}^k \to \mathbf{R}^n$, we can compose $f \circ g$, which is a map from $\mathbf{R}^k$ to $\mathbf{R}^m$. The chain rule expresses the derivative of $f \circ g(x) = f(g(x))$ in terms of the derivatives of f and g.

$\boxed{\frac{\partial}{\partial x_j} f(g(x))_i = \sum_k \frac{\partial}{\partial x_k} f_i(g(x)) \frac{\partial}{\partial x_j} g_k(x)}$ or short $\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$

Both $f'(g(x))$ and $g'(x)$ are matrices and $\cdot$ is the matrix multiplication. The chain rule in higher dimensions looks like the chain rule in one dimension, only that the objects are matrices and the multiplication is matrix multiplication.

EXAMPLE. GRADIENT IN POLAR COORDINATES. In polar coordinates, the gradient is defined as $\nabla f = (f_r, f_\theta/r)$. Using the chain rule, we can relate this to the usual gradient: $d/dr f(x(r,\theta), y(r,\theta)) = f_x(x,y)\cos(\theta) + f_y(x,y)\sin(\theta)$ and $d/(rd\theta) f(x(r,\theta), y(r,\theta)) = -f_x(x,y)\sin(\theta) + f_y(x,y)\cos(\theta)$ means that the length of ∇f is the same in both coordinate systems.

PROOFS OF THE CHAIN RULE.

1. Proof. Near any point, we can approximate f by a linear function L. It is enough to check the chain rule for linear functions $f(x,y) = ax + by - c$ and if $\vec{r}(t) = (x_0 + tu, y_0 + tv)$ is a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv$ and this is the dot product of $\nabla f = (a,b)$ with $\vec{r}\,'(t) = (u,v)$.

2. Proof. Plugging in the definitions of the derivatives and use limits.

WHERE IS THE CHAIN RULE NEEDED? (informal).

While the chain rule is useful in calculations using the composition of functions, the iteration of maps or in doing change of variables, it is also useful for **understanding** some theoretical aspects. Examples:

- In the proof of the fact that **gradients are orthogonal to level surfaces**. (see Wednesday).
- It appears in **change of variable** formulas.
- It will be used in the **fundamental theorem for line integrals** coming up later in the course.
- The chain rule illustrates also the **Lagrange multiplier** method which we will see later.
- In **fluid dynamics**, PDE's often involve terms $u_t + u\nabla u$ which give the change of the velocity in the frame of a fluid particle.
- In **chaos theory**, where one wants to understand what happens after iterating a map f.

Lecture 10: 3/4/2004, DIRECTIONAL DERIVATIVE Math21a

HOMEWORK. 11.5: 22,26,42,56

REVIEW. CHAIN RULE. The chain rule in multivariable calculus is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) .$$

where $\nabla f = (f_x, f_y)$ in two dimensions or $\nabla f = (f_x, f_y, f_z)$ in three dimensions.

DIRECTIONAL DERIVATIVE.
If f is a function of several variables and $\vec{v}$ is a vector, then $\nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$. One writes $\nabla_{\vec{v}} f$ or $D_{\vec{v}} f$.

$$\boxed{D_v f(x, y, z) = \nabla f(x, y, z) \cdot v}$$

It is usually assumed that $\vec{v}$ is a unit vector but we do not insist on that. Using the chain rule, one can write $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$.

EXAMPLE. PARTIAL DERIVATIVES ARE SPECIAL DIRECTIONAL DERIVATIVES.

If $\vec{v} = (1, 0, 0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$.
If $\vec{v} = (0, 1, 0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_y$.
If $\vec{v} = (0, 0, 1)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_z$.

The directional derivative is a generalization of the partial derivatives. Like the partial derivatives, it is a **scalar**.

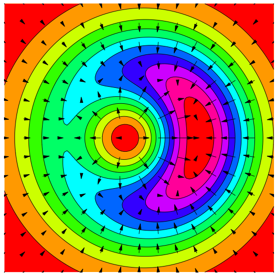
EXAMPLE. DIRECTIONAL DERIVATIVE ALONG A CURVE.

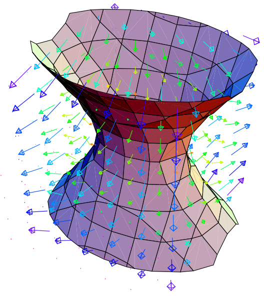
If f is the temperature in a room and $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$, then $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures on the point moving on a curve $\vec{r}(t)$ experiences: the chain rule told us that this is $d/dt f(\vec{r}(t))$.

GRADIENTS AND LEVEL CURVES/SURFACES.

Gradients are orthogonal to level curves and level surfaces.

Every vector $\vec{x} - \vec{x}_0$ in the tangent line satisfies $\nabla f \cdot (\vec{x} - \vec{x}_0) = 0$ and is so orthogonal to ∇f. The same argument holds for surfaces (see Friday).





STEEPEST DECENT. The directional derivative satisfies

$$|D_{\vec{v}} f| \leq |\nabla f||\vec{v}|$$

because $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}|| \cos(\phi)| \leq |\nabla f||\vec{v}|$. The direction $\vec{v} = \nabla f$ is the direction, where f **increases** most, the direction $-\nabla f$ is the direction where f **decreases most**. It is the direction of steepest decent.

IN WHICH DIRECTION DOES f INCREASE? If $\vec{v} = \nabla f$, then the directional derivative is $\nabla f \cdot \nabla f = |\nabla f|^2$. This means that f **increases**, if we move into the direction of the gradient!

EXAMPLE. You are on a trip in a air-shop at $(1, 2)$ and want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x, y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest?

Parameterize the direction by $\vec{v} = (\cos(\phi), \sin(\phi))$. The pressure gradient is $\nabla p(x, y) = (2x, 4y)$. The directional derivative in the ϕ-direction is $\nabla p(x, y) \cdot v = 2\cos(\phi) + 4\sin(\phi)$. This is maximal for $-2\sin(\phi) + 4\cos(\phi) = 0$ which means $\tan(\phi) = 1/2$.

ZERO DIRECTIONAL DERIVATIVE. The rate of change in all directions is zero if and only if $\nabla f(x, y) = 0$: if $\nabla f \neq \vec{0}$, we can choose $\vec{v} = \nabla f$ and get $D_{\nabla f} f = |\nabla f|^2$.

We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x, y), where $\nabla f(x, y) = (0, 0)$ are called **stationary points** or **critical points**. Knowing the critical points is important to understand the function f.

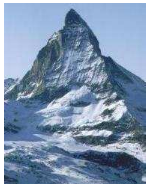
PROPERTIES DIRECTIONAL DERIVATIVE.

$$\begin{aligned} D_v(\lambda f) &= \lambda D_v(f) \\ D_v(f + g) &= D_v(f) + D_v(g) \\ D_v(fg) &= D_v(f)g + f D_v(g) \end{aligned}$$

PROPERTIES GRADIENT

$$\begin{aligned} \nabla(\lambda f) &= \lambda \nabla(f) \\ \nabla(f + g) &= \nabla(f) + \nabla(g) \\ \nabla(fg) &= \nabla(f)g + f\nabla(g) \end{aligned}$$

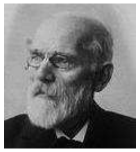
THE MATTERHORN is a popular climbing mountain in the Swiss alps. Its height is 4478 meters (14,869 feet). It is quite easy to climb with a guide. There are ropes and ladders at difficult places. Even so, about 3 people die each year from climbing accidents at the Matterhorn, this does not stop you from trying an ascent. In suitable units on the ground, the height $f(x, y)$ of the Matterhorn is approximated by $f(x, y) = 4000 - x^2 - y^2$. At height $f(-10, 10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the north-east direction $v = (1, -1)$. Calculate the rate of change in that direction. We have $\nabla f(x, y) = (-2x, -2y)$, so that $(20, -20) \cdot (1, -1) = 40$. This is a place, with a ladder, where you climb 40 meters up when advancing 1m forward.



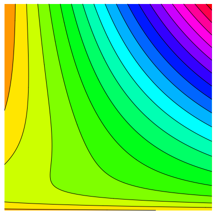
THE VAN DER WAALS (1837-1923) equation for real gases is

$$(p + a/V^2)(V - b) = RT(p, V) ,$$

where $R = 8.314 J/K$mol is a constant called the **Avogadro number**. This law relates the pressure p, the volume V and the temperature T of a gas. The constant a is related to the molecular interactions, the constant b to the finite rest volume of the molecules.



The **ideal gas** law $pV = nRT$ is obtained when a, b are set to 0. The level curves or **isotherms** $T(p, V) = const$ tell much about the properties of the gas. The so called **reduced van der Waals law** $T(p, V) = (p + 3/V^2)(3V - 1)/8$ is obtained by scaling p, T, V depending on the gas. Calculate the directional derivative of $T(p, V)$ at the point $(p, V) = (1, 1)$ into the direction $v = (1, 2)$. We have $T_p(p, V) = (3V - 1)/8$ and $T_V(p, V) = 3p/8 - (9/8)1/V^2 - 3/(4V^3)$. Therefore, $\nabla T(1, 1) = (1/4, 0)$ and $D_v T(1, 1) = 1/5$.

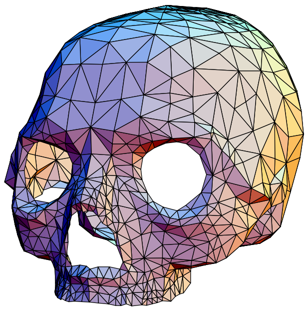


TANGENT LINE. Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b \rangle$ is perpendicular to the level curve $f(x, y) = c$ through (x_0, y_0), the equation for the tangent line is

$$\boxed{ax + by = d,\ a = f_x(x_0, y_0),\ b = f_y(x_0, y_0),\ d = ax_0 + by_0}$$

EXAMPLE. The isotherme in the previous example through $(1, 1)$ has there the tangent $(1/4)x + 0 \cdot y = (1/4)1 + 0 \cdot 1 = 1/4$ which is the horizontal line $x = 1$.

3/5/2004, THE SKULL SURFACE, Oliver Knill , Math21a



Surfaces are usually represented in a triangulated form. where a few points on the surface are given and triples of points form triangles. This illustrates the concept of **linear approximation**. Every triangle is close to the actual surface.

HOW TO PRODUCE THE SKULL SURFACE.

We found on the web a file **skull.dat**, which contains data

```
0 -0.02 0.02
0.29 0.65 0.08
0.4 0.53 0.06
0 -0.02 0.02
0.4 0.53 0.06
0.53 0.41 0.02
0 -0.02 0.02
...
...
```

These data encode points in space: every line is a point, three points in a row represent a triangle in space.

So, the first 3 lines from the datafile represent the triangle $\{P, Q, R\} = \{(0, -0.02, 0.02), (0.29, 0.65, 0.02), (0.4, 0.65, 0.08)\ \}$. In total, the file contains 3729 points which means that there are 1243 triangles encoded.

PLOTTING THE SURFACE.

To the right is a small Mathematica program which produces a surface from these data. The data are first read, then packed into groups of 3 to get the points, then packed into groups of three to get the triangles, then each triple of points is made into a polygon. All these polygons are finally displayed.

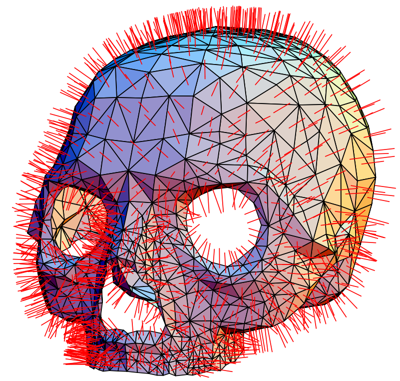
```
A=ReadList["skull.dat",Number];
B=Partition[A,3];
P=Partition[B,3];
U=Map[Polygon,P];
Show[Graphics3D[U],Boxed -> False, ViewPoint -> {0, -3, 0}]
```

PLOTTING THE GRADIENTS.

We can get the gradient vector on each triangle $\{P_1, P_2, P_3\}$ by taking the cross product $n = (P_1 - P_2) \times (P_3 - P_2)$ and adding lines connecting $Q = (P_1 + P_2 + P_3)$ with the point $Q + n$. We used that

Gradients are orthogonal to the level surfaces.

```
A=ReadList["skull.dat",Number];
B=Partition[A,3];
P=Partition[B,3];
U=Map[Polygon,P];
NormedCross[a_,b_]:=Module[{},u=Cross[a,b]; u/Sqrt[u.u]];
NormalVector[{ a_,b_,c_ }]:=Module[{},P1=(a+b+c)/3;
P2=P1-NormedCross[b-a,c-a]/3;Line[{P1,P2}]];
V=Map[NormalVector,P];
Show[Graphics3D[{U,V}],Boxed-> False,ViewPoint-> {0, -3, 0}]
```



Here is the output of the above routine. Fortunately, the points of each triangle were given in such a way that we also know the direction of the normal vector. We have chosen the direction pointing outwards. Through every triangle, one gradient vector is displayed.

Once we have the data in Mathematica, we can also export a file, which a raytracer can read. We could also export to a VRML (Virtual Reality Markup Language) and fly around or through it. The ultimate Haloween experience!

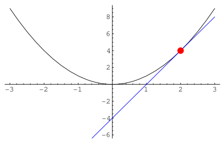
Lecture 11: 3/5/2004, TANGENT PLANES **Math21a**

HOMEWORK. 11.5: 34,38,58,60

REMINDER: TANGENT LINE. Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b\rangle$ is perpendicular to the level curve $f(x, y) = c$ through (x_0, y_0), the equation for the tangent line is

$$\boxed{ax + by = d,\ a = f_x(x_0, y_0),\ b = f_y(x_0, y_0),\ d = ax_0 + by_0}$$

Example: Find the tangent to the graph of the function $g(x) = x^2$ at the point $(2, 4)$. Solution: the level curve $f(x, y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2, 4)$ is obtained by computing the gradient $\langle a, b\rangle = \nabla f(2, 4) = \langle -g'(2), 1\rangle = \langle -4, 1\rangle$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4. Graphs of 1D functions are curves in the plane, you have computed tangents in single variable calculus.

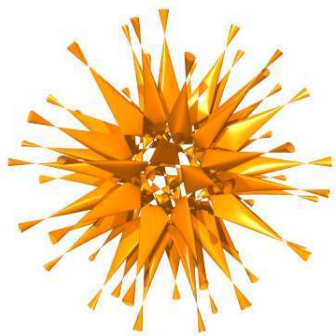


GRADIENT IN 3D. If $f(x, y, z)$ is a function of three variables, then $\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z))$ is called the **gradient** of f.

POTENTIAL AND FORCE. Force fields F in nature often are gradients of a function $U(x, y, z)$. The function U is called a **potential** of F or the potential energy.

EXAMPLE. If $U(x, y, z) = 1/|x|$, then $\nabla U(x, y, z) = -x/|x|^3$. The function $U(x, y, z)$ is the **Coulomb potential** and ∇U is the **Coulomb force**. The gravitational force has the same structure but a different constant. While much weaker, it is more effective because it only appears as an attractive force, while electric forces can be both attractive and repelling.

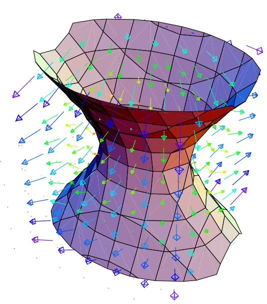
LEVEL SURFACES. If $f(x, y, z)$ is a function of three variables, then $f(x, y, z) = C$ is a surface called a level surface of f. The picture to the right shows the Barth surface $(3 + 5t)(-1 + x^2 + y^2 + z^2)^2\ (-2 + t + x^2 + y^2 + z^2)^2\ 8(x^2 - t^4y^2)(-t^4x^2 + z^2)\ (y^2 - t^4z^2)\ (x^4 - 2x^2y^2 + y^4 - 2x^2z^2 - 2y^2z^2 + z^4) = 0$, where $t = (\sqrt{5} - 1)/2$ is the golden ratio.



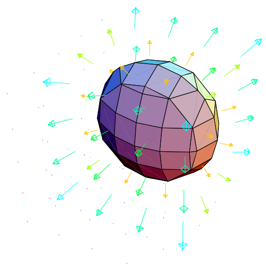
ORTHOGONALITY OF GRADIENT. We have seen that the gradient $\nabla f(x, y)$ is normal to the level curve $f(x, y) = c$. This is also true in 3 dimensions:

$\boxed{\text{The gradient } \nabla f(x, y, z) \text{ is normal to the level surface } f(x, y, z).}$

The argument is the same as in 2 dimensions: take a curve $\vec{r}(t)$ on the level surface. Then $\frac{d}{dt} f(\vec{r}(t)) = 0$. The chain rule tells from this that $\nabla f(x, y, z)$ is perpendicular to the velocity vector $\vec{r}\,'(t)$. Having ∇f tangent to all tangent velocity vectors on the surfaces forces it to be orthogonal.

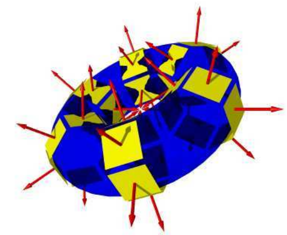


EXAMPLE. The gradient of $f(x, y, z) = x^2 + 2y^2 + z^2$ at a point (x, y, z) is $(2x, 4y, 2z)$. It illustrates well that going into the direction of the gradient **increases** the value of the function.



TANGENT PLANE. Because $\vec{n} = \nabla f(x_0, y_0, z_0) = \langle a, b, c\rangle$ is perpendicular to the level surface $f(x, y, z) = C$ through (x_0, y_0, z_0), the equation for the tangent plane is

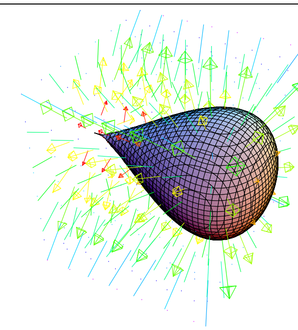
$$\boxed{ax + by + cz = d,\quad (a, b, c) = \nabla f(x_0, y_0, z_0),\ d = ax_0 + by_0 + cz_0.}$$



EXAMPLE. Find the general formula for the tangent plane at a point (x, y, z) of the Barth surface. Just kidding ... Note however that computing this would be no big deal with the help of a computer algebra system like Mathematica. Lets look instead at the quartic surface

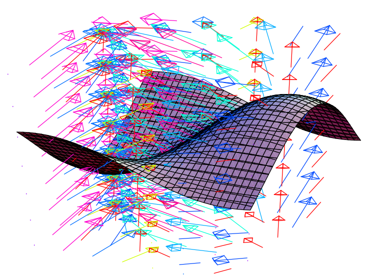
$$f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0$$

which is also called the "piriform" or "pair shaped surface". What is the equation for the tangent plane at the point $P = (2, 2, 2)$? We get $\langle a, b, c\rangle = \langle 20, 4, 4\rangle$ and so the equation of the plane $20x + 4y + 4z = 56$.



EXAMPLE. An important example of a level surface is $g(x, y, z) = z - f(x, y)$ which is the graph of a function of two variables. The gradient of g is $\nabla f = (-f_x, -f_y, 1)$. This allows us to find the equation of the tangent plane at a point.

Quizz: What is the relation between the gradient of f in the plane and the gradient of g in space?

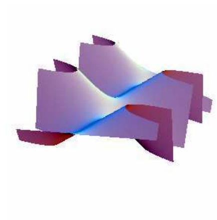


Lecture 12: 3/8/2004, LINEAR APPROXIMATION **Math 21a, O. Knill**

HOMEWORK: 11.6: 4,12,18,40 (for Friday)

LINEAR APPROXIMATION.

$\boxed{\text{1D:}}$ The **linear approximation** of a function $f(x)$ at a point x_0 is the linear function $\boxed{L(x) = f(x_0) + f'(x_0)(x - x_0)}$. The graph of L is tangent to the graph of f.



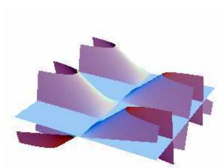
$\boxed{\text{2D:}}$ The **linear approximation** of a function $f(x, y)$ at (x_0, y_0) is

$$\boxed{L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(y - y_0)}$$

The level curve of g is tangent to the level curve of f at (x_0, y_0). The graph of L is tangent to the graph of f.

$\boxed{\text{3D:}}$ The **linear approximation** of a function $f(x, y, z)$ at (x_0, y_0, z_0) by

$$\boxed{\begin{aligned} L(x, y, z) = {} & f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + \\ & f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \end{aligned}}.$$



The level surface of L is tangent to the level surface of f at (x_0, y_0, z_0).

In vector form, the linearization can be written as

$$\boxed{L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)}$$

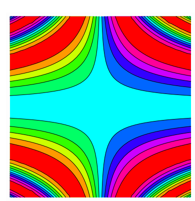
JUSTIFYING THE LINEAR APPROXIMATION. 3 ways to see it:

1) We know the tangent plane to $g(x, y, z) = z - f(x, y)$ at $(x_0, y_0, z + 0 = f(x_0, z_0)$ is $-f_x x - f_y y + z = -f_x x_0 - f_y y_0 + z_0$. This can be read as $z = z_0 + +f_x(x - x_0) + f_y(y - y_0)$. Calling the right hand side $L(x, y)$ shows that the graph of L is tangent to the graph of f at (x_0, y_0).

2) The higher dimensional case can be reduced to the one dimensional case: if $y = y_0$ is fixed and x, then $f(x, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$ is the linear approximation of the function. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. So along two directions, the linear approximations are the best. Together we get the approximation for $f(x, y)$.

3) An other justification is that $\nabla f(\vec{x}_0)$ is orthogonal to the level curve at $\vec{x}_0$. Because $\vec{n} = \nabla f(\vec{x}_0)$ is orthogonal to the plane $\vec{n} \cdot (\vec{x} - \vec{x_0}) = d$ also, the graphs of $L(x, y)$ and $f(x, y)$ have the same normal vector at $(x_0, y_0, f(x_0, y_0))$.

EXAMPLE (2D) Find the linear approximation of the function $f(x, y) = \sin(\pi x y^2)$ at the point $(1, 1)$. The gradient is $\nabla f(x, y) = (\pi y^2 \cos(\pi x y^2), 2y\pi \cos(\pi x y^2))$. At the point $(1, 1)$, we have $\nabla f(1, 1) = (\pi \cos(\pi), 2\pi \cos(\pi)) = (-\pi, 2\pi)$. The linear function approximating f is $L(x, y) = f(1, 1) + \nabla f(1, 1) \cdot (x - 1, y - 1) = 0 - \pi(x - 1) - 2\pi(y - 1) = -\pi x - 2\pi y + 3\pi$. The level curves of G are the lines $x + 2y = const$. The line which passes through $(1, 1)$ satisfies $x + 2y = 3$.



Application: $-0.00943407 = f(1+0.01, 1+0.01) \sim g(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942478$.

EXAMPLE (3D) Find the linear approximation to $f(x, y, z) = xy + yz + zx$ at the point $(1, 1, 1)$.

We have $f(1, 1, 1) = 3$, $\nabla f(x, y, z) = (y + z, x + z, y + x), \nabla f(1, 1, 1) = (2, 2, 2)$. Therefore $L(x, y, z) = f(1, 1, 1) + (2, 2, 2) \cdot (x - 1, y - 1, z - 1) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = 2x + 2y + 2z - 3$.

EXAMPLE (3D). Use the best linear approximation to $f(x, y, z) = e^x \sqrt{y} z$ to estimate the value of f at the point $(0.01, 24.8, 1.02)$.

Solution. Take $(x_0, y_0, z_0) = (0, 25, 1)$, where $f(x_0, y_0, z_0) = 5$. The gradient is $\nabla f(x, y, z) = (e^x \sqrt{y} z, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0, y_0, z_0) = (0, 25, 1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x, y, z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

SECOND DERIVATIVE.

If $f(x, y)$ is a function of two variables, then the matrix $f''(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ is called the **second derivative** or the **Hessian** of f.

For functions of three variables, the Hessian is the $3x3$ matrix $f''(x, y, z) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$ Because for smooth functions, $f_{xy} = f_{yx}, f_{yz} = f_{zy}$, the matrix f'' is **symmetric** (a reflection at the diagonal leaves it invariant).

QUADRATIC APPROXIMATION. (informal)

If F is a function of several variables $\vec{x}$ and $\vec{x}_0$ is a point, then

$$\boxed{Q(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0) + [f''(\vec{x}_0)(\vec{x} - \vec{x}_0)] \cdot (\vec{x} - \vec{x}_0)/2}$$

is called the **quadratic approximation** of $\vec{x}$.

It generalizes the quadratic approximation $L(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2$ of a function of one variables.

EXAMPLE. If the height of a hilly region is given by $f(x, y) = 4000 - \sin(x^2 + y^2)$, find the quadratic approximation of F at $(0, 0)$.

$\nabla f(x, y) = (2x, 2y)\cos(x^2 + y^2)$ so that $\nabla f(0, 0) = (0, 0)$. The linear approximation of F at $(0, 0)$ is $G(x, y) = f(0, 0) = 4000$. The graph of G is a plane tangent to the graph of F.

$$f''(x, y) = \cos(x^2 + y^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$f''(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The quadratic approximation at $(0, 0)$ is $Q(x, y) = 4000 - x^2 - y^2$. The graph of F is an inverted paraboloid.

EXAMPLE: REDUCED VAN DER WAALS LAW $T(p, V) = (p + 3/V^2)(3V - 1)/8$ Find the quadratic approximation of T at $(p, V) = (1, 1)$. We had $T_p(p, V) = (3V - 1)/8$ and $T_V(p, V) = 3p/8 - (9/16)1/V^2 - 3/(4V^3)$ and $\nabla T(1, 1) = (1/4, 0)$. Now, $T_{pp}(p, V) = 0$, $T_{pV} = 3/8$, $T_{Vp} = 3/8$ and $T_{VV} = (9/8)1/V^3 + (9/4)1/V^4$ so that $T''(1, 1) = \begin{bmatrix} 0 & 3/8 \\ 3/8 & 27/8 \end{bmatrix}$. The quadratic approximation at $T = (1, 1)$ is $Q(p, T) = T(1, 1) + T'(1, 1) \cdot (p - 1, T - 1) + [T''(1, 1)(p - 1, T - 1)] \cdot (p - 1, T - 1)/2$.

ERROR OF APPROXIMATION. It follows from Taylors theorem that the error $f(x, y) - L(x, y)$ in a region R near (x_0, y_0) is smaller or equal to $M(|x - x_0| + |y - y_0|)^2/2$, where M is the maximal value of all the matrix entries $f''(x, y)$ in that region R.

TOTAL DIFFERENTIAL. Aiming to estimate the change $\Delta f = f(x, y) - f(x_0, y_0)$ of f for points $(x, y) = (x_0, y_0) + (\Delta x, \Delta y)$ near (x_0, y_0), we can estimate it with the linear approximation which is $L(\Delta x, \Delta y) = f_x(x_0, y_0)\Delta x + f_y \Delta y$. In an old-fashioned notation, one writes also $df = f_x dx + f_y dy$ and calls df the **total differential**. One can **totally avoid** the notation of the **total differential**.

RELATIVE AND ABSOLUTE CHANGE. The **relative change** Δf depends on the magnitude of f. One also defines $\Delta f(x_0, y_0)/f(x_0, y_0)$, the **relative change** of f. A good estimate is $\Delta L(x_0, y_0)/L(x_0, y_0)$.

PHYSICAL LAWS.

Many physical laws are in fact linear approximations to more complicated laws. One could say that a large fraction of physics consists of understand nature with linear laws.

LINEAR STABILITY ANALYSIS.

In physics, complicated situations can occur. Usually, many unknown parameters are present and the only way to analyze the situation theoretically is to assume that things depend linearly on these parameters. The analysis of the linear situation allows then to predict for example the stability of the system with respect to perturbations. Sometimes, the stability of the linearized system will imply the stability of the perturbation.

ERROR ANALYSIS.

Error analysis is based on linear approximation. Assume, you make a measurement of a function $f(a,b,c)$, where a, b, c are parameters. Assume, you know the numbers a, b, c up to accuracy ϵ. How precise do you know the values $f(a,b,c)$? Because $f(a_0+\epsilon_a, b_0+\epsilon_b, c_0+\epsilon_c)$ is about $f(a_0,b_0,c_0)+\nabla f(a_0,b_0,c_0)\cdot(\epsilon_a,\epsilon_b,\epsilon_c)$, the answer is that we know F up to accuracy $|\nabla f(a_0,b_0,c_0)|\epsilon$.

POWER LAWS.

Some laws in physics are given by functions of the form $g(x,y) = x^\alpha y^\beta$. An example is the Cobb-Douglas formula in economics. Such dependence on x or y is called **power law behavior**. If we consider $f = \log(g)$, and introduce $a = \log(x), b = \log(y)$, then this becomes $f(a,b) = \log(g(x,y)) = a\alpha + b\beta$. Power laws become linear laws in a logarithmic scale. But they usually are linear approximations to more complicated nonlinear relations.

ELECTRONICS.

If we apply a voltage difference U at the ends of a resistor R, then a current I flows. The relation $U = RI$ is called **Ohms law**. In logarithmic coordinates $\log(U) = \log(R) + \log(I)$, this is a linear law. In reality, the relation between current, voltage and resistance is more complicated. For example, if the resistor heats up, then its characteristics begin to change. Nonlinear resistors are used for example in synthesizers or in radars. While Ohm's law works **extremely well**, the nonlinear behavior can have important consequences for example to stabilize systems or to protect equipment against over-voltages.

THERMODYNAMICS.

If $l(T)$ is the length of an object with temperature T, then $l(T) = l(T_0) + c(T - T_0)$, where the expansion coefficient c depends on the material. (Trick question: What happens if you heat a ring, does the inner ring become smaller or bigger?). The volume of a hot air balloon and therefore its lift capacity grows like $c(T-T_0)^3$. The law of expansion is only an approximation.

OCEANOGRAPHY.

For oceanographers, it is important to know the water density $\rho(T,S)$ in dependence on the **temperature** T (Kelvin) and the **salinity** S (psu). If we would include the pressure P (Bar), then we had a function $\rho(T,S,P)$ of three variables. Near a specific point (T,S,P) the density can be approximated by a linear function giving a law which is precise enough.

ENGENEERING.

Hooks law tells that the force of a spring is proportional to the length with which it is pulled: $F(l) = c(l - l_0)$, where l_0 is the length when the spring is relaxed. This allows to measure weights or to cushion shocks. However, this law is only good in a certain range. If the spring is pulled too strongly, then more force is needed. Such a nonlinear behavior is needed for example in shock absorbers.

MECHANICS

For small amplitudes, the pendulum motion $\ddot{x} = -g\sin(x)$ can be approximated by $\ddot{x} = -gx$, the harmonic oscillator. Nonlinear (partial) differential equations like $u_{xx} + u_{yy} + u_{zz} = F(x,y,z)$ are often approximated by linear differential equations.

CARTOGRAPHY.

It was well known already to the Greeks that we live on a sphere. On a sphere a triangle however the sum of its angles adds up to more than 180 degrees and every straight line (great circles) crosses every other line at least twice. Despite this, a city map can perfectly assume that the coordinate system is Cartesian. When drawing a plan of a house, an architect can assume that the house stands on a plane (the level curve of the linearization $G(x,y)$ of $F(x,y)$ defining the surface of the earth.)

RELATIVITY

Newton's law tells that $r''(t)$, the acceleration of a particle is proportional to the force F which acts on the mass point: $r''(t) = F/m$. For a constant force and zero initial velocity this implies $r'(t) = tF/m$. This law can not apply for all times, because we can not reach the speed of light with a massive body. In special relativity, the Newton axiom is replaced with $d/dt(r'(t)m(t)) = F$, where the mass $m(t)$ depends on the velocity. This gives $v(t) = (tF/m_0)\frac{1}{\sqrt{1+F^2t^2/(c^2m_0^2)}}$. Linearization at $t = 0$ produces the classical law $v(t) = tF/m_0$.

ECONOMICS.

The mathematician Charles W. Cobb and the economist Paul H. Douglas found in 1928 empirically a formula $F(L,K) = bL^\alpha K^\beta$ giving the total production F of an economic system as a function of the amount of labor L and the capital investment K. This is a linear law in logarithmic coordinates. The formula actually had been found by linear fit of empirical data. In general, the production depends in a more complicated way on labor and capital investment. For example, with increase of labor and investment, logistic constraints will become relevant.

CHEMISTRY.

The ideal gas law $PV = RT$ relates the pressure, the volume and the temperature of an ideal gas using a constant R called the Avogadro number. This law $T = f(P,V)$ is linear in logarithmic scales. This law is only an approximation and has to be replaced by the van der Waals law, which takes into account the molecular interactions as well as the volume of the molecules.

FIRST MIDTERM REVIEW, 3/10/2004 — Math 21a

GENERAL TIPS.

- Review the online quizzes.
- Make list of facts on a sheet of paper.
- Fresh up short-term memory before test.
- Review homework. Find error patterns.
- Ask questions:

> "Ask a question and you're a fool for three minutes; do not ask a question and you're a fool for the rest of your life." - Chinese Proverb

- During the exam: read the questions carefully. Wrong understanding could lead you to solve an other problem:

> There was a college student trying to earn some pocket money by going from house to house offering to do odd jobs. He explained this to a man who answered one door. "How much will you charge to paint my porch?" asked the man. "Forty dollars." "Fine" said the man, and gave the student the paint and brushes. Three hours later the paint-splattered lad knocked on the door again. "All done!", he says, and collects his money. "By the way," the student says, "That's not a Porsche, it's a Ferrari."

MIDTERM TOPICS.

- Properties of dot, cross and triple product
- Orthogonality, parallel, vector projection
- Parametrized Lines and Planes
- Given line and plane, find intersection
- Given plane and plane, find intersection
- Given line and point, find plane
- Given two points, find line
- Given three points, find plane
- Distances: point-line, line-line,point-plane
- Distinguish and analyse curves
- Determine curves from acceleration
- Know Keplers laws, polar form of ellipse
- Recognize functions $f(x,y)$ of two variables.
- Tangent lines, tangent curves
- Distance between two lines
- Distance between two planes
- Angle between two vectors
- Angle between two planes
- Area of parallelogram, triangle in space
- Volume of parallelepiped
- Distinguish contour maps, graphs
- Compute velocity, acceleration, speed
- Integrate from velocity to get position
- Find length of curves
- Level curves, level surfaces
- Directional derivative
- Chain rule
- Implicit differentiation
- Tangent planes

VECTORS.

Two points $P=(1,2,3), Q=(3,4,6)$ define a vector $\vec{v}=\vec{PQ}=\langle 2,2,3\rangle$. If $\vec{v}=\lambda\vec{w}$, then the vectors are **parallel** if $\vec{v}\cdot\vec{w}=0$, then the vectors are called **orthogonal**. For example, $(1,2,3)$ is parallel to $(-2,-4,-6)$ and orthogonal to $(3,-2,1)$. The addition, subtraction and scalar multiplication of vectors is done componentwise. For example: $(3,2,1)-2((1,1,1)+(-1,-1,0))=(3,2,-1)$.

A nonzero vector $\vec{v}$ and a point $P=(x_0,y_0,z_0)$ define a line $\vec{r}(t)=P+t\vec{v}$. Two nonzero, non-parallel vectors $\vec{v},\vec{w}$ and a point P define a plane $P+t\vec{v}+s\vec{s}$. The vector $\vec{n}=\vec{v}\times\vec{w}=(a,b,c)$ is orthogonal to the plane. Points on the line satisfy the symmetric equation $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$. Points on the plane satisfy an equation $ax+by+cz=d$, where $d=ax_0+by_0+cz_0$. Using the dot product for projection and the vector product to get orthogonal vectors, one can solve many geometric problems in 3D.

DOT PRODUCT (is scalar)

$\vec{v}\cdot\vec{w}=\vec{w}\cdot\vec{v}$	commutative
$\lvert\vec{v}\cdot\vec{w}\rvert=\lvert\vec{v}\rvert\lvert\vec{w}\rvert\cos(\alpha)$	angle
$(a\vec{v})\cdot\vec{w}=a(\vec{v}\cdot\vec{w})$	linearity
$(\vec{u}+\vec{v})\cdot\vec{w}==\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$	distributivity
$\{1,2,3\}.\{3,4,5\}$	in Mathematica
$\frac{d}{dt}(\vec{v}\cdot\vec{w})=(\frac{d}{dt}\vec{v})\cdot\vec{w}+(\vec{v}\cdot\frac{d}{dt}\vec{w})$	product rule

CROSS PRODUCT (is vector)

$\vec{v}\times\vec{w}=-\vec{w}\times\vec{v}$	anti-commutative
$\lvert\vec{v}\times\vec{w}\rvert=\lvert\vec{v}\rvert\lvert\vec{w}\rvert\sin(\alpha)$	angle
$(a\vec{v})\times\vec{w}=a(\vec{v}\times\vec{w})$	linearity
$(\vec{u}+\vec{v})\times\vec{w}=\vec{u}\times\vec{w}+\vec{v}\times\vec{w}$	distributivity
Cross[{1,2,3},{3,4,5}]	Mathematica
$\frac{d}{dt}(\vec{v}\times\vec{w})=(\frac{d}{dt}\vec{v})\times\vec{w}+\vec{v}\times(\frac{d}{dt}\vec{w})$	product rule

PROJECTIONS.

Vector projection:

$$\boxed{\mathrm{proj}_{\vec{v}}(\vec{w})=\frac{(\vec{v}\cdot\vec{w})\vec{v}}{|\vec{v}|^2}.}$$

Is a vector parallel to $\vec{w}$.

Scalar projection:

$$\boxed{\mathrm{comp}_{\vec{v}}(\vec{w})=|\mathrm{proj}_{\vec{v}}(\vec{w})|=\frac{|\vec{v}\cdot\vec{w}|}{|\vec{v}|}}$$

the length of the projected vector.

Applications:
- Distance $P+t\vec{v}, Q+s\vec{w}$ is scalar projection of $\vec{PQ}$ onto $\vec{v}\times\vec{w}$.
- Distance $P, Q+t\vec{v}+s\vec{w}$ is scalar projection of $\vec{PQ}$ onto $\vec{n}=\vec{v}\times\vec{w}$.

SURFACES $\{f(x,y,z)=c\}$.

Examples are **graphs**, where $f(x,y,z)=z-g(x,y)=0$ or planes, where $f(x,y,z)=ax+by+cz=c$. Surfaces can be analyzed by looking at intersections with planes parallel to the coordinate planes. For graphs, the traces $f(x,y)=c$ are **contour lines**. Most important fact:

> The gradient $\nabla f(x_0,y_0,z_0)$ is normal to the surface $f(x,y,z)=c$ containing (x_0,y_0,z_0).

SURFACES EXAMPLES

- sphere $x^2+y^2+z^2=1$
- cylinder $x^2+y^2=1$
- ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$
- cone $x^2+y^2-z^2=0$
- plane $ax+by+cz=d$
- one sheeted hyperboloid $x^2+y^2-z^2=1$
- two sheeted hyperboloid $x^2+y^2-z^2=-1$
- paraboloid $x^2+y^2-z=0$
- hyperbolic paraboloid $x^2-y^2-z=0$
- graph of function $g(x,y)-z=0$

can be identified using **traces**, the intersections with planes.

CURVES.

$\vec{r}(t)=(x(t),y(t),z(t))$, $t\in[a,b]$ defines a curve. By differentiation, we obtain the **velocity** $\vec{r}\,'(t)$ and **acceleration** $\vec{r}\,''(t)$. If we integrate the speed $|\vec{r}\,'(t)|$ over the interval $[a,b]$, we obtain the **length** of the curve.

$$\boxed{\int_a^b\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt}$$

Example: $\vec{r}(t)=(1,3t^2,t^3)$, $\vec{r}'(t)=(0,6t,3t^2)$, so that $|\vec{r}'(t)|=3t(4+t^2)$. The length of the curve between 0 and 1 is $\int_0^1 3t(4+t^2)\,dt=6t^2+3\frac{t^4}{4}|_0^1=6\cdot\frac{3}{4}$.

DIRECTIONAL DERIVATIVE.

For any vector $\vec{v}$ and a function $f(x,y)$, define $D_{\vec{v}}f(x,y)=\nabla f(x,y)\cdot\vec{v}$. Unlike in many textbooks:

> We do not divide by $|\vec{v}|$ to compute the directional derivative.

CHAIN RULE.

We have seen that $d/dtf(\vec{r}(t))=\nabla f(\vec{r}(t))\cdot\vec{r}'(t)$. This works also, if $\vec{r}(t,s)$ is a function of two variables:
$f_t(x(t,s),y(t,s))=\nabla f(\vec{r}(t,s))\cdot\vec{r}_t(t,s)$.
$f_s(x(t,s),y(t,s))=\nabla f(\vec{r}(t,s))\cdot\vec{r}_s(t,s)$.

Other variables: $w(u,v)$ function of u,v, where u,v are functions of x and eventually of y:

$$\boxed{\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}}\qquad\boxed{\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}}$$

For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we always have

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{a} \times \vec{c}) \cdot \vec{b}$$

The set of points in the xy-plane which satisfy $x^2 - y^2 = -1$ form a hyperbola.

The velocity vector $\vec{r}\,'(t)$ and the acceleration vector $\vec{r}\,''(t)$ of a curve are always perpendicular.

The length of a curve does not depend on the parameterization chosen for the curve.

The directional derivative of $f(x, y, z) = x^2 + y^2 - z^2$ into the direction $(1, 1, 1)$ at the point $(0, 0, 1)$ is $-2/\sqrt{3}$.

If $|\vec{v} \times \vec{w}| = 0$ for all vectors $\vec{w}$, then $\vec{v} = 0$.

Every vector contained in the line $\vec{r}(t) = \langle 1 + 2t, 1 + 3t, 1 + 4t \rangle$ is parallel to the vector $(1, 1, 1)$.

If the velocity vector $\vec{r}\,'(t)$ of the planar curve $\vec{r}(t)$ is orthogonal to the vector $\vec{r}(t)$ for all times t, then the curve is a circle.

The vectors $\vec{i} + \vec{j}$ and $\vec{k}$ are orthogonal.

If the velocity vector of the curve $\vec{r}(t)$ is never zero and always parallel to a constant vector $\vec{v}$ for all times t, then the curve is a straight line.

If $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are orthogonal, then the vectors $\vec{u}$ and $\vec{v}$ have the same length.

The identity $|\vec{v} \cdot \vec{w}|^2 + |\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2$ holds for all vectors $\vec{v}, \vec{w}$.

The set of points which have distance 1 from a line form a cylinder.

The distance between a point P and a line $L : \vec{r}(t) = Q + t\vec{v}$ is $|\vec{v} \times \vec{PQ}|/|\vec{PQ}|$.

If $w(u, v)$ is a function of two variables u, v and u, v are functions of x, y, then $w_x = w_u u_x + w_y v_x$.

The gradient $\nabla f(x, y, z)$ is always tangent to the surface $f(x, y, z) = c$.

The direction of steepest decent at a point (x, y) is $\nabla f(x, y)$.

The vector projection of a vector $\vec{v}$ onto a vector $\vec{w}$ is $\frac{|\vec{v} \times \vec{w}|}{|\vec{v}|^2}\vec{v}$.

In the Kepler problem, one has $d/dt(\vec{r} \cdot \vec{r}') = 0$.

For any two vectors v, w one has $||v + w||^2 = ||v||^2 + ||w||^2$.

For any two vectors $\vec{v}$ and $\vec{w}$ one has $v \times w = w \times v$.

$$\vec{i} \times \vec{j} - \vec{k} = 0$$

If your thumb $\vec{u}$, your pointing finger $\vec{v}$ and middle finger $\vec{w}$ are perpendicular to each other, and $\vec{u} \times \vec{v}$ points in the direction opposite to $\vec{w}$, then the three fingers belong to the left hand.

If $\vec{r}(t) = Q + t\vec{v}$ is a line and P is a point such that PQ is perpendicular to $\vec{v}$, then the distance from P to the line is equal to the distance from P to the line.

The distance of the two planes $x + y + z = 3$ and $x + y + z = 5$ is $2/\sqrt{3}$.

A level curve of a function $f(x, y)$ is a curve which has no self intersections.

The angle between two planes is equal to the angle between its normal vectors.

The speed of a curve $\vec{r}(t)$ is a vector tangent to the curve.

If the acceleration vector of a curve $\vec{r}(t)$ is parallel to $\vec{r}(t)$ for all times, then the curve must lie in a plane.

The arc length of a curve $\vec{r}(t)$ parameterized on the interval $[a, b]$ is $\int_a^b (x'(t)^2 + y'(t)^2 + z'(t)^2)\, dt$.

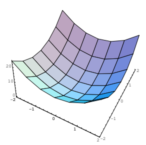
Lecture 14: 3/12/2004, CRITICAL POINTS — Math21a, O. Knill

HOMEWORK: Section 11.7: 8,18,20,22

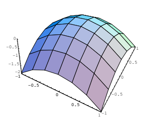
CRITICAL POINTS. A point (x_0, y_0) in a region G is called a **critical point** of $f(x, y)$ if $\nabla f(x_0, y_0) = (0, 0)$. Remarks. Critical points are also called **stationary points**. Critical points are candidates for extrema because at critical points, the directional derivative is zero. It is usually assumed that f is differentiable.

EXAMPLE 1. $f(x, y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x, y) = (4(x^3 - y), 4(y^3 - x))$ with critical points $(0, 0), (1, 1), (-1, -1)$.
EXAMPLE 2. $f(x, y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x, y) = (2x\cos(x^2 + y), \cos(x^2 + y) + 1)$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $(0, \pi), (0, 3\pi), \dots$.
EXAMPLE 3. ("volcano") $f(x, y) = (x^2 + y^2)e^{-x^2-y^2}$. The gradient $\nabla F = (2x - 2x(x^2 + y^2), 2y - 2y(x^2 + y^2))e^{-x^2-y^2}$ vanishes at $(0, 0)$ and on the circle $x^2 + y^2 = 1$. There are ∞ many critical points.
EXAMPLE 4 ("pendulum") $f(x, y) = -g\cos(x) + y^2/2$ is the energy of the pendulum. The gradient $\nabla F = (y, -g\sin(x))$ is $(0, 0)$ for $x = 0, \pi, 2\pi, \dots, y = 0$. These points are equilibrium points, where the pendulum is at rest.
EXAMPLE 5 ("Volterra Lodka") $f(x, y) = a\log(y) - by + c\log(x) - dx$. (This function is left invariant by the flow of the Volterra Lodka differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$ which you might have seen in Math1b.) The point $(c/d, a/b)$ is a critical point.

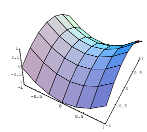
TYPICAL EXAMPLES.



$f(x, y) = x^2 + y^2$

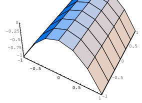


$f(x, y) = -x^2 - y^2$

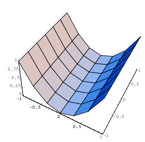


$f(x, y) = x^2 - y^2$

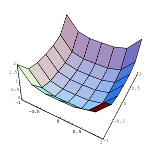
EXAMPLES WITH DISCRIMINANT $D = \det(H) = 0$.



$f(x, y) = x^2$

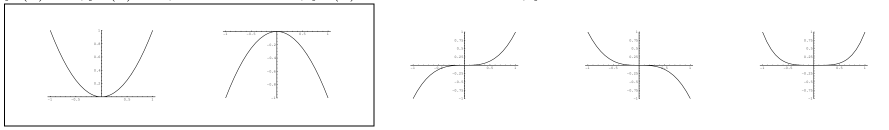


$f(x, y) = -x^2$



$f(x, y) = x^4 + y^4$

CLASSIFICATION OF CRITICAL POINTS IN 1 DIMENSION.
$f'(x) = 0, f''(x) > 0$, local minimum, $f''(x) < 0$ local maximum, $f'' = 0$ undetermined.



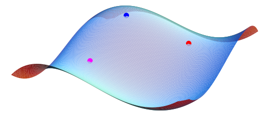
CLASSIFICATION OF CRITICAL POINTS: SECOND DERIVATIVE TEST. Let $f(x, y)$ be a function of two variables with a critical point (x_0, y_0). Define $D = f_{xx}f_{yy} - f_{xy}^2$, called the **discriminant** or **Hessian**. (Remark: With the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ we can write $D = \det(H)$ as a **determinant**.

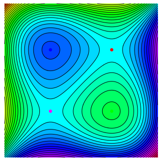
If $D > 0$ and $H_{11} > 0 \Rightarrow$ local minimum (bottom of valley)
If $D > 0$ and $H_{11} < 0 \Rightarrow$ local maximum (top of mountain).
If $D < 0 \Rightarrow$ saddle point (mountain pass).

In the case $D = 0$, we would need higher derivatives to determine the nature of the the critical point.

EXAMPLE. (A "napkin").
The function $f(x, y) = x^3/3 - x - (y^3/3 - y)$ has the gradient $\nabla f(x, y) = (x^2 - 1, -y^2 + 1)$. It is the zero vector at the 4 critical points $(1, 1)$,$(-1, 1)$,$(1, -1)$ and $(-1, -1)$. The Hessian matrix is $H = f''(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.

$H(1, 1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$	$H(-1, 1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$	$H(1, -1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$H(-1, -1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$
$D = -4$ Saddle point	$D = 4, f_{xx} = -2$ Local maximum	$D = 4, f_{xx} = 2$ Local minimum	$D = -4$ Saddle point





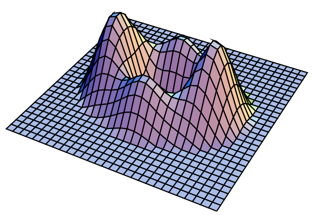
GLOBAL MAXIMA AND MINIMA. To determine the maximum or minimum of $f(x, y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**.

(A point is in the interior of G, if there is a small disc around (x_0, y_0) contained in G. A point is at the boundary of G, if any disc aound (x_0, y_0) contains both points in G and in the complement).

EXAMPLE 5. Find the critical points of $f(x, y) = 2x^2 - x^3 - y^2$. With $\nabla f(x, y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0, 0)$. The Hessian is $H(x, y) = \begin{bmatrix} 4 - 6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0, 0)$, the discriminant is -8 so that this is a saddle point. At $(4/3, 0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum.

WHY DO WE CARE ABOUT CRITICAL POINTS?
- Critical points are candidates for extrema like maxima or minima.
- Knowing all the critical points and their nature tells alot about the function.
- Critical points are physically relevant. Examples are configurations with lowest energy).

A CURIOUS OBSERVATION: (The island theorem) Let $f(x, y)$ be the height on an island. Assume there are only finitely many critical points on the island and all of them have nonzero determinant. Label each critical point with a $+1$ "charge" if it is a maximum or minium, and with -1 "charge" if it is a saddle point. Sum up all the charges and you will get 1, independent of the function. This property is an example of an "index theorem", a prototype for important theorems in physics and mathematics.



CRITICAL POINTS IN PHYSICS. (informal) Most physical laws are based on the principle that the equations are critical points of a functional (in general in infinite dimensions).
- **Newton equations**. (Classical mechanics) A particle of mass m moving in a field V along a path $\gamma : t \mapsto r(t)$ extremizes the integral $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\ dt$. Critical points γ satisfy the Newton equations $mr''(t)/2 - \nabla V(r(t)) = 0$.
- **Maxwell equations**. (Electromagnetism) The electromagnetic field (E, B) extremizes the Integral $S(E, B) = \frac{1}{8\pi}\int(E^2 - B^2)\ dV$ over space time. Critical points are described by the the Maxwell equations in vacuum.
- **Einstein equations** (General relativity) If g is a dot product which depends on space and time, and R is the "curvature" of the corresponding curved space time, then $S(g) = \int\int_R dV(g)$ is a function of g for which critical points g satisfy the Einstein equations in general relativity.

OTHER WAYS TO FIND CRITICAL POINTS. Some ideas: walk in the direction of the gradient until you reach a local maximum or walk backwards to reach a local minima. To find saddle points, consider the shortest path connecting two local minima and take the maximum along this path.

The Mathematician Charles W. Cobb and the economist Paul H. Douglas (picture) found in 1928 empirically a formula $F(L,K) = bL^\alpha K^\beta$ which gives the total production F of an economic system as a function of the amount of labor L and the capital investment K.

By fitting data, they got $b = 1.01, \alpha = 0.75, \beta = 0.25$. By rescaling the production unit we can get $b = 1$ and work with the formula:

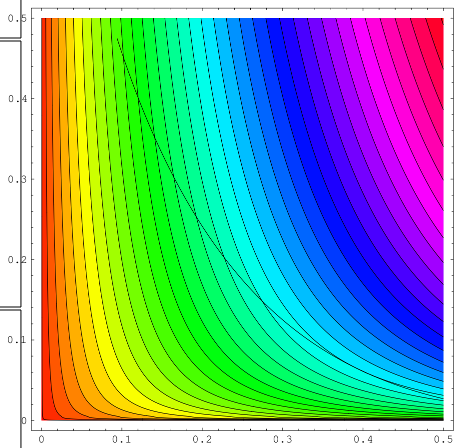
$$F(L,K) = L^{3/4}K^{1/4}$$

Assume that the labor and capital investment are bound by the constraint $G(L,K) = L^{3/4} + K^{1/4} = 1$. Where is the production P maximal under this constraint?

$\nabla F(L,K) =$

$\nabla G(L,K) =$

Solve: $\nabla F(L,K) = \lambda \nabla G(L,K), G(L,K) = 1$:



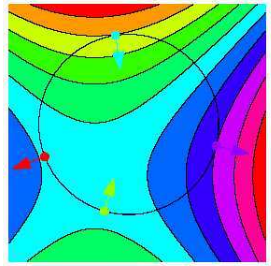
Lecture 15: 3/15/2004, CONSTRAINED EXTREMA **Math21a, O. Knill**

Section 11.7: 32 e,f, 40, Section 11.8: 2,6

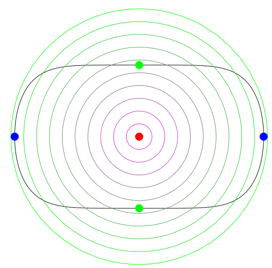
CONSTRAINED EXTREMA. Given a function $f(x,y)$ of two variables and a level curve $g(x,y)=c$. Find the extrema of f on the curve. You see that at places, where the gradient of f is not parallel to the gradient of g, the function f changes when we change position on the curve $g=c$. Therefore we must have a solution of three equations

$$\boxed{\nabla f(x,y)=\lambda \nabla g(x,y), g(x,y)=c}$$

to the three unknowns (x,y,λ). (Additionally: check points with $\nabla g(x,y)=(0,0)$). The constant λ is called the **Lagrange multiplier**. The equations obtained from the gradients are called **Lagrange equations**.



EXAMPLE. To find the shortest distance from the origin to the curve $x^6+3y^2=1$, we extremize $f(x,y)=x^2+y^2$ under the constraint $g(x,y)=x^6+3y^2-1=0$.
SOLUTION.
$\nabla f=(2x,2y), \nabla g=(6x^5,6y)$. $\nabla f=\lambda\nabla g$ gives the system $2x=\lambda 6x^5, 2y=\lambda 6y, x^6+3y^2-1=0$.
$\lambda=1/3, x=x^5$, so that either $x=0$ or 1 or -1. From the constraint equation, we obtain $y=\sqrt{(1-x^6)/3}$. So, we have the solutions $(0,\pm\sqrt{1/3})$ and $(1,0),(-1,0)$. To see which is the minimum, just evaluate f on each of the points. We see that $(0,\pm\sqrt{1/3})$ are the minima.



HIGHER DIMENSIONS. The above constrained extrema problem works also in more dimensions. For example, if $f(x,y,z)$ is a function of three variables and $g(x,y,z)=c$ is a surface, we solve the system of 4 equations

$$\boxed{\nabla f(x,y,z)=\lambda\nabla g(x,y,z), g(x,y)=c}$$

to the 4 unknowns (x,y,z,λ). In n dimensions, we have $n+1$ equations and $n+1$ unknowns $(x_1,\dots,x_n,\lambda)$.

EXAMPLE. Extrema of $f(x,y,z)=z$ on the sphere $g(x,y,z)=x^2+y^2+z^2=1$ are obtained by calculating the gradients $\nabla f(x,y,z)=(0,0,1)$, $\nabla g(x,y,z)=(2x,2y,2z)$ and solving $(0,0,1)=\nabla f=\lambda\nabla g=(2\lambda x,2\lambda y,2\lambda z), x^2+y^2+z^2=1$. $\lambda=0$ is excluded by the third equation $1=2\lambda z$ so that the first two equations $2\lambda x=0, 2\lambda y=0$ give $x=0,y=0$. The 4'th equation gives $z=1$ or $z=-1$. The extrema are the north pole $(0,0,1)$ (maximum) and the south pole $(0,0,-1)$ (minimum).

THE PRINCIPLE OF MAXIMAL ENTROPY. Consider a dice showing i with probability p_i, where $i=1,...,6$. The **entropy** of the probability distribution is defined as $S(\vec{p})=-\sum_{i=1}^6 p_i\log(p_i)$. Find the distribution p which maximizes entropy under the constrained $g(\vec{p})=\sum_{i=1}^6 p_i=1$.

SOLUTION: $\nabla f=(-1-\log(p_1),\dots,-1-\log(p_n))$, $\nabla g=(1,\dots,1)$. The Lagrange equations are $-1-\log(p_i)=\lambda, p_1+...+p_6=1$, from which we get $p_i=e^{-(\lambda+1)}$. The last equation $1=\sum_i\exp(-(\lambda+1))=6\exp(-(\lambda+1))$ fixes $\lambda=-\log(1/6)-1$ so that $p_i=1/6$. The distribution, where each event has the same probability is the distribution of maximal entropy.

REMARK. Maximal entropy means **least information content**. A dice which is fixed (asymmetric weight distribution for example) allows a cheating gambler to gain profit. Cheating through asymmetric weight distributions can be avoided by making the dices transparent.

THE PRINCIPLE OF MINIMAL FREE ENERGY. Assume that the probability that a system is in a state i is p_i and that the energy of the state i is E_i. By a fundamental principle, nature tries to minimize the **free energy** $f(p_1,...,p_n)=-\sum_i(p_i\log(p_i)-E_ip_i)$ when the energies E_i are fixed. The **free energy** is the difference of the **entropy** $S(p)=-\sum_i p_i\log(p_i)$ and the **energy** $E(p)=\sum_i E_ip_i$. The probability distribution p_i satisfying $\sum_i p_i=1$ minimizing the free energy is called the **Gibbs distribution**.

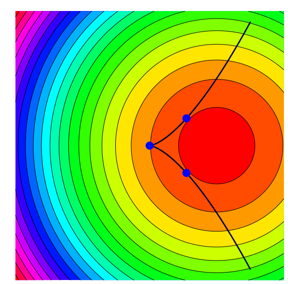
SOLUTION: $\nabla f=(-1-\log(p_1)-E_1,\dots,-1-\log(p_n)-E_n)$, $\nabla g=(1,\dots,1)$. The Lagrange equation are $\log(p_i)=-1-\lambda-E_i$, or $p_i=\exp(-E_i)C$, where $C=\exp(-1-\lambda)$. The additional equation $p_1+\dots+p_n=1$ gives $C(\sum_i\exp(-E_i))=1$ so that $C=1/(\sum_i e^{-E_i})$. The Gibbs solution is $p_i=\exp(-E_i)/\sum_i\exp(-E_i)$. For example, if $E_i=E$, then the Gibbs distribution is the uniform distribution $p_i=1/n$.

MOST ECONOMIC ALUMINUM CAN. You manufacture cylindrical soda cans of height h and radius r. You want for a fixed volume $V(r,h)=h\pi r^2=1$ a minimal surface area $A(r,h)=2\pi rh+2\pi r^2$. With $x=h\pi, y=r$, you need to optimize $f(x,y)=2xy+2\pi y^2$ under the constrained $g(x,y)=xy^2=1$. Calculate $\nabla f(x,y)=(2y,2x+4\pi y), \nabla g(x,y)=(y^2,2xy)$. The task is to solve $2y=\lambda y^2, 2x+4\pi y=\lambda 2xy, xy^2=1$. The first equation gives $y\lambda=2$. Putting that in the second one gives $2x+4\pi y=4x$ or $2\pi y=x$. The third equation finally reveals $2\pi y^3=1$ or $y=1/(2\pi)^{1/3}, x=2\pi(2\pi)^{1/3}$. This means $h=0.54.., r=2h=1.08$
REMARK. Other factors can influence the shape also. For example, the can has to withstand pressure forces up to 100 psi.

WHERE DO CONSTRAINED EXTREMAL PROBLEMS OCCUR. (informal)

- Constraints occur at boundaries. If we want to maximize a function over a region, we have also to look at the extrema at the boundary, where the gradient not necessarily vanishes.
- In physics, we often have conserved quantities (like for example energy, or momentum, or angular momentum), constraints occur naturally.
- In economical contexts, one often wants to optimize things under constraints.
- In mechanics constraints occur naturally (i.e. for robot arms).
- Constrained optimization is important in statistical mechanics, where equilibria have to be obtained under constraints (i.e. constant energy).
- Probability distributions are solutions of constrained problems. For example, the Gaussian distribution (normal distribution) is the distribution on the line with maximal entropy.
- Eigenvalue problems in linear algebra can be interpreted as constrained problems. Maximizing $u\cdot Lu$ under the condition $|u|^2=1$ gives the equation $Lu=\lambda u$ and u has to be an eigenvector. The Lagrange multiplier is an eigenvalue.

DID WE GET ALL EXTREMA? The Lagrange equations $\nabla f(x,y)=\lambda\nabla g(x,y), g(x,y)=c$ do not give all the extrema. We can also have $\nabla g(x,y)=(0,0)$. The parallelity condition also could have been written as $\lambda\nabla f(x,y)=\nabla g(x,y)$, $f(x,y)=x^2+(y-1)^2, g(x,y)=x^2-y^3$. The function f has a local maximum 1 at $(0,0)$ under the constraint $g(x,y)=0$ but the Lagrange equations do not find it. The problem is that the gradient of g vanishes. ∇g is technically parallel to ∇f but there is no λ such that $\nabla f=\lambda\nabla g$ at this point. The reason for this "mistake" (which is present in virtually all calculus text books), is that parallelity of the two gradient is not equivalent to $\nabla f=\lambda\nabla g$ but can also mean $\lambda\nabla f=\nabla g$ with $\lambda=0$.



CAN WE AVOID LAGRANGE? We could extremize $f(x,y)$ under the constraint $g(x,y)=0$ by finding $y=y(x)$ from the later and extremizing the 1D problem $f(x,y(x))$.

EXAMPLE 1. To extremize $f(x,y)=x^2+y^2$ with constraint $g(x,y)=x^4+3y^2-1=0$, solve $y^2=(1-x^4)/3$ and minimize $h(x)=f(x,y(x))=x^2+(1-x^4)/3$. $h'(x)=0$ gives $x=0$. The find the maximum $(\pm1,0)$, we had to maximize $h(x)$ on $[-1,1]$, which occurs at ±1.

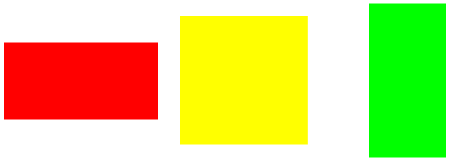
Sometimes, the Lagrange method can be avoided.

EXAMPLE 2. Extremize $f(x,y)=x^2+y^2$ under the constraint $g(x,y)=p(x)+p(y)=1$, where p is a complicated function in x which satisfies $p(0)=0, p'(1)=2$ The Lagrange equations $2x=\lambda p'(x), 2y=\lambda p'(y), p(x)+p(y)=1$ can be solved: with $x=0, y=1, \lambda=1$, however, we can not solve $g(x,y)=1$ for y. Substitution fails:
In general, the Lagrange method is more powerful.

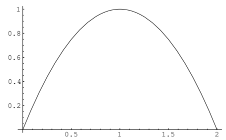
Lecture 16: 4/17/2004, MORE ON EXTREMA **Math21a, O. Knill**

HOMEWORK: Section 11.8: 10,20,24,42

MAXIMAL AREA OF RECTANGLE. We want to extremize the area of a rectangle for which the length of the boundary is fixed 4. If the sides are x and y, then we want to extremize $f(x,y) = xy$ under the constraint $g(x,y) = 2x + 2y = 4$. The Lagrange equations $y = 2\lambda, x = 2\lambda$ show that $x = y$ and so $x = y = 1$.



The last problem could also been solved by substituting $y = 2 - x$ into the area formula $A = xy = x(2-x)$ leading to a one-dimensional extremal problem: maximize $f(x) = x(2-x)$ on the interval $[0,2]$. To do so, we have to find the extrema inside the interval and then consider also the boundary points $x = 0, x = 2$. Again, we get $x = 1$.



VOLUME OF CUBE. Extremize the volume $f(x,y,z) = xyz$ of a box with fixed surface area $xy + yz + xz = 3$. To solve $yz = \lambda(y+z), xz = \lambda(x+z), xy = \lambda(x+y), xy + yz + xz = 1$, take quotients: $z/x = (y+z)/(y+x), z/y = (z+x)/(y+x)$ which gives $z(y+x) = x(y+z), z(y+x) = y(z+x)$ so that either $xz = yz$ or $z = 0$. Similarly, we get $y = z$ or $y = 0$. The solution is $x = y = z = 1$.

AN OTHER SOLUTION. For a solution without Lagrange multipliers, we would plug in $z = (1-xy)/(y+x)$ and try to find the maximum of $f(x,y) = xy(1-xy)/(y+x)$ on the domain $D = \{x > 0, y > 0, xy \leq 1\}$.

We first would have to find critical points inside the region D:

$$f_x(x,y) = -y(1-2xy)/(x+y) - xy(1-xy)/(x+y)^2 = 0$$

$$f_y(x,y) = -x(1-2xy)/(x+y) - xy(1-xy)/(x+y)^2 = 0$$

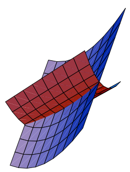
The difference of these two equations gives $(x-y)(1-2xy) = 0$ so that either $x = y$ or $xy = 1/2$. The second case can not give us $f_x = f_y = 0$. The first condition $x = y$ gives $x = y = 1$ which is not inside the region. However, on the boundary $g(x,y) = xy = 1$, the Lagrange equations $\nabla f = \lambda \nabla g$ have a solution with $(x,y) = (1,1)$.

The example illustrates the power of Lagrange multipliers. The substitution method is more complicated.

TWO CONSTRAINTS. (informal) The calculation with Lagrange multipliers can be generalized: if the goal is to optimize a function $f(x,y,z)$ under the constraints $g(x,y,z) = c$, $h(x,y,z) = d$, take the Lagrange equations

$$\boxed{\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d}$$

which are 5 equations for the 5 unknowns (x,y,z,λ,μ). Geometrically the gradient of f is in the plane spanned by the gradients of g and h. (This is the plane orthogonal to the curve $\{g = c, h = d\}$.)

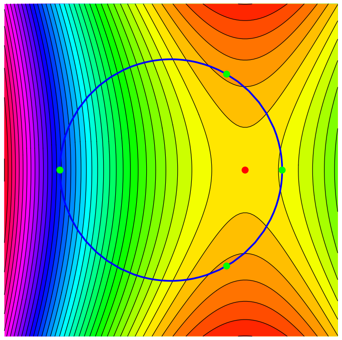


GENERAL PROBLEM. Given a region G whose boundary is given by $g(x,y) = c$. The task to maximize or minimize $f(x,y)$ on G has the following steps:
I) Find extrema inside the region: compute critical points $\nabla f = (0,0)$ and classify them using the second derivative test.
II) Find extrema on the boundary using Lagrange: $\nabla f = \lambda \nabla g, g = c$.
III) Compare the values of the functions obtained in I) and II) to find the maximum or minimum.

EXAMPLE. Extremize $f(x,y) = 3x^2 - 4x - y^2$ on the disc $x^2 + y^2 \leq 1$.
I) Inside the disc. There is only one critical point $(2/3, 0)$. The discriminant D is -6 so that $(2/3,0)$ is a saddle point.
II) On the boundary solve $6x - 4 = 2\lambda x, -2y = 2\lambda y$. There are four solutions: $(1/2, -\sqrt{3}/2), (1/2, +\sqrt{3}/2), (1,0), (-1,0)$.
III) A list of all candidates:

(x,y)	f(x,y)
$(2/3,0)$	-4/3
$(1/2,-\sqrt{3}/2)$	-2
$(1/2,\sqrt{3}/2)$	-2
$(1,0)$	-1
$(-1,0)$	7

reveals that $(-1,0)$ is the maximum and $(1/2, -\pm\sqrt{3}/2)$ are minima.



IN MATHEMATICA.
Here is how a machine solves the above problem. After defining the functions f and g, the machine solves first the equations leading to critical points, and then the Lagrange equations (we put $L = \lambda$).

```
f[x_, y_] := 3x^2 - 4x - y^2
g[x_, y_] := x^2 + y^2 - 1
Solve[{D[f[x, y], x] == 0, D[f[x, y], y] == 0}, {x, y}]
Solve[{D[f[x, y], x] == L * D[g[x, y], x], D[f[x, y], y] == L * D[g[x, y], y], g[x, y] == 0}, {x, y, L}]
```

TRICKY LAGRANGE PROBLEM. Let p and q be positive constants such that $\frac{1}{p} + \frac{1}{q} = 1$. Use the method of Lagrange Multipliers to prove that for any $x > 0$, $y > 0$, the following inequality is true:

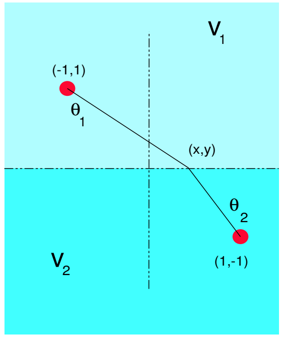
$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

SOLUTION.
We have to find the maximum of $f(x,y) = xy$ ($x > 0$, $y > 0$) under the constraint $\frac{x^p}{p} + \frac{y^q}{q} = c$.)
The Lagrange equations $x = \lambda x^{p-1}, y = \lambda y^{q-1}$ gives $y/x = x^{p-1}/y^{q-1}$ so that $y^q = x^p$.
Plugging this into $x^p/q + y^q = c$ gives $x^p(1/p + 1/q) = c$ or $x = c^{1/p}$ and so $y = c^{1/q}$. The maximal value of $f(x,y) = xy$ is $c^{1/p}c^{1/q} = c^1 = c$. Therefore, everywhere

$$xy = f(x,y) \leq c = x^p/p + y^q/q \, .$$

SNELLS LAW of refraction is the problem to determine the fastest path between two points, if the path crosses a border of two media and the media have different indices of refraction. The law can be derived from Lagrange:

PROBLEM. A light ray travels from $A = (-1,1)$ to the point $B = (1,-1)$ crossing a boundary between two media (air and water). In the air ($y > 0$) the speed of the ray is $v_1 = 1$ (in units of speed of light). In the second medium ($y < 0$) the speed of light is $v_2 = 0.9$. The light ray travels on a straight line from A to a point $P = (x,0)$ on the boundary and on a straight line from P to B. Verify Snell's law of refraction $\sin(\theta_1)/\sin(\theta_2) = v_1/v_2$, where θ_1 is the angle the ray makes in air with the y axis and where θ_2 is the angle, the ray makes in water with the y axis.

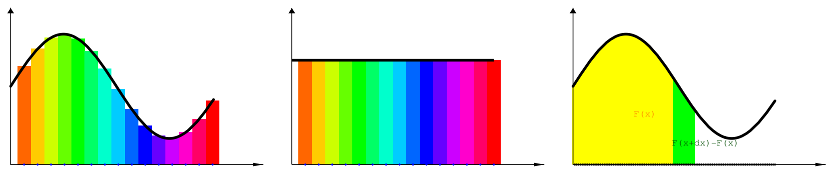


SOLUTION. Minimize $f(x,y) = \sqrt{(-1-x)^2 + y^2}/v_1 + \sqrt{(1-x)^2 + y^2}/v_2 = l_1/v_1 + l_2/v_2$ under the constraint $G(x,y) = y = 0$. The Lagrange equations show that $f_x(x,y) = 0$. This is already Snells law because $f_x = v_1 2(x+1)/(2l_1) + v_2 2(1-x)/(2l_2) = 0$ means $v_1/v_2 = \sin(\theta_1)/\sin(\theta_2)$. If v_1 is larger, then θ_1 is larger.

Lecture 17: 3/19/2004, 2D INTEGRALS **Math21a, O. Knill**

HOMEWORK. Section 12.1: 16,26,30,44. Section 12.4: 44

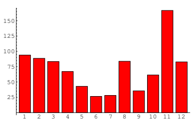
1D INTEGRATION IN 100 WORDS. If $f(x)$ is a continuous function then $\int_a^b f(x)\ dx$ can be defined as a limit of the **Riemann sum** $f_n(x) = \frac{1}{n}\sum_{x_k \in [a,b]} f(x_k)$ for $n \to \infty$ with $x_k = k/n$. This integral divided by $|b-a|$ is the **average** of f on $[a,b]$. The integral can be interpreted as an **signed area** under the graph of f. If $f(x) = 1$, the integral is the **length** of the interval. The function $F(x) = \int_a^x f(y)\ dy$ is called an **anti-derivative** of f. The **fundamental theorem of calculus** states $F'(x) = f(x)$. Unlike the derivative, anti-derivatives can not always be expressed in terms of known functions. An example is: $F(x) = \int_0^x e^{-t^2}\ dt$. Often, the anti-derivative can be found: Example: $f(x) = \cos^2(x) = (\cos(2x)+1)/2, F(x) = x/2 - \sin(2x)/4$.



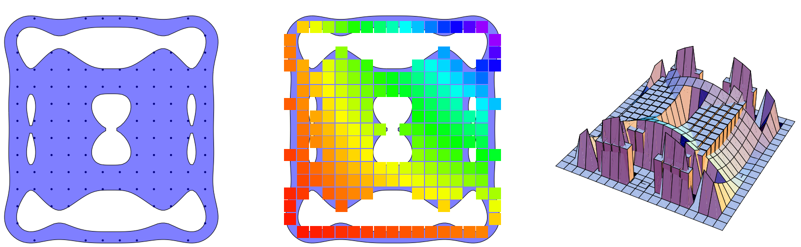
AVERAGES=MEAN. www.worldclimate.com gives the following data for the average monthly rainfall (in mm) for Cambridge, MA, USA (42.38 North 71.11 West,18m Height).

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
93.9	88.6	83.3	67.0	42.9	26.4	27.9	83.8	35.5	61.4	166.8	82.8

The average $860.3/12 = 71.7$ is a Rieman sum integral.



2D INTEGRATION. If $f(x,y)$ is a continuous function of two variables on a region R, the integral $\int_R f(x,y)\ dxdy$ can be defined as the limit $\frac{1}{n^2}\sum_{i,j,x_{i,j}\in R} f(x_i,y_j)$ with $x_{i,j} = (i/n, j/n)$ when n goes to infinity. If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral divided by the area of R is the **average** value of f on R. For many regions, the integral can be calculated as a **double integral** $\int_a^b [\int_{c(x)}^{d(x)} f(x,y)\ dy]dx$. In general, the region must be split into pieces, then integrated seperately.



One can interpret $\int\int_R f(x,y)\ dydx$ as the volume of solid below the graph of f and above R in the $x-y$ plane. (As in 1D integration, the volume of the solid below the x-y plane is counted negatively).

EXAMPLE. Calculate $\int\int_R f(x,y)\ dxdy$, where $f(x,y) = 4x^2y^3$ and where R is the rectangle $[0,1]\times[0,2]$.

$$\int_0^1 [\int_0^2 4x^2y^3\ dy]\ dx = \int_0^1 [x^2y^4|_0^2]\ dx = \int_0^1 x^2(16-0)\ dx = 16x^3/3|_0^1 = \frac{16}{3}\ .$$

FUBINI'S THEOREM. $\boxed{\int_a^b\int_c^d f(x,y)\ dxdy = \int_c^d\int_a^b f(y,x)\ dydx.}$

TYPES OF REGIONS.

$\int\int_R f\ \ dA\ = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ \ dydx$
type I region.

$\int\int_R f\ \ dA\ = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\ \ dxdy$
type II region.

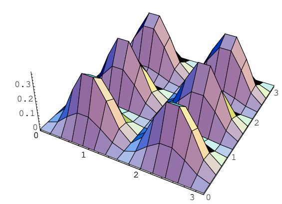
EXAMPLE. Let R be the triangle $1 \geq x \geq 0, 1 \geq y \geq 0, y \leq x$. Calculate $\int\int_R e^{-x^2}\ dxdy$.
ATTEMPT. $\int_0^1 [\int_y^1 e^{-x^2}\ dx]dy$. We can not solve the inner integral because e^{-x^2} has no anti-derivative in terms of elementary functions.
IDEA. Switch order: $\int_0^1 [\int_0^x e^{-x^2}\ dy]\ dx = \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316...$.
A special case of switching the order of integration is **Fubini's theorem.**

If you can't solve a double integral, try to change the order of integration!

QUANTUM MECHANICS. In quantum mechanics, the motion of a particle (like an electron) in the plane is determined by a function $u(x,y)$, the wave function. Unlike in classical mechanics, the position of a particle is given in a probabilistic way only. If R is a region and u is normalized so that $\int |u|^2\ dxdy = 1$, then $\int_R |u(x,y)|^2\ dxdy$ is the **probability**, that the particle is in R.

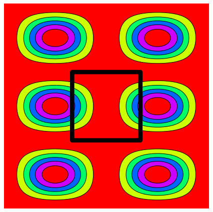
EXAMPLE. Unlike a classical particle, a quantum particle in a box $[0,\pi]\times[0,\pi]$ can have a discrete set of energies only. This is the reason for the name "quantum". If $-(u_{xx}+u_{yy}) = \lambda u$, then a particle of mass m has the energy $E = \lambda\hbar^2/2m$. A function $u(x,y) = \sin(kx)\sin(ny)$ represents a particle of energy $(k^2+n^2)\hbar^2/(2m)$. Let us assume $k=2$ and $n=3$ from now on. Our aim is to find the probability that the particle with energy $13\hbar^2/(2m)$ is in the middle 9'th $R = [\pi/3, 2\pi/3]\times[\pi/3,2\pi/3]$ of the box.
SOLUTION: We first have to normalize $u^2(x,y) = \sin^2(2x)\sin^2(3y)$, so that the average over the whole square is 1:

$$A = \int_0^\pi \int_0^\pi \sin^2(2x)\sin^2(3y)\ dxdy\ .$$



To calculate this integral, we first determine the inner integral $\int_0^\pi \sin^2(2x)\sin^2(3y)\ dx = \sin^2(3y)\int_0^\pi \sin^2(2x)\ dx = \frac{\pi}{2}\sin^2(3y)$ (the factor $\sin^2(3y)$ is treated as a constant). Now, $A = \int_0^\pi (\pi/2)\sin^2(3y)\ dy = \frac{\pi^2}{4}$, so that the **probability amplitude function** is $f(x,y) = \frac{4}{\pi^2}\sin^2(2x)\sin^2(3y)$.

The probability that the particle is in R is slightly smaller than 1/9:



$$\begin{aligned}\frac{1}{A}\int_R f(x,y)\ dxdy &= \frac{4}{\pi^2}\int_{\pi/3}^{2\pi/3}\int_{\pi/3}^{2\pi/3} \sin^2(2x)\sin^2(3y)\ dxdy \\ &= \frac{4}{\pi^2}(4x-\sin(4x))/8|_{\pi/3}^{2\pi/3}(6x-\sin(6x))/12|_{\pi/3}^{2\pi/3} \\ &= 1/9 - 1/(4\sqrt{3}\pi)\end{aligned}$$

The probability is slightly smaller than 1/9.

WHERE DO DOUBLE INTEGRALS OCCUR?
- compute areas.
- compute averages. Examples: average rain fall or average population in some area.
- probabilities. Expectation of random variables.
- quantum mechanics: probability of particle being in a region. - find moment of inertia $\int\int_R (x^2+y^2)\rho(x,y)dxdy$
- find center of mass $(\int\int_R x\rho(x,y)\ dxdy/M, \int\int_R y\rho(x,y)\ dxdy/M)$, with $M = \int\int_R dxdy$.
- compute some 1D integrals.

TRIPLE INTEGRALS are defined similarly and covered later in detail. Fubinis theorem generalizes. $\int\int\int_R 1\ dxdydz$ is a **volume**.

Lecture 18: 7/22/2003, APPLICATIONS **Math21a, O. Knill**

HOMEWORK. Section 12.2: 26,36,44, Section 12.5: 14

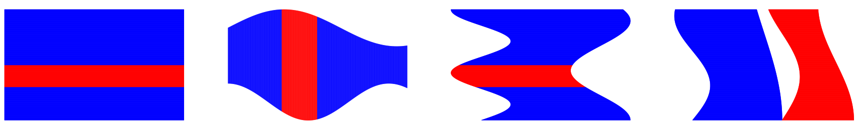
THINGS TO KEEP IN MIND.

- Double integrals can often be evaluated through iterated integrals

$\boxed{\text{"Integrals have layers"}}$.

- $\iint_R 1 \; dxdy = \iint 1 \; dA$ is the **area** of the region R.
- $\iint_R f(x,y) \; dxdy$ is the volume of the solid having the graph of f as the "roof" and the R in the $xy-$ plane as the "floor".

TYPES OF REGIONS.
$\int \int_R f \; dA = \int_a^b \int_c^d f(x,y) \; dydx$ **rectangle**.
$\int \int_R f \; dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \; dydx$ **type I region**.
$\int \int_R f \; dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y) \; dxdy$ **type II region**.
A general region, we try to cut it into pieces, where each piece is a Type I or Type II region.



Rectangle   Type I   Type II   To cut

AREA $A = \iint_R 1 \; dA$
MASS $M = \iint_R \rho(x,y) \; dA$
AVERAGE $\iint_R f(x,y) \; dA/A$.
CENTROID $(\iint_R x \; dA/A, \iint_R y \; dA/A)$.
CTR MASS $(\iint_R x\rho(x,y)dA, \iint_R y\rho(x,y)dA)/M$.
MOMENT OF INERTIA $I = \iint_R (x^2+y^2) \; dA$
RADIUS OF GYRATION $\sqrt{I/M}$

VOLUME $V = \iiint_R 1 \; dV$
MASS $M = \iint_R \rho(x,y,z) \; dV$
AVERAGE $\iiint_R f(x,y,z) \; dV/V$.
CENTROID $(\iiint_R x \; dV/V, \iiint_R y \; dV/V, \iiint_R z \; dV/V)$.
C.O.M. $(\iiint_R x\rho dV, \iiint_R x\rho dV, \iiint_R y\rho dV)/M$.
MOMENT OF INERTIA $I = \iiint_R (x^2+y^2) \; dV$
RADIUS OF GYRATION $\sqrt{I/M}$

AREA OF CIRCLE. To compute the area of the circle of radius r, we integrate

$$A = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \; dydx \; .$$

The inner integral is $2\sqrt{r^2-x^2}$ so that

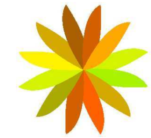
$$A = \int_{-r}^{r} 2\sqrt{r^2-x^2} \; dx \; .$$

This can be solved with a substitution: $x = r\sin(u), dx = r\cos(u)$. With the new bounds $a = -\pi/2, b = \pi/2$ and $\sqrt{r^2-x^2} = \sqrt{r^2 - r^2\sin^2(u)} = r\cos(u)$ we end up with

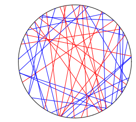
$$A = \int_{-\pi/2}^{\pi/2} 2r^2\cos^2(u) \; du = \int_{-\pi/2}^{\pi/2} r^2(1+\cos(2u)) \; du = r^2\pi \; .$$

MOMENT OF INERTIA. Compute the kinetic energy of a square iron plate $R = [-1,1]\times[-1,1]$ of density $\rho = 1$ (about 10cm thick) rotating around its center with a $6'000 rpm$ (rounds per minute). The angular velocity speed is $\omega = 2\pi \; 6'000/60 = 100 \; 2\pi$. Because $E = \int\int_R (r\omega)^2/2 \; dxdy$, where $r = \sqrt{x^2+y^2}$, we have $\boxed{E = \omega^2 I/2}$, where $I = \int\int_R (x^2+y^2) \; dxdy$ is the **moment of inertia**. For the square, $I = 8/3$. Its energy of the plate is $\omega^2 8/6 = 4\pi^2 100^2 8/6 Joule \sim 0.86 KWh$. You can run with this energy a 60 Watt bulb for 14 hours.

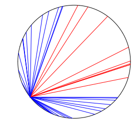
A FLOWER. What is $\int\int_R x^2+y^2 \; dxdy$, where R is a flower obtained by rotating the region enclosed by the curves $y = x^2$ and $y = 2x - x^2$ by adding multiples of the angles $2\pi/12$?
SOLUTION. The moment of inertia of all the petals add up: $I = 12\int_0^1 \int_{x^2}^{2x-x^2} (x^2+y^2) \; dydx = \int_0^1 [x^2y + y^3/3]_{y=x^2}^{y=2x-x^2} \; dx = 1243/210 = 86/35$.



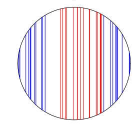
PROBLEM: BERTRAND'S PARADOX (Bertrand 1889)
We throw randomly lines onto the disc. What is the probability that the intersection with the disc is larger than the length $\sqrt{3}$ of the equilateral triangle inscribed in the unit circle?



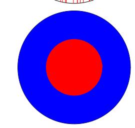
Answer Nr 1: take an arbitrary point P in the disc. The set of lines which pass through that point is parametrized by an angle ϕ. In order that the chord is longer than $\sqrt{3}$, the angle has to fall within an angle of 60° of a total of 180°. The probability is $\boxed{1/3}$.



Answer Nr 2: consider all lines perpendicular to a fixed diameter. The chord is longer than $\sqrt{3}$, when the point of intersection is located on the middle half of the diameter. The probability is $\boxed{1/2}$.

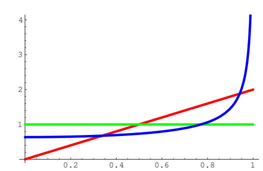


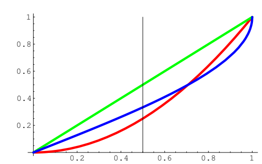
Answer Nr. 3: if the midpoint of the intersection with the disc is located in the disc of radius $1/2$ with area $\pi/4$, then the chord is longer than $\sqrt{3}$. The probability is $\boxed{1/4}$.



The paradox comes from the choice of the probability density function $f(x,y)$. In each case, there is a distribution function $f(x,y)$ which is radially symmetric.
The constant distribution $\boxed{f(x,y) = 1/\pi}$ is obtained when we throw the center of the line into the disc. The disc A_r of radius r has probability r^2/π. The density in the r direction is $2r$.
The distribution $\boxed{f(x,y) = 1/r}$ is obtained when throwing parallel lines. This will put more weight to center. The disc A_r of radius r has probability of A_r is bigger than the area of A_r. The density in the r direction is constant equal to 1.
Lets compute the distribution when we rotate the line around a point at the boundary. We hit a disc A_r of radius r with probability $F(r) = \arcsin(r)2/\pi$. The density in the radial direction is $\boxed{f(r) = 2/(\pi\sqrt{(1-r^2)})}$.

COMPARISON OF THE DENSITY FUNCTIONS. A plot of the radial distribution functions $f(r)$ as well as $F(r)$, the probability of A_r shows why we get different results for $F(1/2)$.





What happens if we **really** do an experiment and throw randomly lines onto a disc? The outcome of the experiment will depend on how the experiment will be performed. If we would do the experiment by hand, we would probably try to throw the center of the stick into the middle of the disc. Since we would aim to the center, the distribution would probably be different from any of the three solutions discussed above.

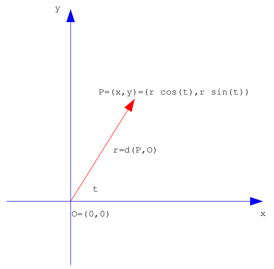
STATISTICS. If $f(x,y)$ is a probability distribution on R: $f(x,y) \geq 0$, $\iint_{\mathbf{R}} f(x,y) \; dA = 1$, then $\mathrm{E}[X] = \int X(x,y)f(x,y) \; dA$ is called the **expectation** of X, $\mathrm{Var}(X) = \mathrm{E}[(X - \mathrm{E}[X])^2]$ is called the **variance** and $\sigma(X) = \sqrt{(\mathrm{Var}[X])}$ the **standard deviation**.

Lecture 19: 3/24/2004, POLAR INTEGRATION — Math21a, O. Knill

HOMEWORK. Section 12.3: 10,20,32,42

POLAR COORDINATES. A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \arctg(y/x)$. We have $x = r\cos(\theta)$, $y = r\sin(\theta)$.

Footnote: Note that $\theta = \arctg(y/x)$ defines the angle θ only up to an addition of π. The points (x, y) and $(-x, -y)$ would have the same θ. In order to get the correct θ, one could take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$ as Mathematica does, where $\pi/2$ is the value when $y/x = \infty$, and add π if $x < 0$ or $x = 0, y < 0$. In Mathematica, you can get the polar coordinates with $(r, \theta) = (\mathrm{Abs}[x + Iy], \mathrm{Arg}[x + Iy])$.



POLAR CURVES. A general polar curve is written as $(r(t), \theta(t))$. It can be translated into x, y coordinates: $x(t) = r(t)\cos(\theta(t)), y(t) = r(t)\sin(\theta(t))$.
POLAR GRAPHS. Curves which are graphs when written in polar coordinates are called **polar graphs**.

EXAMPLE. $r(\theta) = \cos(3\theta)$ is the which belongs to the class of **roses** $r(t) = \cos(nt)$.

EXAMPLE. If $y = 2x + 3$ is a line, then the equation gives $r\sin(\theta) = 2r\cos(\theta) + 3$. Solving for $r(t)$ gives $r(\theta) = 3/(\sin(\theta - 2\cos(\theta))$. The line is also a polar graph.

EXAMPLE. The polar form $r(\theta) = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta)}$ of the ellipse (see Kepler). The ellipse is a polar graph.

POLAR COORDINATES. For many regions, it is better to use polar coordinates for integration:

$$\boxed{\int\int_R f(x, y)\; dxdy = \int\int_R g(r, \theta) r\; drd\theta}$$

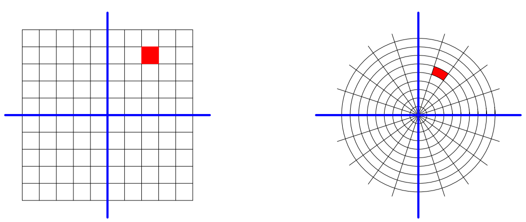
For example if $f(x, y) = x^2 + x^2 + xy$, then $g(r, \theta) = r^2 + r^2\cos(\theta)\sin(\theta)$.

EXAMPLE. We had computed area of the disc $\{x^2 + y^2 \leq 1\}$ using substitution. Was quite a mess. It is easier to do that integral in polar coordinates:

$$\int_0^{2\pi}\int_0^1 r\; drd\theta = 2\pi r^2/2|_0^1 = \pi\;.$$

WHERE DOES THE FACTOR "r" COME FROM?

A small rectangle R with dimensions $d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x, y) plane. It has approximately the area $rd\theta dr$. It is small for small r.

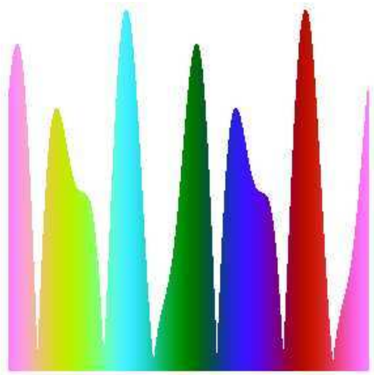


AN OTHER EXPLANATION (for people who have seen some linear algebra). The map translating from polar coordinates to Cartesian coordinates $(x, y) = T(r, \theta) = (r\cos(\theta), r\sin(\theta))$ has as a linear approximation the matrix

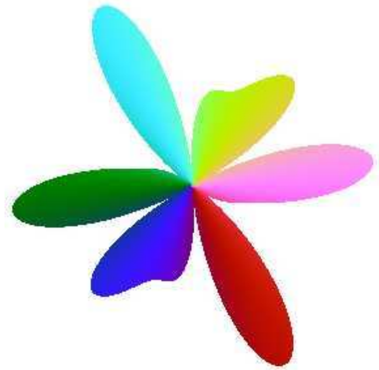
$$DT(x, y) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}$$

which has determinant r. A small rectangle in the (r, θ)-plane of area dA will have in the (x, y) plane the area rdA.

ROSES. We can now integrate over type I or type II regions in the (θ, r) plane. Examples are **roses**: $\{(\theta, r)\ |0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.



The region R in the $\theta - r$ coordinates is a type I region



The region $S = T(R)$ in the $x - y$ coordinates is neither a type I nor a type II region.

EXAMPLE. Find the area of the region $\{(\theta, r(\theta))\ |\ r(\theta) \leq |\cos(3\theta)|\ \}$.

$$\int\int_R y\; dxdy = \int_0^{2\pi}\int_0^{\cos(3\theta)} r\; dr\; d\theta = \int_0^{2\pi}\frac{\cos(3\theta)^2}{2}\; d\theta = \pi/2$$

EXAMPLE. Integrate $f(x, y) = y\sqrt{x^2 + y^2}$ over the semi annulus $R = \{(x, y)\ |\ 1 < x^2 + y^2 < 4, y > 0\ \}$.

Solution.

$$\int_1^2\int_0^{\pi} r\sin(\theta) r\; r\; d\theta dr = \int_1^2 r^3\int_0^{\pi}\sin(\theta)\; d\theta dr = \frac{(2^4 - 1^4)}{4}\int_0^{\pi}\sin(\theta)\; d\theta = 15/2$$

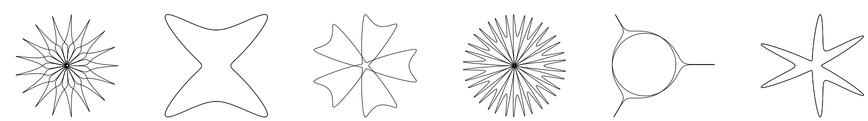
For integration problems, where the region is part of an annulus, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

THE SUPER-CURVE. The Belgian Biologist Johan Gielis came up in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left(\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\cos(\frac{m\phi}{4})|^{n_2}}{b}\right)^{-1/n_3}$$

```
S[m_,n1_,n2_,n3_,a_,b_]:=Module[{},
  r1[t_]:=Abs[Cos[m t/4]/a]^n1;
  r2[t_]:=Abs[Sin[m t/4]/b]^n2;
  R[t_]:=(r1[t]+r2[t])^(1/n3);
  r[t_]:=If[R[t]==0,{0,0},{Cos[t],Sin[t]}/R[t]];
  ParametricPlot[r[t],{t,-2Pi,2Pi},PlotRange->All,
  PlotPoints->1000,Axes->False,AspectRatio->1,Frame->False]
];

RI:=Random[Integer,{3,20}]; RR:=20 Random[];
T:=S[RI,RR,RR,RR,RR,RR]
```

It is called the **super-curve** because it can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes" (see later).
The super-curve generalizes the **superellipse** which had been discussed in 1818 by Lamé and helps to tackle one of the more intractable problems in biology: describing form. A twist: Gielis has patented his discovery described in "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003). To the right you see the Mathematica code.



4/26/2004, 3D INTEGRATION **Math21a, O. Knill**

3D INTEGRATION. If $f(x,y,z)$ is a function of three variables and R is a region in space, then $\int\int\int_R f(x,y,z)\ dxdydz$ is defined as the limit of Riemann sum $\frac{1}{n^3}\sum_{(x_i,y_j,z_k)\in R} f(x_i,y_j,z_k)$ for $n \to \infty$, where $(x_i,y_j,z_k) = (\frac{i}{n},\frac{j}{n},\frac{k}{n})$.

TRIPLE INTEGRALS. As in two dimensions, triple integrals can be evaluated through iterated 1D integrals.

EXAMPLE. Assume R is the box $[0,1]\times[0,1]\times[0,1]$ and $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dx\ dy\ dz .$$

CALCULATION. We start from the core $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then integrate the middle layer: $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally handle the outer layer: $\int_0^1 3x^2 dx$ =1.

WHAT DID WE DO? When we calculate the most inner integral, we fix z and y. The integral is the average of $f(x,y,z)$ along a line intersected with the body. After completing the second integral, we have computed the average on the plane $z = const$ intersected with R. The most outer integral averages all these two dimensional sections.

VOLUME UNDER GRAPH. The volume under the graph of a function $f(x,y)$ and above a region $R = [a,b]\times[c,d]$ is the integral $\int_a^b\int_c^d f(x,y)\ dxdy$. Actually, this is a triple integral:

$$V = \int_a^b\int_c^d\int_0^{f(x,y)} 1\ dzdxdy .$$

An integral of the form $\int_a^b\int_c^d\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\ dzdxdy$ is sometimes also called a **type I** tipple integral.

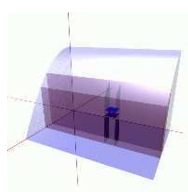
VOLUME OF THE SPHERE. (We will do this more elegantly later). The volume is

$$V = \int\int\int_R dxdydz = \int_{-1}^1 [\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz]dy]dx$$

After computing the inner integral, we have $V = 2\int_{-1}^1[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)^{1/2}dy]dx$.
To resolve the next layer, call $1-x^2 = a^2$. The task is to find $\int_{-a}^a \sqrt{a^2-x^2}\ dx$. Make the substitution $x/a = \sin(u), dx = a\cos(u)$ to write this as $a\int_0^{\arcsin(a/a)}\sqrt{1-\sin^2(u)}a\cos(u)du = a^2\int_0^{\pi/2}\cos^2(u)du = a^2\pi/2$.
We can finish up the last integral

$$V = 2\pi/2\int_{-1}^1(1-x^2)\ dx = 4\pi/3 .$$

MASS OF A BODY. In general, the mass of a body with density $\rho(x,y,z)$ is $\int\int\int_R \rho(x,y,z)\ dV$. For bodies with constant density ρ the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4-x^2$, and the planes $x=0, y=0, y=6, z=0$ if the density of the body is 1.



$$\int_0^2\int_0^6\int_0^{4-x^2} dz\ dy\ dx = \int_0^2\int_0^6 (4-x^2)\ dydx$$
$$= 6\int_0^2(4-x^2)\ dx = 6(4x-x^3/3|_0^2 = 32$$

CENTER OF MASS. Compute the center of mass of the same body. The center of mass is $(24/32, 96/32, 256/180) = (3/4, 3, 8/5)$:

$$\int_0^2\int_0^6\int_0^{4-x^2} x\ dz\ dy\ dx = \int_0^2\int_0^6 x(4-x^2)\ dydx = 6\int_0^2 x(4-x^2)\ dx = 24x^2/2 - 6x^4/4|_0^2 = 24$$

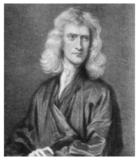
$$\int_0^2\int_0^6\int_0^{4-x^2} y\ dz\ dy\ dx = \int_0^2\int_0^6 y(4-x^2)\ dydx = \int_0^2 18(4-x^2)\ dx = 18(4x-x^3/3)|_0^2 = 96$$

$$\int_0^2\int_0^6\int_0^{4-x^2} z\ dz\ dy\ dx = \int_0^2\int_0^6 (4-x^2)^2/2\ dydx = 6\int_0^2(4-x^2)^2/2\ dx = 3(16x - 8x^3/3 + x^5/5)|_0^2 = 256/5$$

SOME HISTORY OF COMPUTING VOLUMES. How did people come up calculating the volume $\int\int\int_R 1\ dxdydz$ of a body?

Archimedes ((-287)-(-212): Archimedes's method of integration allowed him to find areas, volumes and surface areas in many cases. His method of exhaustion paths the numerical method of integration by Riemann sum. The **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies.

Cavalieri (1598-1647): Cavalieri could determine area and volume using tricks like the **Cavalieri principle**. Example: to get the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z, this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2-r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$.



Newton (1643-1727) and Leibniz(1646-1716): Newton and Leibniz, developed calculus independently. The new tool made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools.

MONTE CARLO COMPUTATIONS. Here is an other way to compute integrals: Suppose we want to calculate the volume of some body R inside the unit cube $[0,1]\times[0,1]\times[0,1]$. The **Monte Carlo method** is to shoot randomly n times onto the unit cube and count the fraction of times, we hit the solid. Here is an experiment with Mathematica and where the body is one eights of the unit ball:

```
R := Random[]; k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k + +], {10000}]; k/10000
```

Assume, we hit 5277 of n=10000 times. The volume so measured is 0.5277. The actual volume of 1/8'th of the sphere is $\pi/6 = 0.524$. For $n \to \infty$ the Monte Carlo computation gives the actual volume.

WHERE CAN TRIPLE INTEGRALS OCCUR?

- Calculation of **volumes** $V = \int\int\int_R 1\ dV$ and **masses** $M = \int\int\int \rho\ dV$.
- Finding **averages** $\int\int\int_R f\ dV / \int\int\int_R 1\ dV$. Examples: average algae concentration in a swimming pool.
- Determining **probabilities**. Example: quantum probability $\int\int\int_R f(x,y,z)^2\ dxdydz$.
- **Moment of inertia** $\int\int\int_R r(x,y,z)^2\rho(x,y,z)\ dV$, where $r(x,y,z)$ is the distance to the axis of rotation.
- **Center of mass** $(\int\int\int_R x\rho\ dV/M, \int\int\int_R y\rho\ dV/M, \int\int\int_R z\rho\ dV/M)$.

3/26/2004, CYLINDRICAL COORDINATES — Math 21a, O. Knill

REMINDER: INTEGRATION POLAR COORDINATES.

$$\int \int_R f(r,\theta) \boxed{r} d\theta dr \; .$$

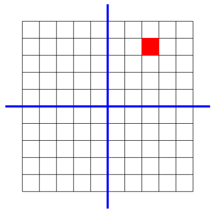
EXAMPLE 1. Area of a disk of radius R

$$\int_0^R \int_0^{2\pi} r \; d\theta dr = 2\pi \frac{r^2}{2}|_0^R = R^2 \pi \; .$$

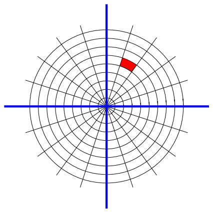
WHERE DOES THE FACTOR "r" COME FROM?

1. EXPLANATION. A small rectangle with dimensions $d\theta dr$ in the (r,θ) plane is mapped to a sector segment in the (x,y) plane. It has approximately the area $r d\theta dr$. It is small for small r.

2. EXPLANATION. The map $(r,\theta) \mapsto (r\cos(\theta, r\sin(\theta))(f(r,\theta), g(r,\theta))$ which changes from Cartesian to polar coordinates has the **Jacobean** is $T' = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{bmatrix} = \begin{bmatrix} f_r & f_\theta \\ g_r & g_\theta \end{bmatrix}$ with determinant $f_r g_\theta - f_\theta g_r = r$. This is a special case of a more general formula which we do not cover in this class.



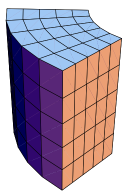
$$T : (r,\theta,z) \mapsto (r\cos(\theta), r\sin(\theta), z)$$



CYLINDRICAL COORDINATES. Use polar coordinates in the x-y plane and leave the z coordinate. Take $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$. The integration factor r is the same as in polar coordinates.

$$\int \int \int_{T(R)} f(x,y,z) \; dxdydz = \int \int \int_R g(r,\theta,z) \boxed{r} dr d\theta dz \; .$$

For example, if $f(x,y,z) = (x^2+y^2) + xz$, then $g(r,\theta,z) = r^2 + r\cos(\theta)z$.



EXAMPLE. Calculate the volume bounded by the parabolic $z = 1-(x^2+y^2)$ and the $x-y$ plane. In cylindrical coordinates, the paraboloid is given by the relation $z = 1 - r^2$:

$$\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} r \; dz d\theta dr = \int_0^1 \int_0^{2\pi} (r - r^3) \; d\theta dr = 2\pi(r^2/2 - r^4/4)_0^1 = \pi \; .$$

USE A GOOD PICTURE! A good conceptual picture not only helps to solve double and triple integral problems. Sometimes, it is even virtually impossible to solve the problem without having a good picture.

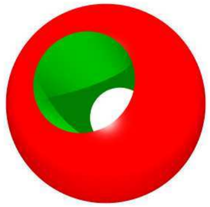
PROBLEM. Find the volume of the solid obtained by taking a sphere $x^2+y^2+z^2 = 1$ into which a hole $x^2+y^2 \leq 1/2$ has been drilled.

SOLUTION.

$$2\pi \int_{1/2}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr = 2\pi \int_{1/2}^1 2r\sqrt{1-r^2} \; dr$$

which is

$$-2\pi \frac{2}{3}(1-r^2)^{3/2}|_{1/2}^1 = \frac{4\pi}{3}\frac{\sqrt{27}}{8} = \pi\sqrt{3}/2$$

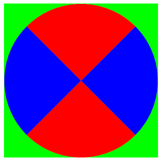


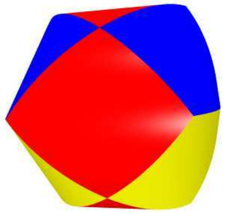
PROBLEM. Find the volume of the intersection of the three cylinders $x^2+y^2 \leq 1$, $x^2+z^2 \leq 1$ and $y^2+z^2 \leq 1$.

SOLUTION.

$$8 \int_{-\pi/4}^{\pi/4} \int_0^1 \sqrt{1 - r^2\sin(\theta)^2} \; r dr d\theta$$

which is $-16/3 + 8\sqrt{2}$.





PROBLEM. Find the volume of the intersection of the two solid cylinders $x^2+y^2 \leq 1$ and $x^2+z^2 \leq 1$.

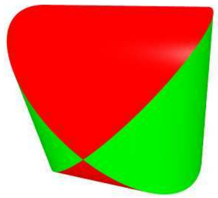
SOLUTION. We compute the volume in one of the 8 octants and multiply by 8 in the end.

$$8 \int_0^1 \int_0^{\pi/2} \sqrt{1 - r^2\cos(\theta)^2} \; r d\theta dr = 16/3 \; .$$

Here is how one would evaluate this integral with Mathematica:

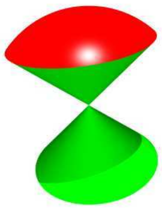
$$8 \;\; \mathrm{Integrate[Sqrt[1 - r^2 Cos[theta]^2]r, \{r, 0, 1\}, \{theta, 0, Pi/2\}]}$$

Note the order in which the integration range in entered the computer algebra system!



PROBLEM. Find $\int\int\int_R z^2 \; dV$, where R of the solid obtained by intersecting $\{1 \leq x^2+y^2+z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2+y^2\}$.

SOLUTION. We split the integral up into a "cone part" $z \in [-\sqrt{2}, \sqrt{2}]$, and the cup part $|z| > \sqrt{2}$ and evaluate each seperatly. The double cone has the volume $2\pi \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^z r \; drdz = 2\pi\sqrt{2}/3$. One cup has the volume $2\pi \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-z^2}} r \; drdz = \pi \int_{\sqrt{2}}^2 (4-z^2) \; dz$. The total volume is $\pi(2-\sqrt{2})16/3$.

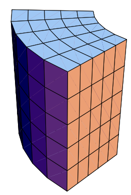


4/5/2004, SPHERICAL INTEGRALS **Math21a, O. Knill**

HOMEWORK: Section 12.6: 14,18,32,34,48,62

REMINDER: POLAR AND CYLINDRICAL COORDINATES.

$$\int \int_R f(r,\theta) \boxed{r} d\theta dr \; .$$



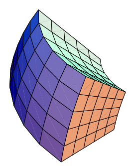
Cylindrical coordinates are obtained by taking polar coordinates in the x-y plane and leave the z coordinate. With $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$. the integration factor r is the same as in polar coordinates.

$$\int \int \int_{T(R)} f(x,y,z) \; dxdydz = \int \int \int_R g(r,\theta,z) \boxed{r} \; drd\theta dz$$

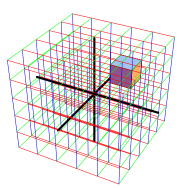
SPHERICAL COORDINATES. Spherical coordinates use ρ, the distance to the origin as well as two angles: θ the polar angle and ϕ, the angle between the vector and the z axis. The coordinate change is

$$T : (x,y,z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \; .$$

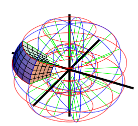
The integration factor can be seen by measuring the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\; d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.



$$\int \int \int_{T(R)} f(x,y,z) \; dxdydz = \int \int \int_R g(\rho,\theta,z) \boxed{\rho^2\sin(\phi)} \; d\rho d\theta d\phi$$



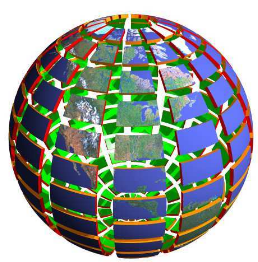
$$T : (\rho,\theta,\phi) \mapsto (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$$



VOLUME OF SPHERE. A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin(\phi) \; d\phi d\theta d\rho \; .$$

The most inner integral $\int_0^{\pi} \rho^2\sin(\phi)d\phi = -\rho^2\cos(\phi)|_0^{\pi} = 2\rho^2$.
The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2 \; d\phi = 4\pi\rho^2$.
The final integral is $\int_0^R 4\pi\rho^2 \; d\rho = 4\pi R^3/3$.



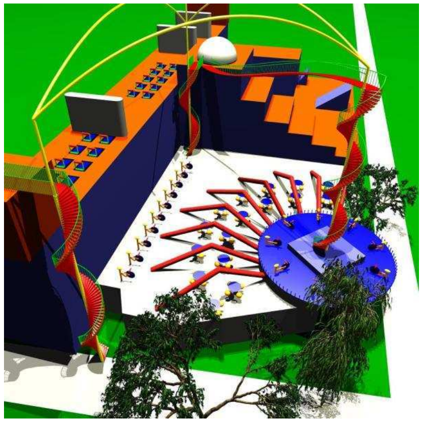
MOMENT OF INERTIA. The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int\int\int_G r(x,y,z)^2 \; dzdydx$, where $r(x,y,z) = R\sin(\phi)$ is the distance from the axes L. For a sphere of radius R we obtain with respect to the z-axis:

$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin^2(\phi)\rho^2\sin(\phi) \; d\phi d\theta d\rho$$

$$= (\frac{1}{3}\sin^3(\phi)_0^{\pi})(\int_0^R \rho^4 \; dr)(\int_0^{2pi} \; d\theta)$$

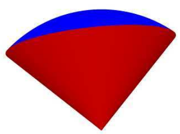
$$= \frac{4}{3}\cdot\frac{R^5}{5}\cdot 2\pi = \frac{8\pi R^5}{15} = \frac{VR^2}{5}$$

If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. Example: the moment of inertia of the earth is $810^{37}kgm^2$. The angular velocity is $\omega = 1/day = 1/(86400s)$. The rotational energy is $810^{37}kgm^2/(7464960000s^2) \sim 10^{28}J = 10^{25}kJ \sim 2.510^{24}kcal$.



How long would you have to run on a treadmill to accumulate this energy if you could make 2'500 kcal/hour? We would have to run 10^{21} hours $= 3.610^{24}$ seconds. Note that the universe is about 10^{17} seconds old. If all the 6 million people in Massachusetts would have run since the big bang on a treadmill, they could have produced the necessary energy to bring the earth to the current rotation. To make classes pass faster, we need to spin the earth more and just to add some more treadmills ... To the right you see a proposal for the science center.

DIAMOND. Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.

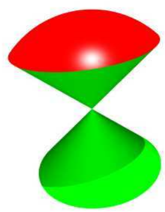


Solution: we use spherical coordinates to find the center of mass $(\overline{x}, \overline{y}, \overline{z})$:
$V = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^2\sin(\phi) \; d\phi d\theta d\rho = \frac{(1-\frac{\sqrt{3}}{2})}{3}2\pi$
$\overline{x} = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\sin^2(\phi)\cos(\theta) \; d\phi d\theta d\rho/V = 0$
$\overline{y} = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\sin^2(\phi)\sin(\theta) \; d\phi d\theta d\rho/V = 0$
$\overline{z} = \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\cos(\phi)\sin(\phi) \; d\phi d\theta d\rho/V = \frac{2\pi}{32V}$

PROBLEM Find $\int\int\int_R z^2 \; dV$ for the solid obtained by intersecting $\{1 \le x^2+y^2+z^2 \le 4\}$ with the double cone $\{z^2 \ge x^2+y^2\}$.

Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2.
In spherical coordinates, the solid R is given by $1 \le \rho \le 2$ and $0 \le \phi \le \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4\cos^2(\phi)\sin(\phi) \; d\phi d\theta d\rho = (\frac{2^5}{5} - \frac{1^5}{5})2\pi(\frac{-\cos^3(\phi)}{3}|_0^{\pi/}$$

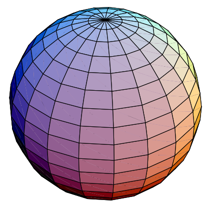


The result for the double cone is $\boxed{4\pi(31/5)(1-1/\sqrt{2}^3)}$.

4/5/2004 VOLUME SPHERE/TORUS **Math21a, O. Knill**

We first calculate the volume of a sphere of radius R in different ways. Then we show how to calculate the volume of the torus in three different ways. The page serves more as an illustration for the variety of tools which are available. We did not cover all the material (yet) to understand all of the methods.

SEVEN WAYS TO COMPUTE THE VOLUME OF THE SPHERE



1) IN RECTANGULAR COORDINATES. The volume is

$$V = \int\int\int_R \, dxdydz = \int_{-L}^{L}\int_{-\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}}\int_{-\sqrt{L^2-x^2-y^2}}^{\sqrt{L^2-x^2-y^2}} \, dzdydx$$

After computing the most inner integral, we have $V = \int_{-L}^{L}\int_{-\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}} 2(L^2-x^2-y^2)^{1/2} \, dydx$. Define $a^2 = L^2-x^2$ and use the integral in the box below:

$$V = \pi \int_{-L}^{L} (L^2-x^2) \, dx = 2\pi L^3 - 2\pi L^3/3 = 4\pi L^3/3 \, .$$

Substitution $\frac{x}{a} = \sin(u), dx = a\cos(u)du$ gives:
$\boxed{\int_{-a}^{a}\sqrt{a^2-x^2} \, dx} = \int_{-\pi/2}^{\pi/2} a\sqrt{1-\sin^2(u)}a\cos(u) \, du = a^2\int_{-\pi/2}^{\pi/2}\cos^2(u)du = \boxed{a^2\pi/2}$.

2) IN CYLINDRICAL COORDINATES. At height z, we parameterize a disc of radius L^2-z^2, so that the integral is

$$\int_{-L}^{L}\int_0^{\sqrt{L^2-z^2}}\int_0^{2\pi} r d\theta dr dz = \int_{-L}^{L}(L^2-z^2)2\pi dz = 4\pi L^3 - 2\pi L^3/3 = 4\pi L^3/3 \, .$$

3) IN SPHERICAL COORDINATES.

$$\int_0^R\int_0^{2\pi}\int_0^{\pi} \rho^2\sin(\phi) \, d\phi d\theta d\rho = 2\pi\int_0^R \rho^2 \, d\rho \int_0^{\pi}\sin(\phi) \, d\phi = 4\pi L^3/3 \, .$$

4) WITH CAVALIERI. Cavalieri cuts the hemisphere at height z to obtain a disc of radius $\sqrt{L^2-z^2}$ with area $\pi(L^2-z^2)$. He looked now at the complement of a cone of height L and radius L which when cut at height z gives a ring of outer radius L and inner radius z. The ring has area $\pi(L^2-z^2)$. Cavalieri concludes (Cavalieri principle) that the volume of that body is the same as the volume of the hemisphere. Since the difference of the volume of the cylinder and the cone which is $\pi L^3 - \pi L^3/3$ the hemisphere has the volume $2\pi L^3/3$ and the sphere has volume $4\pi L^3/3$.

5) CAS. Integrate$[1, \{x, -L, L\}, \{y, -\mathrm{Sqrt}[L^2-x^2], \mathrm{Sqrt}[L^2-x^2]\}, \{z, -\mathrm{Sqrt}[L^2-x^2-y^2], \mathrm{Sqrt}[L^2-x^2-y^2]\}]$

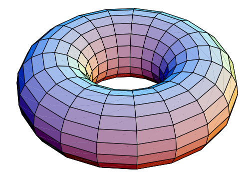
6) IN LAS VEGAS. The Monte Carlo Method is to shoot randomly onto the cube $[-L, L] \times [-L, L] \times [-L, L]$ and see how many times we hit the sphere. Here an experiment with Mathematica:

$$\mathrm{R := (2Random[] - 1); k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k++], \{10000\}]; k/10000}$$

Assume, we hit 5277 of 10000 the measured fraction of the volume of the sphere with the volume of the cube 8 is 0.5277. The volume of 1/8'th of the sphere is $\pi/6 = 0.524$

7) USING GAUSS THEOREM (see later) The vector field $F(x, y, z) = (x, y, z)$ has divergence 3 Gauss theorem tells that $3V$ is the flux of the vector field through the surface which is L times the surface area $4\pi L^2$. Therefore, $V = 4\pi L^3/3$

FIVE WAYS TO COMPUTE THE VOLUME OF THE TORUS



1) WITH TORAL COORDINATES.

$T(r, \theta, \phi) = (x, y, z) = ((b + r\cos(\phi))\cos(\theta), (b + r\cos(\phi))\sin(\theta), r\sin(\phi))$ parameterizes the torus.

The Jacobean is $\det(T') = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r(b + r\cos(\phi))$. The torus is the image of the cube $[0, a] \times [0, 2\pi] \times [0, 2\pi]$ under the map T. The change of variables formula gives

$$\int_0^a\int_0^{2\pi}\int_0^{2\pi} r(b + r\cos(\phi)) \, d\phi d\theta dr = (2\pi)(2\pi)\int_0^a br \, dr = 2\pi^2 a^2 b$$

2) USING CYLINDRICAL COORDINATES.
If we fix the z coordinate, we obtain an annulus with inner radius $b - \sqrt{a^2-z^2}$ and outer radius $b + \sqrt{a^2-z^2}$. This annulus has the area $\pi(b+\sqrt{a^2-z^2})^2 - \pi(b-\sqrt{a^2-z^2})^2$. Therefore, the volume is $4\pi b\int_{-a}^{a}\sqrt{a^2-z^2} \, dz = 4\pi b(\pi a/2) = 2\pi^2 a^2 b$.

3) USING PAPPUS CENTROID THEOREM. "The volume of a solid of revolution generated by the revolution of a region S in the $x-z$ plane around the z axis is equal to the product of the area of S and the arc length $2\pi b$ of the circle on which the center of S moves".
In the case of the torus, the length of the curve is $2\pi b$. The area of the lamina is $A = \pi a^2$. Therefore, the volume is $2\pi^2 a^2 b$.

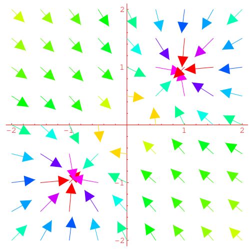
PROOF OF THE CENTROID THEOREM. We use a coordinate change transformation. In Polar coordinates, the lamina S with center of mass (b, c) is parametrized by r and z. Introduce new coordinates $T(u, v) = (u+b, v+c) = (r, z)$ so that $(0, 0)$ is the center of mass in the new coordinates. The Jacobean of this coordinate change is 1. The volume of the solid of revolution is $V = (2\pi)\int\int_S r \, drdz = (2\pi)\int\int_R (u + b) \, dudv = 2\pi b\int\int b \, dudv = 2\pi bA$, where we used that $\int\int_R u \, dudv = 0$ because $(u, v) = (0, 0) = (\int\int_R u \, dudv/A, \int\int_R v \, dudv/A)$ is the center of mass of R.

4) MONTE CARLO AGAIN. Lets assume $b = 2$ and $a = 1$. If (x, y, z) is a random point in $[-3, 3] \times [-3, 3] \times [-1, 1]$ then $(r - 2)^2 + z^2 \leq 1$ is the condition to be in the torus, where $r^2 = x^2 + y^2$.
Assume, we hit 5484 of 100000 the measured fraction of the volume 72 of the box we estimate $72*5484/100000 = 3.94848$ for the actual volume $4\pi^2 = 39.4784$.

5) CAS. Integrate$[r, \{r, 1, 3\}, \{\mathrm{theta}, 0, 2\mathrm{Pi}\}, \{z, -\mathrm{Sqrt}[1 - (r - 2)^2], \mathrm{Sqrt}[1 - (r - 2)^2]\}]$

4/7/2004, VECTOR FIELDS **Math21a, O. Knill**

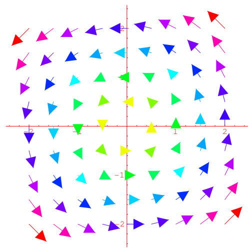
VECTOR FIELDS.



Planar vector field.
A vector field in the plane is a map, which assigns to each point (x, y) in the plane a vector $F(x, y) = (M(x, y), N(x, y))$.

Vector field in space.
A vector field in space is a map, which assigns to each point (x, y, z) in space a vector $F(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$.

PLANAR VECTOR FIELD EXAMPLES.



1) $F(x, y) = (y, -x)$ is a planar vector field which you see in a picture on the right.
2) $F(x, y) = (x - 1, y)/((x - 1)^2 + y^2)^{3/2} - (x + 1, y)/((x + 1)^2 + y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called **dipole field**. It is shown in the picture above.

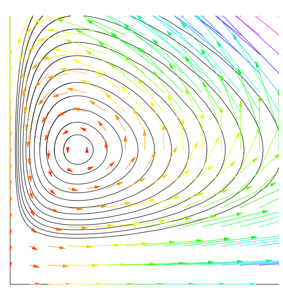
GRADIENT FIELD. 2D: If $f(x, y)$ is a function of two variables, then $F(x, y) = \nabla f(x, y)$ is called a gradient field. The same in 3D: gradient fields are of the form $F(x, y, z) = \nabla f(x, y, z)$.
EXAMPLE. $(2x, 2y, -2z)$ is the vector field which is orthogonal to hyperboloids $x^2 + y^2 - z^2 = const$.

EXAMPLE of a VECTOR FIELD. If $H(x, y)$ is a function of two variables, then $(H_y(x, y), -H_x(x, y))$ is called a **Hamiltonian vector field**. An example is the harmonic Oscillator $H(x, y) = x^2 + y^2$. Its vector field $(H_y(x, y), -H_x(x, y)) = (y, -x)$ is the same as in example 1) above.

WHEN IS A VECTOR FIELD A GRADIENT FIELD (2D)?

$F(x, y) = (M(x, y), N(x, y)) = \nabla f(x, y)$ implies $N_x(x, y) = M_y(x, y)$. If this does not hold at some point, F is no gradient field. We will see next week that the condition $\text{curl}(F) = N_x - M_y = 0$ is also necessary for F to be a gradient field.

EXAMPLE. VECTOR FIELDS IN BIOLOGY.
Let $x(t)$ denote the population of a "prey species" like tuna fish and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) + bx(t)y(t)$ with positive a, b because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $-cy(t) + dxy$, where c, d are positive. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction.

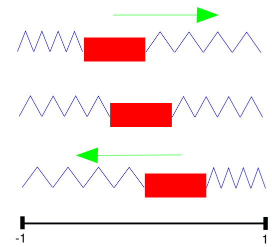


A concrete example is the **Volterra-Lodka system**

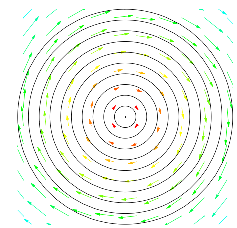
$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \end{aligned}$$

Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $(x, y) = (x(t), y(t))$, there is a curve $r(t) = (x(t), y(t))$ through that point for which the tangent $r'(t) = (x'(t), y'(t)$ is the vector $(0.4x - 0.4xy, -0.1y + 0.2xy)$.

VECTOR FIELDS IN PHYSICS

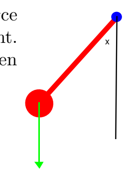


Newton's law $m\vec{r}'' = F$ relates the acceleration $\vec{r}''$ of a body with the force F acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1, 1]$ attached at two springs and the mass is $m = 2$, then the point experiences a force $(-x + (-x)) = -2x$ so that $mx'' = 2x$ or $x''(t) = -x(t)$. If we introduce $y(t) = x'(t)$ of t, then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course y is the velocity of the mass point, so a pair (x, y), thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.



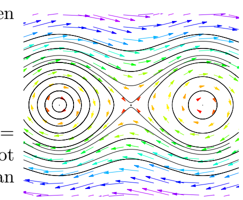
We don't yet know yet the curve $t \mapsto (x(t), y(t))$, but we know the tangents $(x'(t), y'(t)) = (y(t), -x(t))$. In other words, we know a direction at each point. The equation $(x' = y, y' = -x)$ is called a system of ordinary differential equations (ODE). More generally, the problem when studying ODE's is to find solutions $x(t), y(t)$ of equations $x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))$. Here we look for curves $x(t), y(t)$ so that at any given point (x, y), the tangent vector $(x'(t), y'(t))$ is $(y, -x)$. You can check by differentiation that the circles $(x(t), y(t)) = (r\sin(t), r\cos(t))$ are solutions. They form a family of curves. Can you interpret these solutions physically?

VECTOR FIELDS IN MECHANICS



If $x(t)$ is the angle of a pendulum, then the gravity acting on it produces a force $F(x) = -gm\sin(x)$, where m is the mass of the pendulum and where g is a constant. For example, if $x = 0$ (pendulum at bottom) or $x = \pi$ (pendulum at the top), then the force is zero.
The Newton equation "mass times acceleration = Force" gives

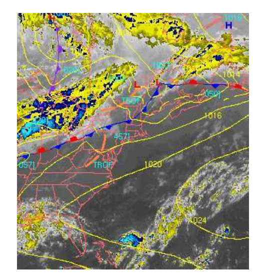
$$\ddot{x}(t) = -g\sin(x(t)) \ .$$



The equation of motion for the pendulum $\ddot{x}(t) = -g\sin(x(t))$ can be written with $y = \dot{x}$ also as

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -g\sin(x(t)) \ .$$

Each possible motion of the pendulum $x(t)$ is described by a curve $r(t) = (x(t), y(t))$. Writing down explicit formulas for $(x(t), y(t))$ is in this case not possible with known functions like sin, cos, exp, log etc. However, one still can understand the curves:
Curves on the top of the picture represent situations where the velocity y is large. They describe the pendulum spinning around fast in the clockwise direction. Curves starting near the point $(0, 0)$, where the pendulum is at a stable rest, describe small oscillations of the pendulum.



VECTOR FIELDS IN METEOROLOGY. On maps like http://www.hpc.ncep.noaa.gov/sfc/satsfc.gif one can see **Isoterms**, curves of constant temperature or pressure $p(x, y) = c$. These are level curves. The wind maps are vector fields. $F(x, y)$ is the wind velocity at the point (x, y). The wind velocity F is not always normal to the **isobares**, the lines of equal pressure p. The scalar pressure field p and the velocity field F depend on time. The equations which describe the weather dynamics are called the **Navier Stokes equations**

$$d/dtF + F \cdot \nabla F = \nu\Delta F - \nabla p + f, \text{div}F = 0$$

(we will see what is Δ, div later.) It is a **partial differential equation** like $u_x - u_y = 0$. Finding solutions is not trivial: 1 Million dollars are given to the person proving that the equations have smooth solutions in space.

4/7/2004, LINE INTEGRALS Math21a,O. Knill

Section 13.1: 16,26, Section 13.2: 14,16,22

LINE INTEGRALS.

2D: If $F(x,y)$ is a vector field in the plane and $\gamma : t \mapsto \vec{r}(t)$ is a curve, then $\boxed{\int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt}$ is called the **line integral** of F along the curve γ.

3D: If $F(x,y,z)$ is a vector field in space and $\gamma : t \mapsto \vec{r}(t)$ is a curve, then $\boxed{\int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt}$ is called the **line integral** of F along the curve γ.

NOTATION. The short-hand notation $\int_\gamma F \; \cdot ds$ is also used. In the literature, where curves are sometimes written as $r(t) = (x(t), y(t), z(t))$ or $r(t)$, the notation $\int_\gamma F \; \cdot dr$ or $\int_\gamma F \; \cdot dr$ appears. For simplicity, we leave out below the arrows above the $r(t)$ and $F(r(t))$ even so they are vectors.

WRITTEN OUT. If $F = (M,N)$ and $\vec{r} = (x(t), y(t))$, we can write $\int_a^b M(x(t),y(t))x'(t) + N(x(t),y(t))y'(t)\; dt$.

MORE NOTATION. One also can write $\int_a^b M(x,y)\; dx + N(x,y)\; dy$. Warning: this later notation is only possible for certain type of curves and rather old fashioned. Avoid it.

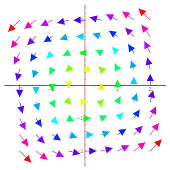
EXAMPLE: **Work**. If $F(x,y,z)$ is a force field, then the line integral $\int_a^b F(r(t)) \cdot r'(t)\; dt$ is called **work**.

EXAMPLE: **Electric potential**. If $E(x,y,z)$ is an electric field, then the line integral $\int_a^b E(r(t)) \cdot r'(t)\; dt$ is called **electric potential**.

EXAMPLE: **Gradient field**. If $F(x,y,z) = \nabla U(x,y,z)$ is a gradient field, then as we will see next hour $\int_a^b F(r(t)) \cdot r'(t)\; dt = U(r(b)) - U(r(a))$. The gradient field has physical relevance. For example, if $U(x,y,z)$ is the pressure distribution in the atmosphere, then $\nabla U(x,y,z)$ is the pressure gradient roughly the wind velocity field.

EXAMPLE 1. Let $\gamma : t \mapsto r(t) = (\cos(t), \sin(t))$ be a circle parametrized by $t \in [0, 2\pi]$ and let $F(x,y) = (-y, x)$. Calculate the line integral $I = \int_\gamma F(r) \cdot dr$.

ANSWER: We have $I = \int_0^{2\pi} F(r(t)) \cdot r'(t)\; dt = \int_0^{2\pi} (-\cos(t), \sin(t)) \cdot (-\cos(t), \sin(t))\; dt = \int_0^{2\pi} \cos^2(t) + \sin^2(t)\; dt = 2\pi$



EXAMPLE 2. Let $r(t)$ be a curve given in polar coordinates as $r(t) = \cos(t), \phi(t) = t$ defined on $[0,\pi]$. Let F be the vector field $F(x,y) = (-xy, 0)$. Calculate the line integral $\int_\gamma F \cdot dr$.

SOLUTION. In Cartesian coordinates, the curve is $r(t) = (\cos^2(t), \cos(t)\sin(t))$. The velocity vector is then $r'(t) = (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t))$. The line integral is

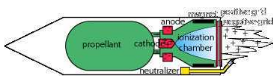
$$\int_0^\pi F(r(t)) \cdot r'(t)\; dt = \int_0^\pi (\cos^4(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t))\; dt$$

$$= \quad -2\int_0^\pi \sin^2(t)\cos^4(t)\; dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8$$

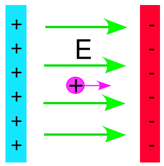
WORK. If F is a **force field** and $r(t)$ a path of a body, then $F(r(t))$ is the force acting on the body. The component of that force in the velocity direction is $G(t) = F(r(t)) \cdot r'(t)/|r'(t)|$. For some small time dt, the body will move a distance $|r'(t)|dt$. In physics, $G(t)ds$ is the amount of work done when traveling this distance. Integrating up gives the total work or energy $W = \int_a^b G(t)|r'(t)|\; dt = \int_a^b = F(r(t)) \cdot r'(t)\; dt$.

> $W = \int_\gamma F\; ds$ is the **energy** gained by a body traveling along the path γ in a force field F.

EXAMPLE. In **ion rockets** (used for example in "deep space" space craft), ionized xenon gas is passed by an electrically charged plate and accelerated to high velocities 30km/s. This (kinetic) energy is built up as work in a force field F which is parallel to an electric field E. Let $\gamma : r(t) = (t, 0, 0), 0 \le t \le L$ be the path a particle travels between the positively charged plate and the negatively charged plate.



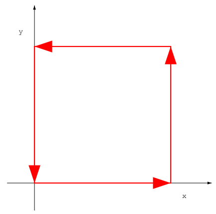
$\int_\gamma E\; dr$ is the voltage difference between the two plates and $\int_\gamma F\; dr$ is the energy difference of the particle. Because $F = cE$, where e is the charge of the ion and the velocity r' of the ions is parallel to the field, we know that ELc is the voltage difference and FL is the energy difference which is $mv^2/2$, where m is the mass and v the velocity of the ion, we could get the electric field strength $E = mv^2/(2Lc)$.



ADDING AND SUBTRACTING CURVES.

If γ_1, γ_2 are curves, then $\gamma_1 + \gamma_2$ denotes the curve obtained by traveling first along γ_1, then along γ_2. One writes $-\gamma$ for the curve γ traveled backwards and $\gamma_1 - \gamma_2 = \gamma_1 + (-\gamma_2)$.

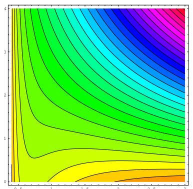
EXAMPLES. If $\gamma_1(t) = (t, 0)$ for $t \in [0,1]$, $\gamma_2(t) = (1, (t-1))$ for $t \in [1,2]$, $\gamma_3(t) = (1-(t-2), 1)$ for $t \in [2,3]$, $\gamma_4(t) = (0, 1-(t-3))$ for $t \in [3,4]$, then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ for $t \in [0,4]$ is the path which goes around a the unit square. The path $-\gamma$ travels around in the clockwise direction.



CALCULATING WITH LINE-INTEGRALS.

- $\int_\gamma F \cdot dr + \int_\gamma G \cdot dr = \int_\gamma (F+G) \cdot dr$.
- $\int_\gamma cF \cdot dr = c\int_\gamma F \cdot dr$.
- $\int_{\gamma_1+\gamma_2} F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr$
- $\int_{-\gamma} F \cdot dr = -\int_\gamma F \cdot dr$.

VOLUME-PRESSURE. Processes involving gases or liquids can be described in a **Volume-pressure diagram**.



Left: V-P diagram
Right: Sadi Carnot

A periodic processes like a refrigerator defines a closed cycle $\gamma : t \mapsto r(t) = (V(t), p(t))$ in the $V-p$ plane. The curve is parameterized by the time t. At a given time the gas has a specific volume $V(t)$ and a specific pressure $p(t)$. Consider the vector field $F(V,p) = (p, 0)$ and a closed curve γ and the line integral $\int_\gamma F\; ds$. Writing it out, we get $\int_0^{2\pi} (p(t), 0) \cdot (V'(t), p'(t))\; dt = \int_0^{2\pi} p(t)V'(t)\; dt = \int_0^{2\pi} p\; dV$.

If the volume of the gas changes under pressure p, then the work on the system is pdV. On the other hand, if the volume is kept constant, then for a gas, one does not do work on the system, when changing the pressure. Processes described by this approximation are called **adiabatic**.

For example, if the volume is decreased under high pressure and increased under low pressure then we do the work $\int_0^{2\pi} p\; dV$. Lets compute that if $r(t) = (2 + \cos(t), 2 + \sin(t))$ for $t \in [0, 2\pi]$ and $F(V,p) = (p, 0)$. $r'(t) = (-\sin(t), \cos(t))$, $F(r(t)) = (2 + \sin(t), 0)$. so that $F(r(t)) \cdot r'(t) = -\sin(t)(2 + \sin(t))$ and $\int_0^{2\pi} F(r(t)) \cdot r'(t)\; dt = -\int_0^{2\pi} \sin^2(t)\; dt = \pi$.

SCALAR LINE INTEGRALS.

If $f(r(t))$ is a function defined on a curve $\gamma : t \mapsto \vec{r}(t)$, then $\boxed{\int_a^b f(\vec{r}(t))|\vec{r}'(t)|\,dt}$ is called the **scalar line integral** of f along the curve γ.

NOTATION. The short-hand notation $\int_\gamma f\,ds$ is also used.
WRITTEN OUT. If $f(x,y)$ is the function and $\vec{r}(t) = (x(t),y(t))$, we can write $\int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$.

In three dimensions, where $\vec{r}(t) = (x(t),y(t),z(t))$, we can write $\int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$.

EXAMPLE. Integrate $f(x,y,z) = x^2+y^2+z^2$ over the path $r(t) = (\cos(t),\sin(t),t)$ from $t=0$ to $t=\pi$. The answer is $\int_0^\pi (1+t^2)\sqrt{2}\,dt = \pi\sqrt{2}+\pi^3\sqrt{2}/3$.

EXAMPLE: Let $\mathbf{r}(t) = \{\cos(t),\sin(t),t^2/2\}$ be the path of a model plane. What is the average height of the plane?

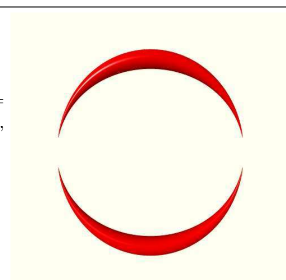
This is not a very clearly formulated question. We want to know

$$\int_0^{2\pi} z(t)|r'(t)|\,dt = \int_0^{2\pi} \frac{t^2}{2}\sqrt{1+t^2}\,dt$$

if we want to know the average height of the path and

$$\int_0^{2\pi} z(t)\,dt = \int_0^{2\pi} t^2/2\,dt$$

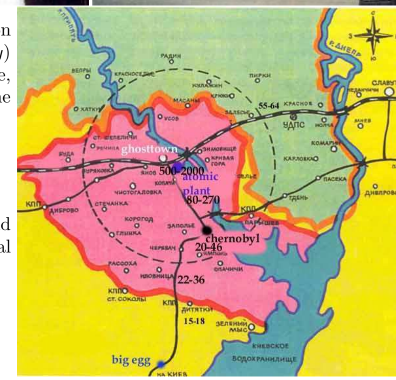
if we want to know the average height per time.



EXAMPLE. A wire $r(t) = (\cos(t),0,\sin(t))$ has thickness $f(r(t)) = \sin^2(t)$ and $t \in [0,2\pi]$ What is the mass of this wire? The mass is, because $r'(t) = 1$:

$$M = \int_0^{2\pi} \sin^2(t)\,dt = \pi\ .$$

EXAMPLE. One of the hits on the web in March 2004 was a photo report of a Russian girl "Elena" who rode with her Kawasaki motorcycle and a camera through the deserted Chernobyl area and left an impressive document on the web. The URL is http://www.angelfire.com/extreme4/kiddofspeed/



Assume Elena picks up radioactive radiation proportional to the radioactivity level $f(x,y)$ and the amount of path covered with the bike, then the total radiation obtained during the time $[0,T]$ is

$$\int_0^T f(r(t))|r'(t)|\,dt\ .$$

This is not realistic. If Elena stops, she would get no radiation increase. The correct integral would rather be

$$\int_0^T f(r(t))\,dt\ .$$

REMARK.

Scalar line integrals should be thought as a generalization of the length integral.

Do not mix it up with the line integral defined by a vector field which we cover next and which is infinitely more important.

The examples above show that dealing with scalar line integrals can be confusing. You are measuring quantities with it which are given **"per distance"** and not quantities **"per time"**. Scalar integrals hardly appear in applications. (They do for example in tomography where the problem is to reconctruct $f(x,y,z)$ from knowing all line integrals along all lines. But also there, it is possible and simpler to avoid them.)

- the application of computing mass is very artificial because mass is a triple integral. All solid bodies, even wires have three dimension.
- In general, for one dimensional situations, the density is constant so that the scalar line integral is actually a usual length integral. Nobody questions the importance of the length integral.
- For "center of mass" or moment of inertia" computations, one better uses triple integrals. Arcs or wires have a nonzero radius and the "simplification" done by computing it with one dimensional integrals **produces an error**. For mass computations, the error is zero by the **Pappus Centroid theorem**. Introducing artificial scalar line integrals is not only confusing, it is also not precise.

The topic has been introduced into calculus text books as a "bridge" to ease the transition from 1D integrals to line integrals but experience shows that this "bridge" unnecessarily complicates things because it introduces a new concept.

Just treat the topic as a "footnote" to the length integral.

4/9/2004, LINE INTEGRALS II — Math21a, O. Knill

Section 13.2, Problems 20,24,40

REVIEW LINE INTEGRALS.

$$\boxed{\int_C F \cdot dr = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt}$$

is called the **line integral** of F along the curve C. If F is a force, then the line integral is work.

A RIDDLE ABOUT WORK.

1) If you accelerate a car from $0m/s$ to $10m/s$ = 2.24miles/hour the kinetic energy needed is mass times $10^2/2 = 50$.

2) If you accelerate a car from $10m/s$ to $20m/s$ = 4.48miles/hour the kinetic energy has increased by $20^2/2 - 10^2/2 = 200 - 50 = 150$.

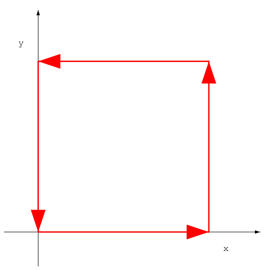
3) Now watch situation 2) from a moving coordinate system which moves with constant $10m/s$. In that system, the car accerates from 0 to 10 meter per second.

Because all physical laws are the same in different coordinate systems (going from one to the other is a Galilei transformation), the energy should be the same when accelerating from 0 to 10 or from 10 to 20.

This is in contradiction to the fact that accelerating the car from 0 to 10 needs three times less energy than accelerating the car from 10 to 20. Why?

The picture above shows the "Smart car", an extremly energy friendly car seen often in Europe, where gas prizes are much higher than here (even now)

EXAMPLE. Compute the line integral for the vector field $F(x, y) = (x^2, y^2)$ along the boundary of a square in the counter clockwise direction.



We split up the path into four paths:
$\gamma_1 : r(t) = (t, 0)$ for $t \in [0, 1]$, $\gamma_2 : r(t) = (1, t)$ for $t \in [0, 1]$, $\gamma_3 : r(t) = (1-t, 1)$ for $t \in [0, 1]$, $\gamma_4 : r(t) = (0, 1-t)$ for $t \in [0, 1]$, then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$.

The integral is then

$$\int_\gamma F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr + \int_{\gamma_3} F \cdot dr + \int_{\gamma_4} F \cdot dr$$

Plugging in $F \cdot dr = F(r(t)) \cdot r'(t)$ gives

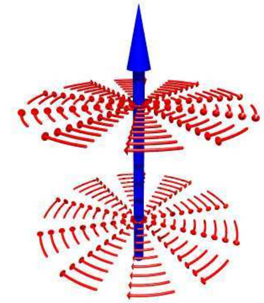
$$\int_0^1 (t^2, 0) \cdot (1, 0)\, dt + \int_0^1 (1, t^2) \cdot (0, 1)\, dt + \int_0^1 ((1-t)^2, 1) \cdot (-1, 0)\, dt + \int_0^1 (0, (1-t)^2) \cdot (0, -1)\, dt$$

and end up with $\int_0^1 2t^2 - 2(1-t)^2\, dt = 0$.

AMPERES LAW. A wire along the z-axes carries a current I. What is the strength $|B|$ of the magnetic field $\vec{B}(x, y, z) = (-By, Bx, 0)$ in distance r from the z-axes if we know that Amère law

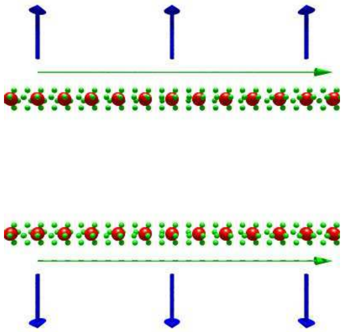
$$\int_C \vec{B} \cdot \vec{dr} = \mu_0 I$$

holds for a closed curve $C : \vec{r}(t) = (r\cos(t), r\sin(t), 0)$ winding in distance r once around the x axes and where μ_0 is a constant.
Remark. The Ampere law actually follows from a basic identity which is one of the Maxwell equations and which we will see later in this course.



Solution: compute $B(r(t)) = (-B\sin(t), B\cos(t), 0)$ and $r'(t) = (-r\sin(t), r\cos(t))$ The line integral is $2\pi r B$ and because we know this is μI, we have $B = \mu I/2\pi r$.

We see that the magnetic field decays like $1/r$. One can actually derive this result also from special relativity: A striking application of the Lorentz transformation is that if you take two wires and let an electric current flow in the same direction, then the distance between the electrons shrinks: the positively charged ions in the wire see a larger electron density than the ion density. The other wires appears negatively "charged" for the ions and attract each other. If the currents flow in different directions and we go into a coordinate system, where the electrons are at rest in the first wire, then the ion density of the ions in the same wire appears denser and also the electron density is denser in the other wire. The two wires then repell each other. The force is proportional to $1/r$ because two charged wires in distance r attract or repell each other with the force $1/r$.



A PERPETUUM MOTION MACHINE. Before entering high school, one of my passions was to construct "perpetuum motion machines". Physics classes in school quickly killed that dream. But still, these machines still fascinate me, especially if they work! Will will see more about these machines next week, but here is a model which allows us to practice line integrals:

Problem: by arranging cleverly charged plates (see lecture), we realize a static electric field $\vec{E} = (0, 0, x)$. We construct a wire along the elliptical path $C : r(t) = (\cos(t), 0, 3\sin(t))$. What is the Voltage

$$V = \int_C \vec{E}(r(t)) \cdot r'(t)\, dt$$

we measure at the wire?

Answer: $\int_0^{2\pi} 3\cos^2(t)\, dt = 3\pi$.

30 second background info in electricity: when an electron is moved around in an electric field along a path C, it gains the potential $\int_C \vec{E} \cdot \vec{dr}$. This potential is also called "voltage" and measured in "Volts". When moving a charge through a voltage differenence, it gains some energy. This energy is proportional to the amount of charge going through the wire as well as the voltage V. If the charge is It which is "current" times "time", then the energy is VIt, which can also be seen as the power VI ("volt times ampere = Watts) multiplied with time. For example, through a light bulb of $100W$, a current of about one ampere flows if the voltage is 110 Volts. If we let the lamp burn for 10 hours, we use the energy $10 \cdot 100Wh$ which is one Watt hour. In Massachusetts, a Kilowatt hour costs about $10 - 15$ cents.

Running a 100 Watt light bulb for 24 hours costs a quarter.

4/11/2004, FUNDAMENTAL THM OF LINE INTEGRALS Math21a, O. Knill

HOMEWORK: Section 13.3: 10,12,38

REVIEW. If F is a vector field and $C: t \mapsto \vec{r}(t)$ is a curve, then $\boxed{\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt}$ is called the **line integral** of F along the curve C.

GRADIENT FIELD. A vector field F is called a **gradient field** if there exists a function f such that $F = \nabla f$. For example, $F(x,y) = (xy^2, yx^2)$ is a gradient field.

FUNDAMENTAL THEOREM OF LINE INTEGRALS. If $F = \nabla f$, then

$$\boxed{\int_a^b F(r(t)) \cdot r'(t)\, dt = f(r(b)) - f(r(a))}$$

In other words, the line integral integral is the potential difference between the end points $r(b)$ and $r(a)$ if F is a gradient field.

EXAMPLE. Let $f(x,y,z)$ be the temperature distribution in a room and let $r(t)$ the path of a fly in the room, then $f(r(t))$ is the temperature, the fly experiences at the point $r(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(r(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end and initial point.

SPECIAL CASES.
$r(t)$ parallel to level curve of f means $d/dt f(r(t)) = 0$ and $r'(t)$ orthogonal to $\nabla f(r(t))$ $r(t)$ orthogonal to level curve means $|d/dt f(r(t))| = |\nabla f||r'(t)|$ and $r'(t)$ parallel to $\nabla f(r(t))$.

PROOF OF THE FUNDAMENTAL THEOREM. Use the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b F(r(t)) \cdot r'(t)\, dt = \int_a^b \nabla f(r(t)) \cdot r'(t)\, dt = \int_a^b \frac{d}{dt} f(r(t))\, dt = f(r(b)) - f(r(a)) .$$

CONSERVATIVE. A field F is called **conservative** if every line integral is independent of paths.

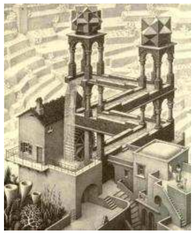
$\boxed{\text{$F$ is a gradient field if and only if it is conservative.}}$

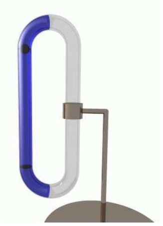
Proof. If $F = \nabla f$, the conservative property follows from the fundamental theorem of line integrals. To see the other direction, chose a point y and for every x a path C_x connecting y with x and define $f(x) = \int_{C_x} F \cdot dr$. The conservative property assures that the result is independent of the chosen path. Note that f is not unique: changing y will add a constant to f.

CLOSED LOOP PROPERTY = ENERGY CONSERVATION. It follows that for a gradient field the lineintegral along any closed curve is zero. Conversely, if this "energy conservation" holds, one has a conservative field.

$\boxed{\text{If F is a gradient field if and only if the line integral along a closed curve is zero if and only if the field is a gradient field.}}$

PERPETUUM MOTION MACHINES. A machine which implements a force field which is not a gradient field is called a **perpetuum mobile**. Mathematically, it realizes a force field for which along some closed loops the energy gain is nonnegative. (By possibly changing the direction, the energy change can be made positive). The first law of thermodynamics forbids the existence of such a machine.
It is informative to stare at some of the ideas people have come up with and to see why they don't work. The drawings of Escher appear also to produce situations, where a force field can be used to gain energy. Escher uses genius graphical tricks however.





THE COMPONENT TEST.
When is a vector field a gradient field? $F(x,y) = \nabla f(x,y)$ implies $F_y(x,y) = F_x(x,y)$. If this does not hold at some point, $F = (M,N)$ is no gradient field. This is called the **component test**. The condition $\mathrm{curl}(F) = N_x - M_y = 0$ implies that the field is conservative if the region satisfies a certain property.

FIND THE POTENTIAL.

PROBLEM 1. Let $F(x,y) = (2xy^2 + 3x^2, 2yx^2)$. Find a potential f of F.

SOLUTION. The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$.

Find g, h from $f(x,y) = \int_0^x M(x,y)\, dx + h(y)$ and $f_y(x,y) = g(x,y)$.

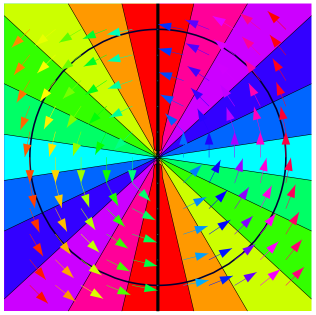
PROBLEM 2. Find values for the constants a,b which make the vector field $F = (M,N) = (ax^3y + by^2, x^4 + yx)$ conservative.
SOLUTION. The curl $N_x - M_y = 4x^3 + y - ax^3$ must be zero. This gives $a = 4$ and $b = 1/2$. The potential is $f(x,y) = (x^4 + y^2x/2)$.

A COUNTER EXAMPLE? Let $F(x,y) = (M,N) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = M$, $f_y = (1/x)/(1+y^2/x^2) = N$. However, the line integral $\int_\gamma F\, ds$, where γ is the unit circle is

$$\int_0^{2\pi} \left(\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2}\right) \cdot (-\sin(t), \cos(t)\, dt$$

which is $\int_0^{2\pi} 1\, dt = 2\pi$. What is wrong?

Solution: note that the potential f as well as the vectorfield F are not smooth everywhere.



SIMPLY CONNECTED. A region R is called **simply connected**, if every curve in R can be contracted to a point in a continuous way and every two points can be connected by a path. A disc is an example of a simply connected region, an annulus is an example which is not.

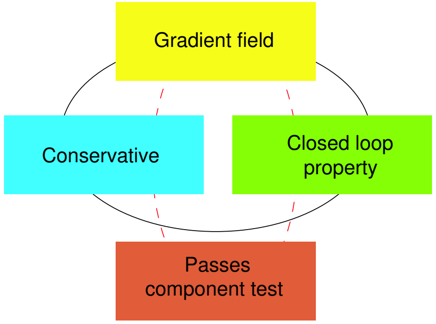
"CONSERVATIVE $\Leftrightarrow$ $\mathrm{curl}(F) = 0$.

$\boxed{\text{If R is a simply connected region, then F is a gradient field if and only if $\mathrm{curl}(F) = 0$ everywhere in R.}}$

We will prove this on Friday.

SUMMARY.

We have three equivalent properties: conservative=gradient field= closed loop property. They imply the component test $\mathrm{curl}(F) = 0$. If the region is simply connected, all four properties are equivalent.



4/14/2004 REVIEW II — Math 21a, O. Knill

MAIN MIDTERM TOPICS.	Extrema of functions of two variables.	Linear approximation.
	Extrema of functions with constraints.	Estimation using linear approximation
	Error of linear approximation	Tangent lines and planes
	Integration in two dimensions	Integration in polar coordinates
	Integration in three dimensions	Integration in spherical coordinates
	Integration in cylindrical coordinates	Application of integrals

LEVELS OF UNDERSTANDING.

I)	KNOW	Know **what** the objects, definitions, theorems, names are. Know jargon and history.
II)	DO	Know **how** to work with the objects. Pursue algorithms.
III)	UNDERSTAND	See different aspects, contexts. **Why** is it done like this?
IV)	APPLY	Extend the theory, apply to new situations, invent new objects. Ask: **Why not...?**

I) Definitions and objects.

CONSTRAINED EXTREMUM of f constrained by $g = c$ are obtained where $\nabla f = \lambda \nabla g, g = c$ of $\nabla(g) = 0$.
CRITICAL POINT. $\nabla f(x,y) = (0,0)$. Is also called **stationary point**.
DOUBLE INTEGRAL. $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx$ is an example of a double integral.
2D POLAR INTEGRAL. $\int\int_R f(r,\phi)\; rdrd\phi$ in polar coordinates.
DISCRIMINANT. $D = f_{xx}(x,y)f_{yy}(x,y) - fxy^2(x,y)$.
LINEAR APPROXIMATION. $L(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0) \cdot (x - x_0, y - y_0)$.
ERROR ESTIMATE. $|L(x,y) - f(x,y)| \leq M/2|(x-x_0)^2 + (y-y_0)^2|$ where $|f_{xx}| < M, |f_{yy}| < M, |f_{xy}| < M$ in rectangle having (x_0,y_0) and (x,y) opposite corners.
LOCAL MAXIMUM. A critical point for which $\det(H(x,y)) > 0, H_{xx}(x,y) < 0$ is a local maximum.
LOCAL MINIMUM. A critical point for which $\det(H(x,y)) > 0, H_{xx}(x,y) > 0$ is a local minimum.
SADDLE POINT. A critical point for which $\det(H(x,y)) < 0$.
SECOND DERIVATIVE TEST. $D < 0 \Rightarrow$ saddle, $D > 0, f_{xx} > 0 \Rightarrow$ min, $D > 0, f_{yy} < 0 \Rightarrow$ max.
ON CLOSED DOMAIN. There is a maximum and a minimum. If it is not inside, it is on the boundary.

II) Algorithms: INTEGRATION OVER A DOMAIN R.

1) Eventually chop the region into pieces which can be parametrized.
2) Start with one variable, say x and find the smallest x-interval $[a,b]$ which contains R.
3) For fixed x, intersect the line $x =$ const with R to determine the y-bounds $[f(x), g(x)]$.
4) Evaluate the integral $\int_a^b \left[\int_{f(x)}^{g(x)} f(x,y)\right]\, dydx$.
5) Solve the double integral by 1D integration starting from inside.
6) In case of problems with the integral, try to switch the order of integration. (Go to 2) and start with y).

EXAMPLE. Integrate x^2y^2 over the triangle $x + y/2 \leq 3, x > 0, y > 1$. The triangle is contained in the strip $0 \leq x \leq 3$. The x-integration ranges over the interval $[0,3]$. For fixed x, we have $y \geq 1$ and $y \leq 2(3-x)$ which means that the y bounds are $[0, 2(3-x)]$. The double integral is $\int_0^3 \int_1^{6-2x} x^2y^2\; dydx$.

II) Algorithms: FINDING EXTREMA ON A DOMAIN R with boundary.

1) First look for all stationary points $\nabla f(x,y)$ in the interior of R.
2) Eventually classify the points in the interior by looking at D, $f_{xx}(x,y)$ at the critical points.
3) Locate the critical points at the boundary by solving $\nabla F(x,y) = \lambda \nabla G(x,y), G(x,y) = c$.
4) List the values of F evaluated at all the points found in 1) and 3) and compare them.

EXAMPLE. Find the maximum of $F(x,y) = x^2 - y^2 - x^4 - y^4$ on the domain $x^4 + y^4 \leq 1$.
$\nabla F(x,y) = (2x - 4x^3, -2y - 4y^3)$. The critical points inside the domain are obtained by solving $2x - 4x^3 = 0, -2y - 4y^3 = 0$ which means $x = 0, x = 1/4$, $y = 0, y = -1/4$. We have four points $P_1 = (0,0), P_2 = (1/\sqrt{2}, 0), P_3 = (0, -1/\sqrt{2}), P_4 = (1/\sqrt{2}, -1/\sqrt{2})$.
The critical points on the boundary are obtained by solving the Lagrange equations $(2x - 4x^3, -2y - 4y^3) = \lambda(4x^3, 4y^3), x^4 + y^4 = 1$. Solutions (see below) are $P_5 = (0,-1), P_6 = (0,1), P_7 = (-1,0), P_8 = (1,0)$. A list of function values $F(P_1) = 0, F(P_2) = 1/2 - 1/4, F(P_3) = -1/2 - 1/4, F(P_4) = -2/4, F(P_5) = -2, F(P_6) = -2, F(P_7) = 0, F(P_8) = 0$ shows that P_2 in the interior is the maximum. Indeed, the Hessian at this point is $H = \mathrm{diag}(-1,-2)$ which has positive determinant and negative H_{11}.

II) Algorithms: SOLVE LAGRANGE PROBLEMS.
1) Write down equations neatly.
2) See whether some variable can be eliminated easily.
3) If some variable can be eliminated easily, go back to 1) using one variable less and repeat.
4) Try to combine, rearrange, simplify the equations. The system might not have an algebraic solution.

$$\begin{aligned} 2x - 4x^3 &= 4\lambda x^3 \\ -2y - 4y^3 &= 4\lambda y^3 \\ x^4 + y^4 &= 1 \end{aligned} \qquad \begin{aligned} 2x - (4+4\lambda)x^3 &= 0 \\ -2y - (4+4\lambda)y^3 &= 0 \\ x^4 + y^4 &= 1 \end{aligned} \qquad \begin{aligned} x = 0 \quad \text{or} \quad & x = 1/\sqrt{2+2\lambda} \\ y = 0 \quad \text{or} \quad & y = -1/\sqrt{2+2\lambda} \\ x^4 + y^4 &= 1 \end{aligned}$$

If x=0, then y=1, or y=-1, if $y = 0$ then $x = 1$ or $x = -1$. There are 4 critical points.

III) Understanding: Try to answer questions like:
What can happen at a critical point if the discriminant is zero at this point?
What is the geometric meaning of f_{xx}?
What does it mean if the Lagrange multiplier is zero?
Is there f with two saddles as critical points and no other critical points?

IV) Apply to new situations.
Problem solving and creativity skills are acquired best by "doing" it, by pondering over new questions, working on specific problems. Nevertheless, there is also theoretical help: for example, Polyas's book "how to solve it" gives some general advise on "how to solve problems". Abstract: **"How to solve a problem?"**:

1) Understand the problem.
2) Think of a plan by solving subproblems. Connect with older problems.
3) Walk along the plan while controlling each step.
4) Check the result. Is the result obvious? Is the method useful for other problems?

PRO MEMORIAM. Solve integration problems with the help of good figures!

polar coordinates	cylindrical coordinates	spherical coordinates
$\int\int_R g(r,\theta)\, \boxed{r}\, d\theta dr$	$\int\int\int_R g(r,\theta,z)\, \boxed{r}\, drd\theta dz$	$\int\int\int_R g(\rho,\theta,z)\, \boxed{\rho^2\sin(\phi)}\, d\rho d\theta d\phi$
$x = r\cos(\theta)$	$x = r\cos(\theta)$	$x = \rho\sin(\phi)\cos(\theta)$
$y = r\sin(\theta)$	$y = r\sin(\theta)$	$y = \rho\sin(\phi)\sin(\theta)$
	$z = z$	$z = \rho\cos(\phi)$

One uses these formulas to express the function $f(x,y,z)$ in the new coordinates as a new function g like in

$f(x,y) = (x^2+y^2)^2 + xy$	$g(r,\theta) = r^4 + r^2\cos(\theta)\sin(\theta)$
$f(x,y,z) = x^2 + y^2 + z^2 - x^3$	$g(\rho,\theta,\phi) = \rho^2 - \rho^3\cos^3(\theta)\sin^3(\phi)$

Application	2D Integral
Area	$A = \int\int_R 1\; dA$
Mass	$M = \int\int_R \delta(x,y)\; dA$
Average	$\int\int_R f(x,y)\; dA/A$.
Centroid	$(\int\int_R x\; dA/A, \int\int_R y\; dA/A)$.
Center of mass	$(\int\int_R x\delta(x,y)dA/M, \int\int_R y\delta(x,y)dA/M)$.
Moment of inertia	$I = \int\int_R (x^2+y^2)\; dA$ (with respect to origin)
Radius of gyration	$\sqrt{I/M}$

Application	3D Integral
Volume	$V = \int\int\int_R 1\; dV$
Mass	$M = \int\int\int_R \delta(x,y,z)\; dV$
Average	$\int\int\int_R f(x,y,z)\; dV/V$.
Centroid	$(\int\int\int_R x\; dV/V, \int\int\int_R y\; dV/V, \int\int\int_R z\; dV/V)$.
Center of mass	$(\int\int\int_R x\delta\; dV/M, \int\int\int_R x\delta\; dV/M, \int\int\int_R y\delta\; dV/M)$.
Moment of inertia	$I = \int\int\int_R (x^2+y^2)\; dV$ (with respect to z-axes.)
Radius of gyration	$\sqrt{I/M}$

HOMEWORK: section 13.4: 4,22,24,26

LINE INTEGRALS (RECALL). If $\vec{F}(x,y) = (M(x,y), N(x,y))$ is a vector field and $C : r(t) = (x(t), y(t)), t \in [a,b]$ is a curve, then

$$\int_\gamma F\ ds = \int_a^b F(x(t), y(t)) \cdot (x'(t), y'(t))\ dt$$

is called the **line integral of** F **along** γ. Its helpful to think of the integral as the work of a force field F along C. It is positive if "the force is with you", negative, if you have to "fight against the force", while going along the path C.

THE CURL OF A 2D VECTOR FIELD. The **curl** of a 2D vector field $F(x,y) = (M(x,y), N(x,y))$ is defined as the scalar field

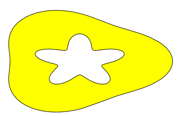
$$\text{curl}(F)(x,y) = N_x(x,y) - M_y(x,y)\ .$$

INTERPRETATION. curl(F) measures the **vorticity** of the vector field. One can write $\nabla \times F = \text{curl}(F)$ for the curl of F because the 2D cross product of (∂_x, ∂_y) with $F = (M,N)$ is $N_x - M_y$.

EXAMPLES.
1) $F(x,y) = (-y, x)$. $\text{curl}(F)(x,y) = 2$.
2) $F(x,y) = \nabla f$, (conservative field = gradient field = potential) Because $M(x,y) = f_x(x,y), N(x,y) = f_y(x,y)$, we have $\text{curl}(F) = N_x - M_y = f_{yx} - f_{xy} = 0$.

GREEN'S THEOREM. (1827) If $F(x,y) = (M(x,y), N(x,y))$ is a vector field and R is a region which has as a boundary a piecewise smooth closed curve C traversed in the direction so that the region R is "to the left". Then

$$\int_C F \cdot ds = \int\int_R \text{curl}(F)\ dxdy$$



Note: for a region with holes, the boundary consists of many curves. They are always oriented so that R **is to the left**.

GEORGE GREEN (1793-1841) was one of the most remarkable physicists of the nineteenth century. He was a self-taught mathematician and miller, whose work has contributed greatly to modern physics. Unfortunately, we don't have a picture of George Green.

SPECIAL CASE. If F is a gradient field, then both sides of Green's theorem are zero:

$\int_C F \cdot ds$ is zero by the fundamental theorem for line integrals.	$\int\int_R \text{curl}(F) \cdot dA$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(f)) = 0$.

The fact that $\text{curl}(\text{grad}(f)) = 0$ can be checked directly but it can also be seen from $\nabla \times \nabla f$ and the fact that the cross product of two identical vectors is 0. One just has to treat ∇ as a vector.

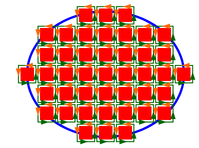
APPLICATION: CALCULATING LINE INTEGRALS. Sometimes, the calculation of line integrals is harder than calculating a double integral. Example: calculate the line integral of $F(x,y) = (x^2 - y^2, 2xy) = (M,N)$ along the boundary of the rectangle $[0,2] \times [0,1]$. Solution: $\text{curl}(F) = N_x - M_y = 2y - 2y = -4y$ so that $\int_C F\ dr = \int_0^2 \int_0^1 4y\ dydx = 2y^2|_0^1 x|_0^2 = 4$.

Remark. One can easily find examples, where the calculation of the line integral is not possible in closed form directly, but where Green allows to do it nevertheless.

WHERE IS THE PROOF? (Quote: General Hein in "Final Fantasy").

To prove Green's theorem, look first at a small square $R = [x, x+\epsilon] \times [y, y+\epsilon]$. The line integral of $F = (M,N)$ along the boundary is $\int_0^\epsilon M(x+t, y)dt + \int_0^\epsilon N(x+\epsilon, y+t)\ dt - \int_0^\epsilon M(x+t, y+\epsilon)\ dt - \int_0^\epsilon N(x, y+t)\ dt$. (Note also that this line integral measures the "circulation" at the place (x,y).)

Because $N(x+\epsilon, y) - N(x,y) \sim N_x(x,y)\epsilon$ and $M(x, y+\epsilon) - M(x,y) \sim M_y(x,y)\epsilon$, the line integral is $(M_x - N_x)\epsilon^2$ is about the same as $\int_0^\epsilon \int_0^\epsilon \text{curl}(F)\ dxdy$. All identities hold in the limit $\epsilon \to 0$.
To prove the statement for a general region R, we chop it into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the rectangles is a Riemann sum approximation of the double integral.



APPLICATION: CALCULATING DOUBLE INTEGRALS.

Sometimes, the reverse is true and it is harder to calculate the double integral. An example is to determine the area of a polygon with sides $(x_1, y_1), ..., (x_n, y_n)$ (see problem 21, section 13.4 in the book). In that case, there is a closed formula for the area: $A = \frac{1}{2} \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i)$, something a computer can do evaluate very fast and which does not involve any integration. This formula is used for example in computer graphics.

APPLICATION: FINDING THE CENTROID OF A REGION.

See homework. Green's theorem allows to express the coordinates of the centroid

$$(\int\int_R x\ dA/A, \int\int_R y\ dA/A)$$

as line integrals. One just has to find the right vector fields for each coordinate. For example, to verify

$$\int\int_R x\ dA = \int_C F\ dr$$

take the vector field $F\ dr = x^2 dy$.

APPLICATION: AREA FORMULAS.

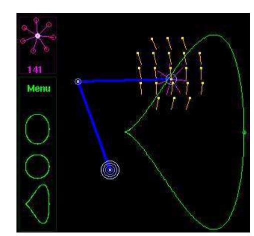
The vector fields $F(x,y) = (M,N) = (-y, 0)$ or $F(x,y) = (0, x)$ have vorticity $\text{curl}(F(x,y)) = 1$. The right hand side in Green's theorem is the **area** of R:

$$\boxed{\text{Area}(R) = \int_C -y\ dx = \int_C x\ dy}$$

EXAMPLE. Let R be the region under the graph of a function $f(x)$ on $[a,b]$. The lineintegral around the boundary of R is 0 from $(a,0)$ to $(b,0)$ because $F(x,y) = 0$ there. The lineintegral is also zero from $(b,0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a,0)$ because $N = 0$. The line integral on $(t, f(t))$ is $-\int_a^b ((-y(t), 0) \cdot (1, f'(t))dt = \int_a^b f(t)\ dt$. Green's theorem assures that this is the area of the region below the graph.

APPLICATION. THE PLANIMETER.

The planimeter is a mechanical device for measuring areas: in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engeneering to measure the size of profiles. There is a vector field F associated to a planimeter (put a vector of length 1 orthogonally to the arm). One can prove that F has vorticity 1. The planimeter calculates the line integral of F along a given curve. Green's theorem assures it is the area.



The picture to the right shows a Java applet which allows to explore the planimeter (from a CCP module by O. Knill and D. Winter, 2001).
To explore the planimeter, visit the URL http://www.math.duke.edu/education/ccp/materials/mvcalc/green/

GREEN'S THEOREM is extremely useful in physics.
It belongs to the most advanced topics in calculus. If you master it, you have so to speek the black belt in calculus. Of course, it needs some time to learn.

THERMODYNAMICS. When studing gases or liquids, one draws often a $P - V$ **diagram** (evenso one draws V usually as the x-axis). The volume in the x-axis and the pressure in the y axis.

For periodic (running of a car, the pump in a refridgeratore), one gets a closed curve $\gamma : t \mapsto r(t) = (V(t), p(t))$ in the $V - p$ plane. The curve is parameterized by the time t. At a given time, the gas has a specific volume $V(t)$ and a specific pressure $p(t)$.

Consider the vector field $F(V, p) = (p, 0)$ and a closed curve γ. What is $\int_\gamma F \ ds$? Writing it out, we get $\int_0^T (p(t), 0) \cdot (V'(t), p'(t) \ dt = \int_0^T p(t)V'(t)dt = \int_0^T pdV$. The curl of $F(V, p)$ is -1. You see that the integral $-\int_0^T pdV$ is the area of the region enclosed by the curve.

Where is the physics? If the volume of a gas changes by constant pressure, then the work on the system is pdV. On the other hand, if the volume is held constant, then for a gas, one does no work on the system, when changing the pressure.

EXAMPLE: ELECTRO ENGINE. Let us look at a cyclic process, where the volume is decreased under low pressure and increased under high pressure. It is clear that the gas does some work in this case. How much is it? Well, it is $\int_0^T pdV$ for a curve which goes clockwise around a closed region R. Green's theorem tells us that the work done is the area of the region R.

ELECTROMAGNETISM. In the plane (flatland), the electric field is a vector field $E = (E_1, E_2)$, while the magnetic field is a scalar field. One of the 2D Maxwell equation is $\mathrm{curl}(E) = -\frac{1}{c}\frac{d}{dt}B$, where c is the speed of light.

Consider a region R bounded by a wire γ. Greens theorem tells us that $d/dt \int\int_R B(t)dxdy$ is the lineintegral of E around the boundary. But $\int_\gamma Eds$ is a voltage. A change of the magnetic field produces a voltage. This is the **dynamo**.

This statement will become much more clear and useful when we lift Green to three dimensions (which will be called the theorem of Stokes).

FLUID DYNAMICS. If v is the velocity distribution of a fluid in the plane, then $\omega(x, y) = \mathrm{curl}(v)(x, y)$ is called the **vorticity** of the fluid.

If $v = \nabla U$ (potential flows), then the fluid is called **irrotational**. As you can check, then $\omega = \mathrm{curl}(v) = 0$ by Green's theorem.

The integral $\int\int_R \omega dxdy$ is called the vortex flux through R. Green's theorem tells that this flux is related to the circulation on the boundary.

In water, you could ideally measure the amount of vorticity inside a region R by measureing the work, you have to do by swimming around the boundary of R.

A RELATED THEOREM. If we rotate the vector field $F = (P, Q)$ by $\pi/2$ we get a new vector field $G = (-Q, P)$ and the integral $\int_\gamma F \ \cdot ds$ becomes a **flux** of G through the boundary of R. Introducing $\mathrm{div}(F) = (P_x + Q_y)$ we see that $\mathrm{curl}(F) = \mathrm{div}(G)$. Greem's theorem now becomes

$$\int\int_R \mathrm{div}(G) \ dxdy = \int_\gamma G \cdot dn \ ,$$

where $dn(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vecgtor $r'(x, y)$ at a point (x, y).
This new theorem has a generalisation to three dimensions too. It is called Gauss theorem or divergence theorem.

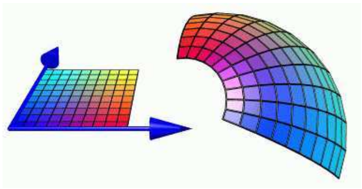
HOMEWORK. Section 13.5: 2,4,10, Section 13.6: 20

SURFACE AREA

$$\boxed{\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv}$$

is the area of the surface.

INTEGRAL OF A SCALAR FUNCTION ON A SURFACE. If S is a surface, then $\int\int_S f(x,y)\; dS/Area$ should be an average of f on the surface. If $f(x,y) = 1$, then $A = \int\int_S \; dS$ should be the area of the surface. If S is the image of $\vec{r}$ under the map $(u,v) \mapsto \vec{r}(u,v)$, then $dS = |\vec{r}_u \times \vec{r}_v| \; dudv$.



Given a surface $S = \vec{r}(R)$, where R is a domain in the plane and where $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$. The surface integral of $f(u,v)$ on S is defined as $\int\int_S f \; dS = \int\int_R f(u,v)|\vec{r}_u \times \vec{r}_v| \; dudv$.

INTERPRETATION. If $f(x,y)$ measures a quantity then $\int\int_S f \; dS/\int\int_S 1 \; dS$ is the average of the function f on S. Analoguously to the scalar line ingegral $\int_a^b f(r(t)|r'(t)| \; dt$, the scalar surface integral should be considered a **footnote** to the surface area integral only.

EXPLANATION OF $|\vec{r}_u \times \vec{r}_v|$. The vector $\vec{r}_u$ is a tangent vector to the curve $u \mapsto \vec{r}(u,v)$, when v is fixed and the vector $\vec{r}_v$ is a tangent vector to the curve $v \mapsto \vec{r}(u,v)$, when u is fixed. The two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A little rectangle spanned by $[u, u+du]$ and $[v, v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r} + \vec{r}_u]$ and $[\vec{r}, \vec{r} + \vec{r}_v]$.

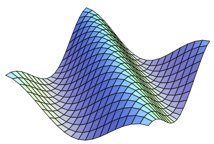
A simple case: consider $\vec{r}(u,v) = (2u, 3v, 0)$. This surface is part of the xy-plane. The parameter region R just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = (0,0,6)$ and we see for example that the area of $\vec{r}(R)$ is 6 times the area of R.

POLAR COORDINATES. If we take $\vec{r}(u,v) = (u\cos(v), u\sin(v), 0)$, then the rectangle $[0,R] \times [0,2\pi]$ is mapped into a flat surface which is a disc in the xy-plane. In this case $\vec{r}_u \times \vec{r}_v = (\cos(v), \sin(v), 0) \times (-u\sin(v), u\cos(v), 0) = (0,0,u)$ and $|\vec{r}_u \times \vec{r}_v| = u = r$. We can explain the integration factor r in polar coordinates as a special case of a surface integral.

THE AREA OF THE SPHERE.

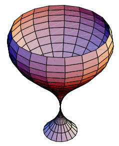
The map $\vec{r} : (u,v) \mapsto (L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v))$ maps the rectangle $R : [0,2\pi] \times [0,\pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1 \; dS = \int_0^{2\pi}\int_0^{\pi} L^2 \sin(v) \; dvdu = 4\pi L^2$.

SURFACE AREA OF GRAPHS. For surfaces $(u,v) \mapsto (u,v,f(u,v))$, we have $\vec{r}_u = (1,0,f_u(u,v))$ and $\vec{r}_v = (0,1,f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1+f_u^2+f_v^2}$. The area of the surface above a region R is $\boxed{\int\int_R \sqrt{1+f_u^2+f_v^2} \; dA}$



EXAMPLE. The surface area of the paraboloid $z = f(x,y) = x^2 + y^2$ is (use polar coordinates) $\int_0^{2\pi} \sqrt{1+4r^2} r \; drd\theta = 2\pi(2/3)(1+4r^2)^{3/2}/8|_0^1 = \pi(5^{3/2}-1)/6$.

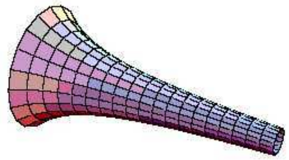
AREA OF SURFACES OF REVOLUTION. If we rotate the graph of a function $f(x)$ on an interval $[a,b]$ around the x-axis, we get a surface parameterized by $(u,v) \mapsto \vec{r}(u,v) = (v, f(v)\cos(u), f(v)\sin(u))$ on $R = [0,2\pi] \times [a,b]$ and is called a **surface of revolution**. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v| \; dudv = 2\pi \int_a^b |f(v)|\sqrt{1+f'(v)^2} \; dv$.



EXAMPLE. If $f(x) = x$ on $[0,1]$, we get the surface area of a cone: $\int_0^{2\pi}\int_0^1 x\sqrt{1+1} \; dvdu = 2\pi\sqrt{2}/2 = \pi\sqrt{2}$.

P.S. In computer graphics, surfaces of revolutions are constructed from a few prescribed points $(x_i, f(x_i))$. The machine constructs a function (**spline**) and rotates

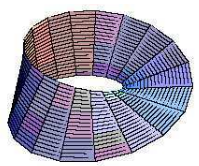
GABRIEL'S TRUMPET. Take $f(x) = 1/x$ on the interval $[1,\infty)$.
Volume: The volume is (use cylindrical coordinates in the x direction): $\int_1^\infty \pi f(x)^2 \; dx = \pi \int_1^\infty 1/x^2 \; dx = \pi$.
Area: The area is $\int_0^{2\pi}\int_1^\infty 1/x\sqrt{1+1/x^4} \; dx \geq 2\pi \int_1^\infty 1/x \; dx = 2\pi \log(x)|_0^\infty = \infty$.



The Gabriel trumpet is a surface of finite volume but with infinite surface area! You can fill the trumpet with a finite amount of paint, but this paint does not suffice to cover the surface of the trumpet!

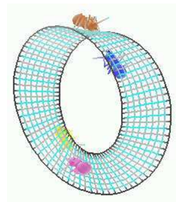
Question. How long does a Gabriel trumpet have to be so that its surface is $500cm^2$ (area of sheet of paper)? Because $1 \leq \sqrt{1+1/x^4} \leq \sqrt{2}$, the area for a trumpet of length L is between $2\pi\int_1^L 1/x \; dx = 2\pi\log(L)$ and $\sqrt{2}2\pi\log(L)$. In our case, L is between $e^{500/(\sqrt{2}2\pi)} \sim 2 * 10^{24}cm$ and $e^{500/(2\pi)} \sim 4 * 10^{34}cm$. Note that the universe is about 10^{26} cm long (assuming that the universe expanded with speed of light since 15 Billion year). It could not accommodate a Gabriel trumpet with the surface area of a sheet of paper.

MÖBIUS STRIP. The surface $\vec{r}(u,v) = (2+v\cos(u/2)\cos(u), (2+v\cos(u/2))\sin(u), v\sin(u/2))$ parametrized by $R = [0,2\pi] \times [-1,1]$ is called a **Möbius strip**.
The calculation of $|\vec{r}_u \times \vec{r}_v| = 4+3v^2/4+4v\cos(u/2)+v^2\cos(u)/2$ is straightforward but a bit tedious. The integral over $[0,2\pi] \times [-1,1]$ is 17π.



QUESTION. If we build the Moebius strip from paper. What is the relation between the area of the surface and the weight of the surface?

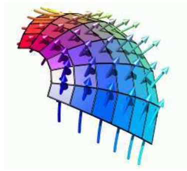
REMARKS.
1) An OpenGL implementation of an Escher theme can be admired with **"xlock -inwindow -mode moebius"** in X11.
2) A patent was once assigned to the idea to use a Moebius strip as a **conveyor belt**. It would last twice as long as an ordinary one.



SURFACE AREA OF IMPLICIT SURFACES. If $f(x,y,z) = c$ is an implicit surface **which is a graph over a region** R **in the xy-plane** (add the graph assumption to the box on page 1095 in the Thomas), then the surface area is $\int\int_R |\nabla f|/|f_z| \; dA$ if $f_z \neq 0$ on R. This follows from the formula in the case of the graph and the fact that $\frac{\sqrt{f_x^2+f_y^2+f_z^2}}{|f_z|} = \sqrt{(f_x/f_z)^2+(f_y/f_z)^2+1} = \sqrt{g_x^2+g_y^2+1}$, if $z = g(x,y)$ by implicit differentation.

4/19/2004, FLUX INTEGRALS — Math21a, O. Knill

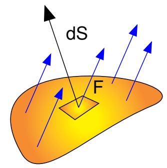
FLUX INTEGRAL OF A VECTOR FIELD THROUGH SURFACE. If S is a surface, and F is a vector field in space we define an integral $\int\int_S F \cdot dS$ which is called the **flux** of F through the surface S.



DEFINITION. Given a surface $S = r(R)$, where R is a domain in the plane and where $r(u,v) = (x(u,v), y(u,v), z(u,v))$ is the parameterization of the surface. The **flux integral** of F through S is defined as the double integral

$$\boxed{\int\int_S F \cdot dS = \int\int_R F(r(u,v)) \cdot (r_u \times r_v)\, dudv}$$

The notation $dS = (r_u \times r_v)\, dudv$ means an infinitesimal vector in the normal direction at the surface.



INTERPRETATION. If F = fluid velocity field, then $\int\int_S F \cdot dS$ is the flux of fluid passing through S.

EXAMPLE. Let $F(x,y,z) = (0,1,z^2)$ and let S be the sphere with

$$r(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

, $r_u \times r_v = \sin(v) r$ so that $F(r(u,v)) = (0,1,\cos^2(v))$ and

$$\int_0^{2\pi}\int_0^{\pi} \sin(v)(0,1,\cos^2(v)) \cdot (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))\, dudv\ .$$

The integral is $\int_0^{2\pi}\int_0^{\pi} \sin^2(v)\sin(u) + \cos^3(v)\sin(v)\, dudv = 0$.

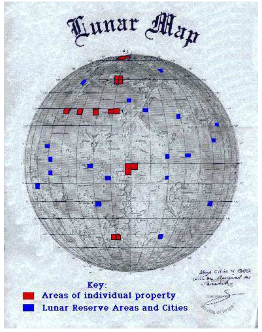
Look at the vector field. Most flux passes in at the south pole, most flux passes out at the north pole. However, there is some symmetry and what enters in the south leaves in the north. This explains why the total flux is zero.

WHAT IS THE FLUX INTEGRAL? Because $n = r_u \times r_v / |r_u \times r_v|$ is a unit vector normal to the surface and on the surface, $F \cdot n$ is the normal component of the vector field with respect to the surface, we can write $\int\int_S F \cdot dS = \int\int F \cdot n\, dS$. The function $F \cdot n$ is the scalar projection of of F in the normal direction. Whereas the formula $\int\int 1\, dS$ gave the area of the surface with $dS = |r_u \times r_v| dudv$, the flux integral weights each area element dS with the normal component of the vector field $f(u,v) = (F(r(u,v)) \cdot n(r(u,v))$.

BIKING IN THE RAIN.
I experience (like last Monday) that during rain one gets soaked more by the rain when biking fast then by biking slow or walking foot. Why? For a biker who drives with speed V along the x axis, the rain is a fluid with velocity $F = (-V, 0, W)$, where W is the speed of the rain drops falling along the z-axis. If the biker is a rectangular box $[0,1/2] \times [0,1] \times [1/4,1/3]$ we can calculate the flux through that surface. Parameterize the front by $r(u,v) = (0,u,v)$ in the yz-plane. The water soaked up in unit time by the front clothes is the flux of the rain through that surface. We know $r_u \times r_v = (1,0,0)$ and $\int\int_S F \cdot dS = \int_0^{1/2}\int_0^1 (-V,0,W) \cdot (1,0,0)\, dudv = -V/2$. The flux through the side surface is zero (if the rain falls vertically) and the flux through the top is constant, namely $W/12$. The total flux is $W/12 - V/2$. You catch more rain if you drive faster.

PROPERTY ON THE MOON. More than two years ago, in lack of a better present idea, I bought property on the moon for family (For 30 Us-Dollars one could get 1'777.58 Acres)) at the Lunar Embassy (www.moonshop.com). An acre is now sold for 19.99 Dollars plus 1.51 Dollars lunar tax. You can also buy an entire "city" for 4500 dollars now.

The coordinates of the parcel are $34E, 26N$. The moon has a radius of r=1737 km. The surface is $3.810^6 km^2$. Multiply this by 247 gives about 10 billion (10^{10}) acres (more than an acre for each person on the world). The moon surface is parameterized by $r(u,v) = (r\cos(u)\sin(v), r\sin(u)\sin(v), r\cos(v))$ where u is the latitude and $\pi/2 - v$ is the latitude. If we look at the moon from the direction $(1,0,0)$, how big does the area of a real estate $S = r([a,b] \times [c,d])$ appear?



The flux of the vector field $(1,0,0)$ through the surface is $\int\int_S F \cdot dS = \int_a^b\int_c^d r^2\cos(u)\sin^2(v) dvdu$.
This is also the area we **see**. You notice that the visible area is small for $\cos(u)$ small or $\sin(v)$ small, which is near $90E, 90W$ or near the poles.

EXAMPLE. Calculate the flux of the vector field $F(x,y,z) = (1,2,3z)$ through the paraboloid $z = x^2 + y^2$ lying above the region $x^2 + y^2 \leq 1$.

Parameterize it using polar coordinates $r(u,v) = (u\cos(v), u\sin(v), u^2)$ where $r_u \times r_v = (-2u^2\cos(v), -2u^2\sin(v), u)$ and $F(r(u,v)) = (1,2,3u^2)$. We get $\int_S F \cdot dS = \int_0^{2\pi}\int_0^1 (-2u^2\cos(v) - 4u^2\sin(v) + 3u^3) dudv = 1$.

WHERE CAN FLUX INTEGRALS APPEAR?

- **Fluid dynamics** The flux of the velocity vector field of a membrane through a surface is the volume of the fluid which passes through that surface in unit time. Often one looks at incompressible fluids which means div$(v) = 0$. In this case, the amount of fluid which passes through a closed surface like a sphere is zero because what enters also has toe leave the surface.
- **Electromagnetism** The change of the flux of the magnetic field through a surface is related to the voltage at the boundary of the surface. We will see why one of the Maxwell equations div$(B) = 0$ means that there are no **magnetic monopoles** and see how divergence is related to flux.
- **Magnetohydrodynamics** Magnetohydrodynamics deals with electrically charged "fluids" like plasmas in the sun or solar winds etc. The velocity field of the wind induces a current and the flux of the field trough a surface can be used to calculate the magnetic field.
- **Gravity** We will see that the flux of a field through a closed surface like the sphere is related to the amount of source inside the the surface. In gravity, where F is the gravitational field, the flux of the gravitational field through a closed surface is proportional to the mass inside the surface. Mass is a "source for gravitational field".
- **Thermodynamics**. The heat flux through a surface like the walls of a house are related to the amount of heat which is produced inside the house. A badly isolated house needs has a large heat flux and needs a large source of heating.
- **Particle physics**. When studying the collision of particles, one is interested in the relation between the incoming flux and out-coming fluxes depending on the direction. The key word is: "differential cross-section".
- **Optics**. Measurements of "light fluxes" are important for lasers, astronomy, photography.

4/23/2004, STOKES THEOREM **Math21a, O. Knill**

HOMEWORK. Section 13.7: 6,12,16,20

REMINDERS. The **curl** of a vector field F is

$$\mathrm{curl}(M,N,P) = \nabla \times F = (P_y - N_z, M_z - P_x, N_x - M_y) \; .$$

The flux integral of a vector field F through a surface $S = r(R)$ is defined as

$$\int\int_R F(r(u,v)) \cdot (r_u \times r_v) \; dudv$$

The line integral of a vector field F along a curve $C = r([a,b])$ is given as

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \; dt \; .$$

The picture shows a tornado near Cordell, Oklahoma. Date: May 22, 1981. Photo Credit: NOAA Photo Library, NOAA Central Library. The tornado points into the direction of the field curl(F), where F is the velocity of the air. Twisters cause annually about 80 deaths in the US. The 17 Illinois twisters last Tuesday killed 8 people

STOKES THEOREM. Let S be a surface with boundary curve C and let F be a vector field. Then

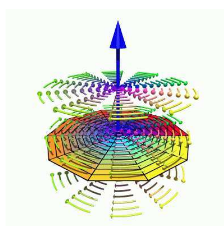
$$\boxed{\int\int_S \mathrm{curl}(F) \; \cdot dS = \int_C F \cdot dr \; .}$$

Note: the orientation of C is such that if you walk along the surface (head into the direction of the normal $r_u \times r_v$), then the surface to your left.

EXAMPLE. Let $F(x,y,z) = (-y,x,0)$ and let S be the upper semi hemisphere, then $\mathrm{curl}(F)(x,y,z) = (0,0,2)$. The surface is parameterized by $r(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ on $R = [0,2\pi] \times [0,\pi/2]$ and $r_u \times r_v = \sin(v) r(u,v)$ so that $\mathrm{curl}(F)(x,y,z) \cdot r_u \times r_v = \cos(v)\sin(v)2$. The integral $\int_0^{2\pi} \int_0^{\pi/2} \sin(2v) \; dvdu = 2\pi$.

The boundary C of S is parameterized by $r(t) = (\cos(t), \sin(t), 0)$ so that $dr = r'(t)dt = (-\sin(t), \cos(t), 0)dt$ and $F(r(t)) \; r'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C F \cdot dr$ along the boundary is 2π.

SPECIAL CASE: GREEN'S THEOREM. If S is a surface in the $x-y$ plane and $F = (M,N,0)$ has zero z component, then $\mathrm{curl}(F) = (0,0,N_x - M_y)$ and $\mathrm{curl}(F) \cdot dS = (N_x - M_y) \; dxdy$.



PROOF OF STOKES THEOREM.
For a surface which is flat, Stokes theorem can be seen with Green's theorem. If we put the coordinate axis so that the surface is in the xy-plane, then the vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of curl(F). The reason is that the third component $N_x - M_y$ of $\mathrm{curl}(F) = (P_y - N_z, M_z - P_x, N_x - M_y)$ is the two dimensional curl: $F(r(u,v)) \cdot (0,0,1) = N_x - M_y$. If C is the boundary of the surface, then $\int\int_S F(r(u,v)) \cdot (0,0,1) \; dudv = \int_C F(r(t)) \; r'(t)dt$.

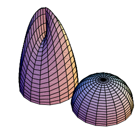
For a general surface, we approximate the surface by a mesh of small parallelepipeds. When summing up line integrals along all these parallelepipeds, the lineintegrals inside the surface cancel and only the integral along the boundary remain. On the other hand, the sum of the fluxes of the curl through boundary adds up to the flux through the surface.

DISCOVERY OF STOKES THEOREM **Stokes theorem was found by Ampère in 1825.** George Gabriel Stokes: (1819-1903) was probably inspired by work of Green and rediscovers the identity around 1840.

George Gabriel Stokes

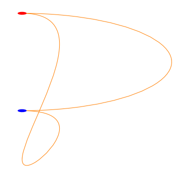
André Marie Ampere

EXAMPLE. Calculate the flux of the curl of $F(x,y,z) = (-y,x,0)$ through the surface parameterized by $r(u,v) = (\cos(u)\cos(v), \sin(u)\cos(v), \cos^2(v) + \cos(v)\sin^2(u + \pi/2))$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.



For every surface bounded by C the flux of curl(F) through through the surface is the same. The flux of the curl of a vector field through a surface S depends only on the boundary of S.

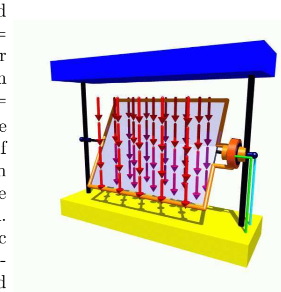
Compare this with the earlier statement:



For every curve between two points A, B the line integral of grad(f) along C is the same. The line integral of the gradient of a function of a curve C depends only on the end points of C.

BIOT-SAVARD LAW. A magnetic field B in absence of an electric field satisfies a Maxwell equation $\mathrm{curl}(B) = (4\pi/c)j$, where j is the current and c is the speed of light. How do we get the magnetic field B, when the current is known? Stokes theorem can give the answer: take a closed path C which bounds a surface S. The line integral of B along C is the flux of curl(B) through the surface. By the Maxwell equation, this is proportional to the flux of j through that surface. Simple situation. Assume j is contained in a wire of thickness r which we align on the z-axis. To measure the magnetic field at distance $R > r$ from the wire, we take a curve $C : r(t) = (R\cos(t), R\sin(t), 0)$ which bounds a disc S and measure $2\pi RB = \int_C B \cdot ds = \int\int_S \mathrm{curl}(B) \; dS = \int\int_S 4\pi/cj \; dS = 4\pi J/c$, where J is the total current passing through the wire. The magnetic field satisfies $B = 2J/(cR)$. This is called the Biot-Savard law.

THE DYNAMO, FARADEY'S LAW. The electric field E and the magnetic field B are linked by a Maxwell equation $\mathrm{curl}(E) = -\frac{1}{c}\dot{B}$. Take a closed wire C which bounds a surface S and consider $\int\int_S B \cdot dS$, the flux of the magnetic field through S. Its change can be related with a voltage using Stokes theorem: $d/dt \int\int_S B \cdot dS = \int\int_S \dot{B} \cdot dS = \int\int_S -c \; \mathrm{curl}(E) \cdot dS = -c\int_C E \; ds = U$, where U is the voltage measured at the cut-up wire. It means that if we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire, we get an electric voltage. That is what happens in a power-generator like an alternator in a car. In Dynamo implementations, the wire is turned inside a fixed magnet.



HOMEWORK. Section: 13.3: 34,38, p.1138: 30,32,34

curl: $\text{curl}(M,N,P) = (P_y - N_z, M_z - P_x, N_x - M_y)$.
gradient: $\text{grad}(f) = (f_x, f_y, f_z)$.
Fundamental theorem of line integrals
curve $r(t) : [a,b] \to C$ with boundary $\{r(a), r(b)\}$.

$$\int_a^b \text{grad}(f)(r(t)) \cdot r'(t)\, dt = f(r(b)) - f(r(a))$$

Stokes theorem:
surface $r(u,v) : R \to S$ with boundary $r(t) : [a,b] \to C$

$$\int\int_R \text{curl}(F)(r(u,v)) \cdot (r_u \times r_v) dudv = \int_a^b F(r(t)) \cdot r'(t)\, dt$$

REMINDERS:

CURL: $\boxed{\text{curl}(F) = \nabla \times F}$

GRAD: $\boxed{\text{grad}(f) = \nabla f}$

FTL: $\boxed{\int_C \text{grad}(f) \cdot dr = f(B) - f(A)}$

STOKES: $\boxed{\int\int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr}$

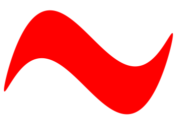
CURL(GRAD)=0: $\boxed{\nabla \times \nabla f = \vec{0}}$

CONSERVATIVE FIELDS. F is **conservative** if is a gradient field $F = \nabla f$. The fundamental theorem of line integral implies that the line integral along closed curves is zero and that line integrals are path independent. We have also seen that $F = \text{grad}(f)$ implies that F has zero curl. If we know that F is conservative, how do we compute f? If $F = (M,N,P) = (f_x, f_y, f_z)$, we got f by integration. There is an other method which you do in the homework in two dimensions: to get the potential value f, find a path C_P from the origin to the point $P = (x,y,z)$ and compute $\int_C F \cdot dr$. Because line integrals are path independent, the fundamental theorem of line integrals gives $\int_C F \cdot dr = f(P)$.

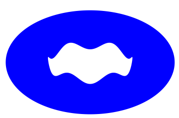
CONNECTED. A region is called **connected** if one can connect any two points in the region with a path.

SIMPLY CONNECTED. A region is called **simply connected** if it is connected and every path in the region can be deformed to a point within the region.

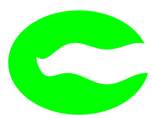
EXAMPLES 2D:



Simply connected.



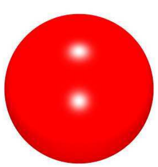
Not simply connected.



Simply connected.

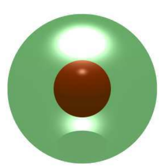
Not simply connected.

EXAMPLES 3D:

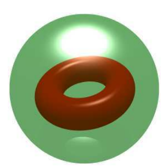


A solid ball is simply connected.

A solid torus is not simply connected.



The complement of a solid ball is simply connected.

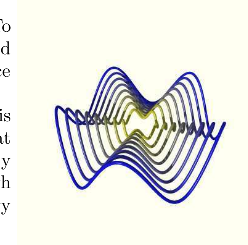


The complement of a solid torus is not simply connected.

THEOREM.

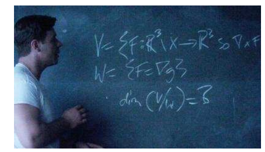
In a simply connected region D, a vector field F is conservative if and only if $\text{curl}(F) = \vec{0}$ everywhere inside D.

Proof. We already know that $F = \nabla f$ implies $\text{curl}(F) = \vec{0}$. To show the converse, we verify that the line integral along any closed curve C in D is zero. This is equivalent to the path independence and allows the construction of the potential f with $F = \nabla f$.
By assumption, we can deform the curve to a point: if $r_0(t)$ is the original curve and $r_1(t)$ is the curve $r_1(t) = P$ which stays at one point, define a parametrized surface S by $r(t,s) = r_s(t)$. By assumption, $\text{curl}(F) = \vec{0}$ and therefore the flux of $\text{curl}(F)$ through S is zero. By Stokes theorem, the line integral along the boundary C of the surface S is zero too.



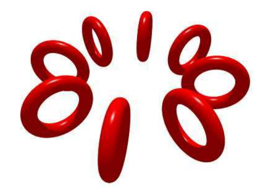
THE NASH PROBLEM. Nash challenged his multivariable class in the movie "A beautiful mind" with a problem, where the region is not simply connected.

Find a region X of $\mathbf{R}^3$ with the property that if V is the set of vector fields F on $\mathbf{R}^3 \setminus X$ which satisfy $\text{curl}(F) = 0$ and W is the set of vector fields F which are conservative: $F = \nabla f$. Then, the space V/W should be 8 dimensional.



You solve this problem as an inclass exercise (ICE). The problem is to find a region D in space, in which one can find 8 different closed paths C_i so that for every choice of constants $(c_1, ..., c_8)$, one can find a vector field F which has zero curl in D and for which one has $\int_{C_1} F\, dr = c_1, ..., \int_{C_8} F\, dr = c_8$.

One of the many solutions is cut out 8 tori from space. For each torus, there is a vector field F_i (a vortex ring), which has its vorticity located inside the ring and such that the line integral of a path which winds once around the ring is 1. The vector field $F = c_1F_1 + ... + c_8F_8$ has the required properties.



CLOSED SURFACES. Surfaces with no boundaries are called **closed surfaces**. For example, the surface of a donought, or the surface of a sphere are closed surfaces. A half sphere is not closed, its boundary is a circle. Half a doughnut is not closed. Its boundary consists of two circles.

THE ONE MILLION DOLLAR QUESTION. One of the Millenium problems is to determine whether any three dimensional space which is simply connected is deformable to a sphere. This is called the **Poincare conjecture**.

LINEINTEGRAL IN HIGHER DIMENSIONS. Line integrals are defined in the same way in higher dimensions. $\int_C F \cdot dr$, where $\cdot$ is the dot product in d dimensions and $dr = r'(t)dt$.

CURL IN HIGHER DIMENSIONS. In d-dimensions, the curl is the field $\text{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j$ with $\binom{d}{2}$ components. In 4 dimensions, it has 6 components. In 2 dimensions it has 1 component, in 3 dimensions, it has 3 components.

SURFACE INTEGRAL IN HIGHER DIMENSIONS. In d dimensions, a surface element in the ij-plane is written as dS_{ij}. The flux integral of the curl of F through S is defined as $\int\int \text{curl}(F) \cdot dS$, where the dot product is $\sum_{i<j} \text{curl}(F)_{ij} dS_{ij}$. If S is given by a map X from a domain R in the 2-plane to R^d, $U = \partial_u X$ and $V = \partial_v X$ are tangent vectors to that plane and $dS_{ij}(u,v) = (U_iV_j - U_jV_i)\, dudv$.

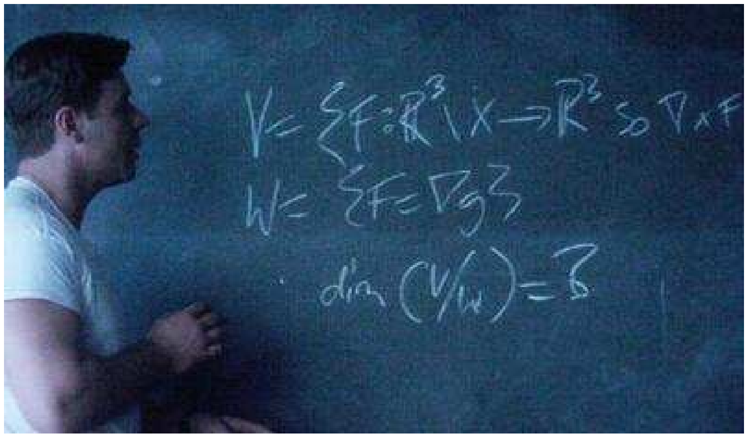
STOKES THEOREM IN HIGHER DIMENSIONS. If S is a two dimensional surface in d-dimensional space and C is its boundary, then $\int\int_S \text{curl}(F) \cdot dS = \int_C F \cdot ds$.

ICE: NASH'S PROBLEM Math 21a

In this ICE, you solve Nash's problem, he gave to a multivariable calculus class. Remember Nash saying in the movie "A beautiful mind":

> "This problem here will take some of you many months to solve, for others amoung you it might take a life time".

You will solve it here in 10 minutes ...



NASH's PROBLEM. Find a subset X of $\mathbf{R}^3$ with the property that if V is the set of vector fields F on $\mathbf{R}^3 \setminus X$ which satisfy $\text{curl}(F) = 0$ and W is the set of vector fields F which are conservative: $F = \nabla f$. Then, the space V/W should be 8 dimensional.

Remark. The meaning of the last sentence means that there should be 8 vectorfields F_i which are not gradient fields and which have vanishing curl outside X. Furthermore, you should not be able to write any of the 8 vectorfields as a sum of multiples of the other 7 vector fields.

Here is a two dimensional version of the problem:

2D VERSION OF NASH's PROBLEM.

If $X = \{0\}$ and V is the set of vector fields F on $\mathbf{R}^2 \setminus X$ which satisfy $\text{curl}(F) = 0$ and W is the set of vector fields F which are gradient fields, then $\dim(V/W) = 1$.

The vector field F is $F(x, y) = (-y/(x^2 + y^2), x/(x^2 + y^2))$ has vanishing curl in $\mathbf{R}^2 \setminus X$ but the line integral around the origin is 2π. It is not a gradient field.
Now: What set X would you have to take to get $\dim(V/W) = 8$?

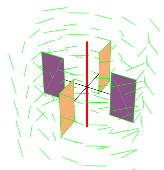
The SOLUTION OF THE 3D VERSION OF NASH's PROBLEM can be obtained directly from the solution of the 2D version. How?

4/28/2004, CURL AND DIV — Math21a, O. Knill

CURL (3D). The curl of a 3D vector field $F = (P, Q, R)$ is defined as the 3D vector field

$$\mathrm{curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) .$$

CURL (2D). Recall, the curl of a 2D vector field $F = (P, Q)$ is $Q_x - P_y$, a scalar field.



WANTED! Is there a multivariable calculus book in which the above wheel is **not** shown? The wheel indicates the curl vector if F is thought of as a wind velocity field. As we will see later the direction in which the wheel turns fastest, is the direction of curl(F). The wheel could actually be used to measure the curl of the vector field at each point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl.

DIV (3D). The **divergence** of $F = (P, Q, R)$ is the scalar field $\mathrm{div}(P, Q, R) = \nabla \cdot F = P_x + Q_y + R_z$.
DIV (2D). The **divergence** of a vector field $F = (P, Q)$ is $\mathrm{div}(P, Q) = \nabla \cdot F = P_x + Q_y$.

NABLA CALCULUS. With the "vector" $\nabla = (\partial_x, \partial_y, \partial_z)$, we can write $\mathrm{curl}(F) = \nabla \times F$ and $\mathrm{div}(F) = \nabla \cdot F$. This is both true in 2D and 3D.

LAPLACE OPERATOR. $\Delta f = \mathrm{div}\,\mathrm{grad}(f) = f_{xx} + f_{yy} + f_{zz}$ can be written as $\nabla^2 f$ because $\nabla \cdot (\nabla f) = \mathrm{div}(\mathrm{grad}(f))$. One can extend this to vectorfields $\Delta F = (\Delta P, \Delta Q, \Delta R)$ and writes $\nabla^2 F$.

IDENTITIES. While direct computations can verify the identities to the left, they become evident with Nabla calculus from formulas for vectors like $\vec{v} \times \vec{v} = \vec{0}, \vec{v} \cdot \vec{v} \times \vec{w} = 0$ or $u \times (v \times w) = v(u \cdot w) - (u \cdot v)w$.

$\mathrm{div}(\mathrm{curl}(F)) = 0$.	$\nabla \cdot \nabla \times F = 0$.
$\mathrm{curl}\,\mathrm{grad}(F) = \vec{0}$	$\nabla \times \nabla F = \vec{0}$.
$\mathrm{curl}(\mathrm{curl}(F)) = \mathrm{grad}(\mathrm{div}(F) - \Delta(F))$.	$\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F$.

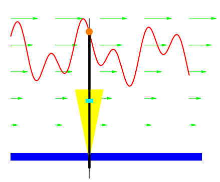
QUIZZ. Is there a vector field G such that $F = (x + y, z, y^2) = \mathrm{curl}(G)$? Answer: no because $\mathrm{div}(F) = 1$ is incompatible with $\mathrm{div} I \mathrm{curl}(G)) = 0$.

ADDENDA TO GREEN'S THEOREM. Green's theorem, one of the advanced topics in this course is useful in physics. We have already seen the following applications

- Simplify computation of double integrals
- Simplify computation of line integrals
- Formulas for centroid of region
- Formulas for area
- Justifying, why mechanical integrators like the planimeter works.

APROPOS PLANIMETER.

The **cone planimeter** is a mechanical instrument to find the antiderivative of a function $f(x)$. It uses the fact that the vector field $F(x, y) = (y, 0)$ has $\mathrm{curl}(F) = -1$. By Greens theorem, the line integral around the type I region bounded by 0 and the graph of $f(x)$ in the counter clockwise direction is $\int_a^x f(x)\ dx$. The planimeter determines that line integral.



THERMODYNAMICS. Gases or liquids are often described in a $P - V$ **diagram**, where the volume in the x-axis and the pressure in the y axis. A periodic process like the pump in a refrigerator leads to closed curve $\gamma : r(t) = (V(t), p(t))$ in the $V - p$ plane. The curve is parameterized by the time t. At a given time, the gas has volume $V(t)$ and a pressure $p(t)$. Consider the vector field $F(V, p) = (p, 0)$ and a closed curve γ. What is $\int_\gamma F\ ds$? Writing it out, we get $\int_0^t (p(t), 0) \cdot (V'(t), p'(t)\ dt = \int_0^t p(t)V'(t)dt = \int_0^t pdV$. The curl of $F(V, p)$ is -1. You see by Green's theorem the integral $-\int_0^t pdV$ is the area of the region enclosed by the curve.

MAXWELL EQUATIONS (in homework you assume no current $j = 0$ and charges $\rho = 0$. c = speed of light.

$\mathrm{div}(B) = 0$	No monopoles	there are no magnetic monopoles.
$\mathrm{curl}(E) = -\frac{1}{c}B_t$	Faraday's law	change of magnetic flux induces voltage
$\mathrm{curl}(B) = \frac{1}{c}E_t + \frac{4\pi}{c}j$	Ampère's law	current or change of E produces magnetic field
$\mathrm{div}(E) = 4\pi\rho$	Gauss law	electric charges are sources for electric field

2D MAXWELL EQUATIONS? In space dimensions d different than 3 the electromagnetic field has $d(d+1)/2$ components. In 2D, the magnetic field is a scalar field and the electric field $E = (P, Q)$ a vector field. The 2D Maxwell equations are $\mathrm{curl}(E) = -\frac{1}{c}\frac{d}{dt}B, \mathrm{div}(E) = 4\pi\rho$. Consider a region R bounded by a wire γ. Green's theorem tells us that $d/dt \int\int_R B(t)\ dxdy$ is the line integral of E around the boundary. But $\int_\gamma Eds$ is a voltage. A change of the magnetic field produces a voltage. This is the **dynamo** in 2D. We will see the real dynamo next week in 3D, where electromagnetism works (it would be difficult to generate a magnetic field in flatland). If v is the velocity distribution of a fluid in the plane, then $\omega(x, y) = \mathrm{curl}(v)(x, y)$ is the **vorticity** of the fluid. The integral $\int\int_R \omega dxdy$ is called the **vortex flux** through R. Green's theorem assures that this flux is related to the circulation on the boundary.

FLUID DYNAMICS. v velocity, ρ density of fluid.

Continuity equation	$\dot{\rho} + \mathrm{div}(\rho v) = 0$	no fluid get lost
Incompressibility	$\mathrm{div}(v) = 0$	incompressible fluids, in 2D: $v = \mathrm{grad}(u)$
Irrotational	$\mathrm{curl}(v) = 0$	irrotation fluids

A RELATED THEOREM. If we rotate the vector field $F = (P, Q)$ by 90 degrees $= \pi/2$ we get a new vector field $G = (-Q, P)$. The integral $\int_\gamma F \ \cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with the length of dr. With $\mathrm{div}(F) = (P_x + Q_y)$, we see that $\mathrm{curl}(F) = \mathrm{div}(G)$. Green's theorem now becomes

$$\int\int_R \mathrm{div}(G)\ dxdy = \int_\gamma G \cdot dn ,$$

where $dn(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vector $r'(x, y)$ at (x, y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. **It is in two dimensions just Green's theorem disguised.** There are only 2 basic integral theorems in the plane: Green's theorem and the FTLI.

PREVIEW. Green's theorem is of the form $\int_R F' = \int_{\delta R} F$, where F' is a "derivative" and δR is a "boundary". There are 2 such theorems in dimensions 2, three theorems in dimensions 3, four in dimension 4 etc. In the plane, Green's theorem is the second one besides the fundamental theorem of line integrals FTLI. In three dimensions, there are two more theorems beside the FTLI: Stokes and Gauss Theorems which we will see in the next week.

dim	dim(R)	theorem
1D	1	Fund. thm of calculus

2D	1	Fund. thm of line integrals
2D	2	Green's theorem

dim	dim(R)	theorem
3D	1	Fundam. thm of line integrals
3D	2	Stokes theorem
3D	3	Gauss theorem

$1 \mapsto 1$	f'	derivative
$1 \mapsto 2$	∇f	gradient
$2 \mapsto 1$	$\nabla \times F$	curl

$1 \mapsto 3$	∇f	gradient
$3 \mapsto 3$	$\nabla \times F$	curl
$3 \mapsto 1$	$\nabla \cdot F$	divergence

4/28/2004, DIVERGENCE THEOREM   Math21a, O. Knill

HOMEWORK. Section 13.8: 6,10,14,30

DIV. The **divergence** of a vector field F is $\text{div}(P,Q,R) = \nabla \cdot F = P_x + Q_y + R_z$. It is a scalar field.
The flux integral of a vector field F through a surface $S = r(R)$ was defined as $\int\int_S F \cdot dS = \int\int_R F(r(u,v)) \cdot r_u \times r_v \, dudv$.
Recall also that the integral of a scalar function f on a region R is $\int\int\int_R f \, dV = \int\int\int_G f(x,y,z) \, dxdydz$.

GAUSS THEOREM or DIVERGENCE THEOREM. Let G be a region in space bounded by a surface S and let F be a vector field. Then

$$\int\int\int_G \text{div}(F) \, dV = \int\int_S F \cdot dS \, .$$

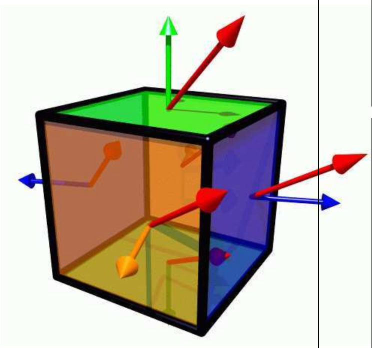
The orientation of S is such that the normal vector $r_u \times r_v$ points **outside** of G.

EXAMPLE. Let $F(x,y,z) = (x,y,z)$ and let S be sphere. The divergence of F is constant 3 and $\int\int\int_G \text{div}(F) \; dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\int\int_S r \cdot r_u \times r_v \; dudv = \int\int_S |r(u,v)|^2 \sin(v) \, dudv = \int_0^\pi \int_0^{2\pi} \sin(v) \, dudv = 4\pi$ also.

CONTINUITY EQUATION. If ρ is the density of a fluid and let v be the velocity field of the fluid, then by conservation of mass, the flux $\int\int_S v \cdot dS$ of the fluid through a closed surface S bounding a region G is $d/dt \int\int\int_G \rho dV$, the change of mass inside G. But this flux is by Gauss theorem equal to $\int\int\int_G \text{div}(v) \; dV$. Therefore, $\int\int\int_G \dot{\rho} - \text{div}(v) \; dV = 0$. Taking a very small ball G around a point (x,y,z), where $\int\int\int_G f \; dV \sim f(x,y,z)$ gives $\dot{\rho} - \text{div}(v)$. This is called the **continuity equation**.

EXAMPLE. What is the flux of the vector field $F(x,y,z) = (2x, 3z^2 + y, \sin(x))$ through the box $G = [0,3] \times [0,2] \times [-1,1]$?
Answer: Use the divergence theorem: $\text{div}(F) = 2$ and so $\int\int\int_G \text{div}(F) \; dV = 2 \int\int\int_G dV = 2\text{Vol}(G) = 24$.
Note: Often, it is easier to evaluate a three dimensional integral than a flux integral because the later needs a parameterization of the boundary, which requires the calculation of $r_u \times r_v$ etc.

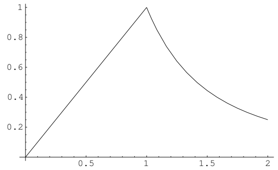
PROOF OF THE DIVERGENCE THEOREM. Consider a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. Call the sides orthogonal to the x axis **x**-boundaries etc. The flux of $F = (P,Q,R)$ through the x-boundaries is $[F(x+dx,y,z) \cdot (1,0,0) + F(x,y,z) \cdot (-1,0,0)] dydz = P(x+dx,y,z) - P(x,y,z) = P_x dxdydz$. Similarly, the flux through the y-boundaries is $P_y dydxdz$ and the flux through the z-boundary is $P_z dzdxdy$. The total flux through the boundary of the box is $(P_x + P_y + P_z) dxdydz = \text{div}(F) dxdydz$.



For a general body, approximate it with a union of small little cubes. The sum of the fluxes over all the little cubes is sum of the fluxes through the sides which do not touch an other box (fluxes through touching sides cancel). The sum of all the infinitesimal fluxes of the cubes is the flux through the boundary of the union. The sum of all the $\text{div}(F) dxdydz$ is a Riemann sum approximation for the integral $\int\int\int_G \text{div}(F) dxdydz$. In the limit, where dx, dy, dz goes to zero, we obtain Gauss theorem.

VOLUME CALCULATION. Similarly as the planimeter allowed to calculate the area of a region by passing along the boundary, the volume of a region can be determined as a flux integral. Take for example the vector field $F(x,y,z) = (x,0,0)$ which has divergence 1. The flux of this vector field through the boundary of a region is the volume of the region. $\int\int_{\delta G} (x,0,0) \cdot dS = \text{Vol}(G)$.

GRAVITY INSIDE THE EARTH. How much do we weight deep in earth at radius r from the center of the earth? (Relevant in the movie "The core") The law of gravity can be formulated as $\text{div}(F) = 4\pi\rho$, where ρ is the mass density. We assume that the earth is a ball of radius R. By rotational symmetry, the gravitational force is normal to the surface: $F(x) = F(r)x/||x||$. The flux of F through a ball of radius r is $\int\int_{S_r} F(x) \cdot dS = 4\pi r^2 F(r)$. By the **divergence theorem**, this is $4\pi M_r = 4\pi \int\int\int_{B_r} \rho(x) \; dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2 \rho r^3/3 = 4\pi r^2 F(r)$ for $r < R$ and $(4\pi)^2 \rho R^3/3 = 4\pi r^2 F(r)$ for $r \geq R$. Inside the earth, the gravitational force $F(r) = 4\pi\rho r/3$. Outside the earth, it satisfies $F(r) = M/r^2$ with $M = 4\pi R^3 \rho/3$.



WHAT IS THE BOUNDARY OF A BOUNDARY? The fundamental theorem for lineintegral, Green's theorem, Stokes theorem and Gauss theorem are all of the form $\int_A dF = \int_{\delta A} F$, where dF is a derivative of F and δA is a boundary of A. They all generalize the fundamental theorem of calculus. There is some similarity in how d and δ behave:

f scalar field	$ddf = \text{curl grad}(f) = 0$	S surface in space	δS is union of closed curves	$\delta\delta S = \emptyset$
F vector field	$ddF = \text{div curl}(F) = 0$	G region in space	δG is a closed surface	$\delta\delta G = \emptyset$

The question when $\text{div}(F) = 0$ implies $F = \text{curl}(G)$ or whether $\text{curl}(F) = 0$ implies $G = \text{grad}(G)$ is interesting. We look at it Friday.

STOKES AND GAUSS. **Stokes theorem was found by Ampere in 1825.** George Gabriel Stokes: (1819-1903) was probably inspired by work of Green and rediscovers the identity around 1840. **Gauss theorem was discovered 1764 by Joseph Louis Lagrange.** Carl Friedrich Gauss, who formulates also Greens theorem, rediscovers the divergence theorem in 1813. Green also rediscovers the divergence theorem in 1825 not knowing of the work of Gauss and Lagrange.

Carl Friedrich Gauss

George Gabriel Stokes

Joseph Louis Lagrange

André Marie Ampere

GREEN IDENTITIES. If G is a region in space bounded by a surface S and f, g are scalar functions, then with $\Delta f = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$, one has as a direct consequence of Gauss theorem the **first and second Green identities** (see homework)

$$\boxed{\int\int\int_G (f\Delta g + \nabla f \cdot \nabla g) \, dV = \int\int_S f \nabla g \cdot dS} \qquad \boxed{\int\int\int_G (f\Delta g - g\Delta f) \, dV = \int\int_S (f\nabla g - g\nabla f) \cdot dS}$$

These identities are useful in electrostatics. Example: if $g = f$ and $\Delta f = 0$ and either $f = 0$ on the boundary S or ∇f is orthogonal to S, then Green's first identity gives $\int\int\int_G |\nabla f|^2 \; dV = 0$ which means $f = 0$. This can be used to prove uniqueness for the **Poisson equation** $\Delta h = 4\pi\rho$ when applying the identity to the difference $f = h_1 - h_2$ of two solutions with either **Dirichlet boundary conditions** ($h = 0$ on S) or von Neumann boundary conditions (∇h orthogonal to S).

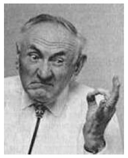
GAUSS THEOREM IN HIGHER DIMENSIONS. If G is a n-dimensional "hyperspace" bounded by a $(n-1)$ dimensional "hypersurface" S, then $\int_G \text{div}(F) dV = \int_S F \cdot dS$.

DIV. In dimension d, the divergence is defined $\text{div}(F) = \nabla \cdot F = \sum_i \partial F_i / \partial x_i$. The proof of the n-dimensional divergence theorem is done as in three dimensions.

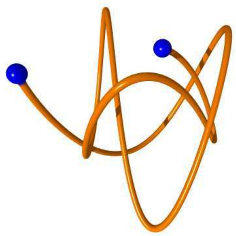
By the way: Gauss theorem in two dimensions is just a version of Green's theorem. Replacing $F = (P,Q)$ with $G = (-Q,P)$ gives $\text{curl}(F) = \text{div}(G)$ and the flux of G through a curve is the lineintegral of F along the curve. Green's theorem for F is identical to the 2D-divergence theorem for G.

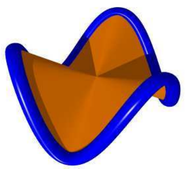
Problems: 1,2,3,4,5 in Homework (see homework handout).

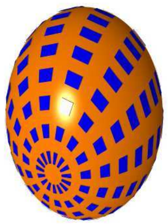
ZWICKY'S VIEW. Fritz Zwicky was an astronomer who also devised the **morphological method** for creativity. The method is one of many tools to solve problems which require new ideas. Zwicky was highly creative himself. The idea of the supernovae, dark matter and many pulsar discoveries can be attributed to him. In 1948 for example, Zwicky suggested to use extraterrestrial sources to reconstruct the universe. This should begin with changing other planets, moons and asteroids by making them inhabitable and to change their orbits around the sun in order to adjust their temperature. He even suggested to alter the fusion of the sun to make displace our own space solar system towards an other planetary system. Zwickies **morphological method** is to organize ideas in boxes like a spreadsheet. For example, if we wanted to make order in the zoo of integral theorems we have seen now, we would one coordinate to display the dimension, in which we work and the second coordinate the maximal dimension along which we integrate in the theorem.



	1	2	3
1	Fundamental theorem of calculus	-	-
2	Fundamental theorem of line integrals	Greens theorem	-
3	Fundamental theorem of line integrals	Stokes theorem	Gauss theorem







Fundamental thm of line integrals. We integrate over 1 and 0 dimensional objects. The 0 dimensional object has no boundary.

Green and Stokes theorems. We integrate over 2 and 1 dimensional objects. The 1 dimensional object is a closed curve and has no boundary.

Divergence theorem. We integrate over 3 and 2 dimensional objects. The 2 dimensional object is a closed surface and has no boundary.

One could build now for each of the differential operators grad, curl and div such a matrix and see whether there is a theorem. Many combinations do not make sense like integrating the curl over a three dimensional object. But there is a theorem, which seem have escaped: if we integrate in two dimensions the $\mathrm{div}(F) = M_x + N_y$ over a region R, then there this can be written as an integral over the boundary. There is indeed such a theorem, but it is just a version of Greens theorem in disguise (turn the vector field by 90 degrees and replace the line by a 2D version of the flux integral). It can be seen as a special case of the divergence theorem in three dimensions and it does not make sense to put it on the footing of the other theorems. Anyway, the morphological method was used in management like in Ciba, it is today just one of a variety of methods to come up with new ideas.

IDENTITIES. While direct computations can verify the identities to the left, they become evident with **Nabla calculus** from formulas for vectors like $\vec{v} \times \vec{v} = \vec{0}, \vec{v} \cdot \vec{v} \times \vec{w} = 0$ or $u \times (v \times w) = v(u \cdot w) - (u \cdot v)w$.

$\mathrm{div}(\mathrm{curl}(F)) = 0$.
$\mathrm{curl}(\mathrm{grad}(F)) = \vec{0}$
$\mathrm{curl}(\mathrm{curl}(F)) = \mathrm{grad}(\mathrm{div}(F) - \Delta(F))$.

$\nabla \cdot \nabla \times F = 0$.
$\nabla \times \nabla F = \vec{0}$.
$\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F$.

QUIZZ. Is there a vector field G such that $F = (x + y, z, y^2) = \mathrm{curl}(G)$? Answer: no because $\mathrm{div}(F) = 1$ is incompatible with $\mathrm{divcurl}(G)) = 0$.

BOXED OVERVIEW. All integral theorems are of the form $\int_R F' = \int_{\delta R} F$, where F' is a "derivative" and δR is a "boundary". There are 2 such theorems in dimensions 2, three theorems in dimensions 3, four in dimension 4 etc.

dim	dim(R)	theorem
1D	1	Fund. thm of calculus

2D	1	Fund. thm of line integrals
2D	2	Green's theorem

dim	dim(R)	theorem
3D	1	Fundam. thm of line integrals
3D	2	Stokes theorem
3D	3	Divergence theorem

$1 \mapsto 1$	f'	derivative
$1 \mapsto 2$	∇f	gradient
$2 \mapsto 1$	$\nabla \times F$	curl

$1 \mapsto 3$	∇f	gradient
$3 \mapsto 3$	$\nabla \times F$	curl
$3 \mapsto 1$	$\nabla \cdot F$	divergence

MAXWELL EQUATIONS. c is the speed of light.

$\mathrm{div}(B) = 0$	No monopoles	there are no magnetic monopoles.
$\mathrm{curl}(E) = -\frac{1}{c}B_t$	Faraday's law	change of magnetic flux induces voltage
$\mathrm{curl}(B) = \frac{1}{c}E_t + \frac{4\pi}{c}j$	Ampère's law	current or change of E produces magnetic field
$\mathrm{div}(E) = 4\pi\rho$	Gauss law	electric charges are sources for electric field

MAGNETOSTATICS: $\mathrm{curl}(B) = 0$ so that $B = \mathrm{grad}(f)$. But since also $\mathrm{div}(B) = 0$, we have

$$\boxed{\Delta f = \mathrm{div}(\mathrm{grad}(f)) = 0}$$

ELECTROSTATICS. $\mathrm{curl}(E) = 0$ so that $E = \mathrm{grad}(f)$. But since also $\mathrm{div}(E) = 0$, we have

$$\boxed{\Delta f = \mathrm{div}(\mathrm{grad}(f)) = 0}$$

Static electric and magnetic fields have a harmonic potential.

FLUID DYNAMICS. v velocity, ρ density of fluid.

Continuity equation	$\dot{\rho} + \mathrm{div}(\rho v) = 0$	no fluid get lost
Incompressibility	$\mathrm{div}(v) = 0$	incompressible fluids
Irrotational	$\mathrm{curl}(v) = 0$	no vorticity
Potential fluids	$v = \mathrm{grad}(f), \Delta(f) = 0$	incompressible, irrotational fluids

Incompressible, irrotational fluid velocity fields have a potential which is harmonic

HARMONIC FUNCTIONS. A function $f(x, y, z)$ is called **harmonic**, if $\Delta f(x, y, z) = 0$. Harmonic functions play an important role in physics. For example time independent solutions of the Schrödinger equation $i\hbar f_t = \Delta f$ or the heat equation $f_t = \Delta f$ are harmonic. In fluid dynamics, incompressible and irrotational fluids with velocity distribution v satisfy $v = \mathrm{grad}(f)$ and since $\mathrm{div}(v) = 0$, also $\Delta f = 0$. Harmonic functions have properties which can be seen nicely using the divergence theorem.

MAXIMUM PRINCIPLE. A harmonic function in D can not have a local maximum inside D.

MEAN VALUE PROPERTY. The average of a harmonic function on a sphere of radius r around a point P is equal to the value of the function at the point P.

You deal with such problems in the homework. They are direct consequences of the divergence theorem.

THE PIGEON PROBLEM. A farmer drives a closed van through the countryside. Inside the van are dozens of pigeons. The van weights 200 pounds, the pigeon, an other 100 pounds. The driver has to pass a bridge, which can only sustain 250 pounds. The driver has an idea: if all the pigeon would fly while crossing the bridge, a passage would be possible. Question: does the bridge hold? What happens if the cage is open?

The flight of the pigeon produces a wind distribution in the cage. The question is whether the flying pigeon will still produces a weight on the van.

Since the middle ages, people were interested in the gravity of earth. Originally, the motivation was to figure out how life would be like in hell ... Nowadays, the interest has sparked by science fiction movies like "the core", where people travel to the core of the earth.

GRAVITY INSIDE THE EARTH. How much do we weight deep in earth at radius r from the center of the earth? (Relevant in the movie "The core") The law of gravity can be formulated as

$$\boxed{\mathrm{div}(F) = 4\pi\rho}$$

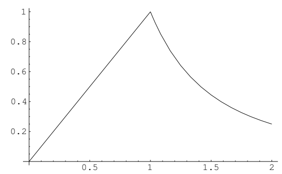
where ρ is the mass density.
We assume that the earth is a ball of radius R. By rotational symmetry, the gravitational force is normal to the surface:

$$F(x) = F(r)x/||x|| \, .$$

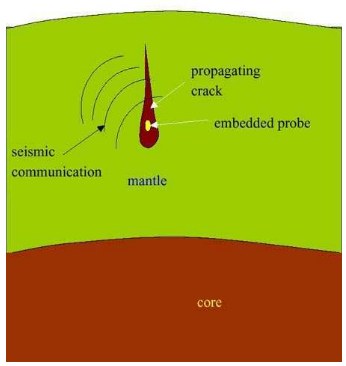
The flux of F through a ball of radius r is

$$\int\int_{S_r} F(x) \cdot dS = 4\pi r^2 F(r)$$

By the **divergence theorem**, this is $4\pi M_r = 4\pi \int\int\int_{B_r} \rho(x) \, dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2 \rho r^3/3 = 4\pi r^2 F(r)$ for $r < R$ and $(4\pi)^2 \rho R^3/3 = 4\pi r^2 F(r)$ for $r \geq R$.
Inside the earth, the gravitational force $F(r) = 4\pi\rho r/3$. Outside the earth, it satisfies $F(r) = M/r^2$ with $M = 4\pi R^3 \rho/3$.



We have seen above, how the gravitational field behaves inside a homogeneous body. This would have been a difficult computation without the divergence theorem.
Since we know more about the interior of other stars then the interior of our earth, people still think about sending probes to the earth center. One of the first proposals was by Zwicky who wanted to use tunnels through the earth for travel purposes. One of the recent proposals is to send liquid metal down a crack and let this propagate.



A nuclear explosion or earthquake might reveal the Earth's innards

MICHAEL HOPKIN

Abstract: Cracking idea gets to the core Astronomer says time is ripe for an unmanned journey to the centre of the Earth.

We have sent probes to Mars, the outer planets and even beyond the farthest reaches of the Solar System - but never to the centre of our own world. Now a planetary scientist has a radical plan to redress the balance.

Outer space is trillions upon trillions of times bigger than the Earth's interior, and yet we know more about what's out there than we do about what's under our feet, says David Stevenson of the California Institute of Technology in Pasadena.

The Earth's core is "crammed with interesting stuff", he says. For example, swirling currents of liquid metal 3,000-5,000 kilometers below the surface are thought to generate the planet's magnetic field.

So Stevenson suggests we blast a crack in the Earth's surface and pour in thousands of tonnes of molten iron. The iron-filled crack would burrow downwards, closing up behind itself as it descends, and reaching the bowels of the planet in about a week.

Concealed in this seething blob of iron would be a grapefruit-sized probe containing instruments to measure temperature, electrical conductivity and chemical composition. The device would send data as sound waves to a surface detector, which would filter the signal from the Earth's natural rumblings.

Starting the crack would be the tricky part, says Stevenson. He calculates that it would take a blast equivalent to several megatonnes of TNT, an earthquake of magnitude 7 on the Richter scale, or a nuclear device such as those already possessed by many nations.

Getting hold of the iron would also be an impressive logistical feat - Stevenson reckons that the experiment would need to commandeer all of the world's iron foundries for anything from an hour to a week. He is confident, however, that the molten reservoir's sheer size would stop it solidifying before it is poured into the ground.

It's an interesting proposition, comments Allan Rubin, who studies the Earth's structure at Princeton University in New Haven, Connecticut, but it does make some rather blithe assumptions. "We don't even know for definite that the lower mantle can fracture," he points out.

Stevenson claims that the cost of his plan would be small beans compared with the riches lavished on space exploration. But he offers prospective investors no guarantee of success: "Frankly, I would be surprised if this really works," he admits.

References

1. Stevenson, D. J. Mission to Earth's core - a modest proposal. Nature, 423, 239 - 240, (2003).

Source: Nature News Service / Macmillan Magazines Ltd, 2003

REVIEW BEYOND VECTOR FIELDS **Math21a,O.Knill**

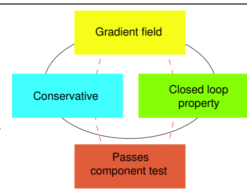
INTEGRATION.
Line integral: $\int_C F \cdot ds = \int_a^b F(r(t)) \cdot r'(t)\, dt$
Surface integral $\int\int_S f dS = \int_a^b \int_c^d f(r(u,v))|r_u(u,v) \times r_v(u,v)|\, dudv$
Flux integral: $\int\int_S F \cdot dS = \int_a^b \int_c^d F(r(u,v)) \cdot r_u(u,v) \times r_v(u,v)\, dudv$
Double integral: $\int\int_R f\, dA = \int_a^b \int_c^d f(x,y)\, dxdy$.
Triple integral: $\int\int\int_R f\, dV = \int_a^b \int_c^d \int_o^p f(x,y,z)\, dxdydz$.

Area $\int\int_R 1 dA = \int\int_R 1\, dxdy$
Length $\int_a^b |r'(t)|\, dt$
Surface area $\int\int 1 dS = \int\int |r_u \times r_v|\, dudv$
Volume $\int\int\int_B 1\; dV = \int\int\int_B 1\, dxdydz$

DIFFERENTIATION.
Derivative: $f'(t) = \dot{f}(t) = d/dt f(t)$.
Partial derivative: $f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)$.
Gradient: $\mathrm{grad}(f) = (f_x, f_y, f_z)$
Curl in 2D: $\mathrm{curl}(F) = \mathrm{curl}((M,N)) = N_x - M_y$
Curl in 3D: $\mathrm{curl}(F) = \mathrm{curl}(M,N,P) = (P_y - N_z, M_z - P_x, N_x - M_y)$
Div: $\mathrm{div}(F) = \mathrm{div}(M,N,P) = M_x + N_y + P_z$.

IDENTITIES.
$\mathrm{div}(\mathrm{curl}(F)) = 0$
$\mathrm{curl}(\mathrm{grad}(f)) = (0,0,0)$
$\mathrm{div}(\mathrm{grad}(f)) = \Delta f$.

NOTIONSS OF CONSERVATIVE. 1) Gradient: $F = \mathrm{grad}(f)$.
2) Closed curve property: $\int_C F \cdot dr = 0$ for any closed curve.
3) Conservative: C_i paths from A to B, then $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$.
4) Mixed derivative property: $\mathrm{curl}(F) = 0$ in simply connected regions.



TOPOLOGY. 1) **Interior** of region D: points which have a neighborhood contained in D.
2) **Boundary** of a curve: endpoints of curve, **boundary** of 2D region D: curves which bound the region, **boundary** of a solid D: surfaces which bound the solid.
3) **Simply connected** region D: a closed curve in D can be deformed within the interior of D to a point.
4) **Closed curve** Curve without boundary.
5) **Closed surface** surface without boundary.

LINE INTEGRAL THEOREM. If $C : r(t) = (x(t), y(t), z(t)), t \in [a,b]$ is a curve and f is a function either in 3D or the plane. Then

$$\int_C \nabla f \cdot ds = f(r(b)) - f(r(a))$$

CONSEQUENCES.
1) For closed curves the line integral $\int_C \nabla f \cdot ds$ is zero.
2) Gradient fields are **conservative**: if $F = \nabla f$, then the line integral between two points P and Q is path independent.
3) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus** $\int_a^b f'(x)\, dx = f(b) - f(a)$
4) The theorem justifies the name **conservative** for gradient vector fields.
5) The term "potential" was coined by George Greeen (1783-1841).

PROBLEM. Let $f(x,y,z) = x^2 + y^4 + z$. Find the line integral of the vector field $F(x,y,z) = \nabla f(x,y,z)$ along the path $r(t) = (\cos(5t), \sin(2t), t^2)$ from $t = 0$ to $t = 2\pi$.

SOLUTION. $r(0) = (1,0,0)$ and $r(2\pi) = (1,0,4\pi^2)$ and $f(r(0)) = 1$ and $f(r(2\pi)) = 1 + 4\pi^2$. FTLI gives $\int_C \nabla f\, ds = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

GREEN'S THEOREM. If R is a region with boundary C and $F = (M,N)$ is a vector field, then

$$\int\int_R \mathrm{curl}(F)\, dA = \int_C F \cdot ds$$

REMARKS.
1) Useful to swap 2D integrals to 1D integrals or the other way round.
2) The curve is oriented in such a way that the region is to your left.
3) The region has to has piecewise smooth boundaries (i.e. it should not look like the Mandelbrot set).
4) If $C : t \mapsto r(t) = (x(t), y(t))$, the line integral is $\int_a^b (M(x(t), y(t)), N(x(t), y(t)) \cdot (x'(t), y'(t))\, dt$.
5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
6) If $\mathrm{curl}(F) = 0$ in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
7) Taking $F(x,y) = (-y, 0)$ or $F(x,y) = (0,x)$ gives **area formulas**.

PROBLEM. Find the line integral of the vector field $F(x,y) = (x^4 + \sin(x) + y, x + y^3)$ along the path $r(t) = (\cos(t), 5\sin(t) + \log(1 + \sin(t)))$, where t runs from $t = 0$ to $t = \pi$.

SOLUTION. $\mathrm{curl}(F) = 0$ implies that the line integral depends only on the end points $(0,1), (0,-1)$ of the path. Take the simpler path $r(t) = (-t, 0), t = [-1,1]$, which has velocity $r'(t) = (-1, 0)$. The line integral is $\int_{-1}^1 (t^4 - \sin(t), -t) \cdot (-1, 0)\, dt = -t^5/5|_{-1}^1 = -2/5$.

REMARK. We could also find a potential $f(x,y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\mathrm{grad}(f) = F$. Again, we get $f(0,-1) - f(0,1) = -1/5 - 1/5 = -2/5$.

STOKES THEOREM. If S is a surface in space with boundary C and F is a vector field, then

$$\int\int_S \mathrm{curl}(F) \cdot dS = \int_C F \cdot ds$$

REMARKS.
1) Stokes theorem implies Greens theorem if F is z independent and S is contained in the z-plane.
2) The orientation of C is such that if you walk along C and have your head in the direction, where the normal vector $r_u \times r_v$ of S, then the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903). 4) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S. This is analogue to the fact that the line integral of a gradient field only depends on the end points of the curve.
5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

PROBLEM. Compute the line integral of $F(x,y,z) = (x^3 + xy, y, z)$ along the polygonal path C connecting the points $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$.

SOLUTION. The path C bounds a surface $S : r(u,v) = (u,v,0)$ parameterized by $R = [0,2] \times [0,1]$. By Stokes theorem, the line integral is equal to the flux of $\mathrm{curl}(F)(x,y,z) = (0,0,-x)$ through S. The normal vector of S is $r_u \times r_v = (1,0,0) \times (0,1,0) = (0,0,1)$ so that $\int\int_S \mathrm{curl}(F)\; dS = \int_0^2 \int_0^1 (0,0,-u) \cdot (0,0,1)\, dudv = \int_0^2 \int_0^1 -u\, dudv = -2$.

GAUSS THEOREM. If S is the boundary of a region B in space with boundary S and F is a vector field, then

$$\int\int\int_B \mathrm{div}(F)\, dV = \int\int_S F \cdot dS$$

REMARKS.
1) Gauss theorem is also called **divergence theorem**.
2) Gauss theorem can be helpful to determine the flux of vector fields through surfaces.
3) Gauss theorem was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
4) For divergence free vector fields F, the flux through a closed surface is zero. Such fields F are also called **incompressible** or **source free**.

PROBLEM. Compute the flux of the vector field $F(x,y,z) = (-x, y, z^2)$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.
SOLUTION. By Gauss theorem, the flux is equal to the triple integral of $\mathrm{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z\, dxdydz = (3-0)(2-(-1))(4-1) = 27$.

HOMEWORK: Problems 6,7,8 in additional homework handout.

SUMMARY. The FTLI, Green, Stokes and Gauss theorem generalize the fundamental theorem of calculus and are all of the form $\boxed{\int_{dR} F = \int_R dF}$, where dR is the boundary of R and dF is a derivative of F.

INTEGRATION.

Line integral: $\int_C F \cdot ds = \int_a^b F(r(t)) \cdot r'(t)\, dt$
Surface integral $\int\int_S f dS = \int_a^b \int_c^d f(r(u,v))|r_u(u,v) \times r_v(u,v)|\, dudv$
Flux integral: $\int\int_S F \cdot dS = \int_a^b \int_c^d F(r(u,v)) \cdot r_u(u,v) \times r_v(u,v)\, dudv$
Double integral: $\int\int_R f\, dA = \int_a^b \int_c^d f(x,y)\, dxdy$.
Triple integral: $\int\int\int_R f\, dV = \int_a^b \int_c^d \int_o^p f(x,y,z)\, dxdydz$.

Area $\int\int_R 1 dA = \int\int_R 1\, dxdy$
Surface area $\int\int 1 dS = \int\int |r_u \times r_v|\, dudv$
Volume $\int\int\int_B 1\ dV = \int\int\int_B 1\, dxdydz$

DIFFERENTIATION.

Derivative: $f'(t) = \dot{f}(t) = d/dt f(t)$.
Partial derivative: $f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)$.
Gradient: $\mathrm{grad}(f) = (f_x, f_y, f_z)$
Curl in 2D: $\mathrm{curl}(F) = \mathrm{curl}((M,N)) = N_x - M_y$
Curl in 3D: $\mathrm{curl}(F) = \mathrm{curl}(M,N,P) = (P_y - N_z, M_z - P_x, N_x - M_y)$
Div: $\mathrm{div}(F) = \mathrm{div}(M,N,P) = M_x + N_y + P_z$.

IDENTITIES.
$\mathrm{div}(\mathrm{curl}(F)) = 0$
$\mathrm{curl}(\mathrm{grad}(f)) = (0,0,0)$

LINE INTEGRAL THEOREM. If $C : r(t) = (x(t), y(t), z(t)), t \in [a,b]$ is a curve and f is a function either in 3D or the plane. Then

$$\boxed{\int_\gamma \nabla f \cdot ds = f(r(b)) - f(r(a))}$$

CONSEQUENCES.
1) If the curve is closed, then the line integral $\int_\gamma \nabla f \cdot ds$ is zero.
2) If $F = \nabla f$, the line integral between two points P and Q does not depend on the chosen path.

REMARKS.
1) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus**

$$\boxed{\int_a^b f'(x)\, dx = f(b) - f(a)}$$

2) The theorem justifies the name **conservative** for gradient vector fields.
3) In physics, f is the **potential energy** and ∇f a **force**. The theorem says that such forces lead to **energy conservation**.
4) The term "potential" was coined by George Greeen (1783-1841).

PROBLEM. Let $f(x,y,z) = x^2 + y^4 + z$. Find the line integral of the vector field $F(x,y,z) = \nabla f(x,y,z)$ along the path $r(t) = (\cos(5t), \sin(2t), t^2)$ from $t = 0$ to $t = 2\pi$.

SOLUTION. $r(0) = (1,0,0)$ and $r(2\pi) = (1,0,4\pi^2)$ and $f(r(0)) = 1$ and $f(r(2\pi)) = 1 + 4\pi^2$. Applying the fundamental theorem gives $\int_\gamma \nabla f\, ds = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

GREEN'S THEOREM. If R is a region with boundary γ and $F = (M,N)$ is a vector field, then

$$\boxed{\int\int_R \mathrm{curl}(F)\, dA = \int_\gamma F \cdot ds}$$

REMARKS.
1) Useful to swap 2D integrals to 1D integrals or the other way round.
2) The curve is oriented in such a way that the region is to your left.
3) The region has to has piecewise smooth boundaries (i.e. it should not look like the Mandelbrot set).
4) If $\gamma : t \mapsto r(t) = (x(t), y(t))$, the line integral is $\int_a^b (M(x(t),y(t)), N(x(t),y(t)) \cdot (x'(t), y'(t))\, dt$.
5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).

CONSEQUENCES.
1) If $\mathrm{curl}(F) = 0$ in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
2) Taking $F(x,y) = (-y, 0)$ gives an **area formula** $\mathrm{Area}(R) = \int -y\, dx$. Similarly $\mathrm{Area}(A) = \int x dy$.

PROBLEM. Find the line integral of the vector field $F(x,y) = (x^4 + \sin(x) + y, x + y^3)$ along the path $r(t) = (\cos(t), 5\sin(t) + \log(1 + \sin(t)))$, where t runs from $t = 0$ to $t = \pi$.

SOLUTION. $\mathrm{curl}(F) = 0$ implies that the line integral depends only on the end points $(0,1), (0,-1)$ of the path. Take the simpler path $r(t) = (-t, 0), t = [-1,1]$, which has velocity $r'(t) = (-1,0)$. The line integral is $\int_{-1}^1 (t^4 - \sin(t), -t) \cdot (-1,0)\, dt = -t^5/5|_{-1}^1 = -2/5$.

REMARK. We could also find a potential $f(x,y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\mathrm{grad}(f) = F$. Again, we get $f(0,-1) - f(0,1) = -1/5 - 1/5 = -2/5$.

STOKES THEOREM. If S is a surface in space with boundary γ and F is a vector field, then

$$\boxed{\int\int_S \mathrm{curl}(F) \cdot dS = \int_\gamma F \cdot ds}$$

REMARKS.
1) Stokes theorem reduces to Greens theorem if F is z independent and S is contained in the z-plane.
2) The orientation of γ is such that if you walk along γ and have your head in the direction, where the normal vector $r_u \times r_v$ of S points, then you have the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903).

CONSEQUENCES.
1) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S. This is analogue to the fact that the line integral of a gradient field only depends on the end points of the curve.
2) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

PROBLEM. Compute the line integral of $F(x,y,z) = (x^3 + xy, y, z)$ along the polygonal path γ connecting the points $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$.

SOLUTION. The path γ bounds a surface $S : r(u,v) = (u,v,0)$ parameterized by $R = [0,2] \times [0,1]$. By Stokes theorem, the line integral is equal to the flux of $\mathrm{curl}(F)(x,y,z) = (0,0,-x)$ through S. The normal vector of S is $r_u \times r_v = (1,0,0) \times (0,1,0) = (0,0,1)$ so that $\int\int_S \mathrm{curl}(F)\, dS = \int_0^2 \int_0^1 (0,0,-u) \cdot (0,0,1)\, dudv = \int_0^2 \int_0^1 -u\, dudv = -2$.

GAUSS THEOREM. If S is the boundary of a region B in space with boundary S and F is a vector field, then

$$\boxed{\int\int\int_B \mathrm{div}(F)\, dV = \int\int_S F \cdot dS}$$

REMARKS.
1) Gauss theorem is also called **divergence theorem**.
2) Gauss theorem can be helpful to determine the flux of vector fields through surfaces.
3) Gauss theorem was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.

CONSEQUENCE.
For divergence free vector fields F, the flux through a closed surface is zero. Such fields F are also called **incompressible** or **source free**.

PROBLEM. Compute the flux of the vector field $F(x,y,z) = (-x, y, z^2)$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.
SOLUTION. By Gauss theorem, the flux is equal to the triple integral of $\mathrm{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z\, dxdydz = (3-0)(2-(-1))(4-1) = 27$.

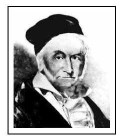
5/3/2003, INTEGRAL THEOREMS: PEOPLE & GOSSIP **Math21a, O. Knill**

AMPÈRE. André Marie Ampère (1775-1836) made contributions to the theory of electricity and magnetism. Stokes theorem was found by Ampère in 1825.

Gossip:

- While still only 13 years old Ampère submitted his first paper to the Académie de Lyon. Amperès father died during the French Revolution. He went to the guillotine writing to Ampère's mother from his cell: "I desire my death to be the seal of a general reconciliation between all our brothers; I pardon those who rejoice in it, those who provoked it, and those who ordered it.... "
- Ampère had a difficult time with his daughter. She married one of Napoleon's lieutenants who was an alcoholic. The marriage soon was in trouble. Ampère's daughter fled to her father's house and Ampère allowed later her husband to live with him also. This proved a difficult situation, led to police intervention and much unhappiness for Ampère.

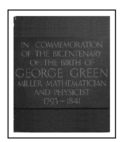
GAUSS. Carl Friedrich Gauss (1777-1855) worked in a wide variety of fields both in mathematics and physics like number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. Gauss theorem was discovered 1764 by Joseph Louis Lagrange.



Gossip:

- At the age of seven, while starting elementary school his potential was noticed almost immediately. His teacher, Büttner, and his assistant, Martin Bartels, were amazed when Gauss summed up the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers each pair summing to 101.
- It seems that Gauss knew already about non-Euclidean geometry at the age of 15.
- Gauss married twice. Both wives died, the first from the birth of a child, the second after a long illness.

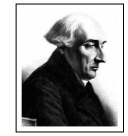
GREEN. George Green (1793-1841) discovered Green's theorem in 1827. Green also rediscovers the divergence theorem in 1825 not knowing of the work of Gauss and Lagrange.



Gossip:

- Green studied mathematics on the top floor of his fathers mill, entirely on his own. The years between 1823 and 1828 were not easy for Green: As well as having a full time job in the mill, two daughters were born, the one in 1824 mentioned above, and a second in 1827. Between these two events his mother had died in 1825 and his father was to die in 1829. Yet despite the difficult circumstances and despite his flimsy mathematical background, Green published one of the most important mathematical works of all time in 1828.
- Green was the son of a baker: from a short obituary in the Nottingham Review which showed they knew little of his life and less of the importance of his work. "... Through Thomson, Maxwell, and others, the general mathematical theory of potential developed by an obscure, self-taught millers son would lead to the mathematical theories of electricity underlying twentieth-century industry.

LAGRANGE. Joseph-Louis Lagrange (1736-1813) worked in all fields of analysis and number theory and analytical and celestial mechanics. Gauss theorem was discovered in 1764 by Joseph Louis Lagrange.



Gossip:

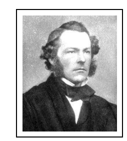
- Lagrange is usually considered a French mathematician. But some refer to him as an Italian mathematician because Lagrange was born in Turin. Only the intervention of Lavoisier saved Lagrange during the French revolution from the guillotine because a law was established arresting all foreign born people. Lavoisier himself was guillotined shortly after.
- Lagrange first wife was his cousin. They had no children. In fact, Lagrange had told d'Alembert in this letter that he did not wish to have children. His first wife died after years of illness.
- Napoleon named Lagrange to the Legion of Honor and Count of the Empire in 1808. On April 3, 1813 Lagrange was named "grand croix of the Ordre Impérial de la Réunion". He died a week later.

OSTROGRADSKI. Mikhail Vasilevich Ostrogradski (1801-1862) worked on integral calculus, mathematical physics like hydrodynamics and partial differential equations. He worked also in algebra and differential equations, theoretical mechanics and wrote several textbooks.

Gossip:

- Ostrogradski wanted actually to pursue a military carrier. Financial considerations however led him to a career in the civil service which needed a university education. He studied physics and mathematics. He successfully finished his exams but the minister of religious affairs and national education refused to confirm the decision. Ostrogradski therefore never got a degree.
- He left Russia to study in Paris, between 1822 and 1827 Ostrogradski attended lectures by Laplace, Fourier, Legendre, Poisson and Cauchy.

STOKES. George Gabriel Stokes (1819-1903) established the science of hydrodynamics with his law of viscosity describing the velocity of a small sphere through a viscous fluid. Stokes discovery of Stokes theorem (around 1840) was probably inspired by work of Green.



Gossip:

- In 1857 Stokes had to give up his fellowship at Pembroke College because fellows at Cambridge had then to be unmarried. Later, after a change in the rules in 1862 Stokes was able to take up the fellowship at Pembroke again.
- While Stokes became engaged, he used to express his feelings to his bride Mary Susanna Robinson in rather mathematical terms: "I too feel that I have been thinking too much of late, but in a different way, my head running on divergent series, the discontinuity of arbitrary constants ... ". Such words did not reveal the love that Mary hoped to find in them and when Stokes wrote her a 55 page (!) letter about the duty he felt towards her, she came close to calling off the wedding.

SOURCES. The sources include E.T. Whittaker's *A History of the Theories of Ether and Electricity* and the MacTutor History of Mathematics http://turnbull.dcs.st-and.ac.uk/history/.

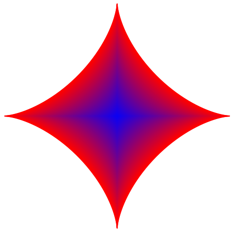
5/5/2004 INTEGRAL THEOREM PROBLEMS — Math21a, O. Knill

HOMEWORK. Problems 9,10,11 on the homeworksheet.

SUMMARY. This is a collection of problems on line integrals, Green's theorem, Stokes theorem and the divergence theorem. Some of them are more challenging.

LINE INTEGRALS GREEN THEOREM.

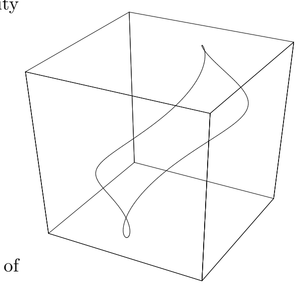
The curve $r(t) = (\cos^3(t), \sin^3(t))$ is called a **hypocycloid**. It bounds a region R in the plane.
a) Calculate the line integral of the vector field $F(x, y) = (x, y)$ along the curve.
b) Find the area of the hypocycloid.



a) Because $\mathrm{curl}(F) = 0$ the result is zero by Green's theorem.
b) Use the vector field $F(x, y) = (0, x)$ which has $\mathrm{curl}(F) = 1$. The line integral is $\int_0^{2\pi} F(r(t)) \cdot r'(t)\ dt = \int_0^{2\pi} \cos^3(t) 3 \sin^2(t) \cos(t)\ dt = \int_0^{2\pi} 3\cos^4(t)\sin^2(t)\ dt = 3\pi/8$. (To compute the integral, use that $8\cos^4(t)\sin^2(t) = \cos(2t)\sin^2(2t) + \sin^2(2t)$).

LENGTH OF CURVE AND LINE INTEGRALS.

Assume $C : t \mapsto r(t)$ is a closed path in space and $F(r(t))$ is the unit tangent vector to the curve (that is a vector parallel to the velocity vector which has length 1).
a) What is $\int_C F\ dr$?
b) Can F be a gradient field?



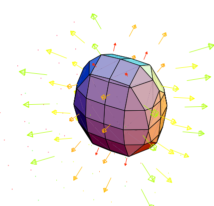
Answer:
a) $F(r(t)) = r'(t)/|r'(t)|$. By definition of the line integral,

$$\int_C F(r(t)) \cdot r'(t)\ dt = \int_a^b \frac{r'(t)}{|r'(t)|} \cdot r'(t)\ dt = \int_a^b |r'(t)|\ dt\ ,$$

which is the length of the curve.
b) No: If F were a gradient field, then by the fundamental theorem of line integrals, we would have that the line integral along a closed curve is zero. But because this is the length of the curve, this is not possible.

SURFACE AREA AND FLUX.

Assume $S : (u, v) \mapsto r(u, v)$ is a closed surface in space and $F(r(u, v))$ is the unit normal vector on S (which points in the direction of $r_u \times r_v$).
a) What is $\int\int_S F \cdot dS$?
b) Is it possible that F is the curl of an other vector field?
c) Is it possible that $\mathrm{div}(F) = (0, 0, 0)$ everywhere inside the surface.

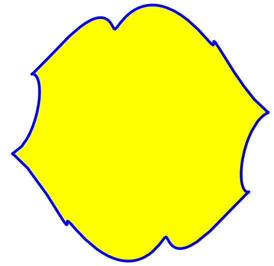


Answer:
a) $F(r(u, v)) = (r_u \times r_v)/|r_u \times r_v|$. By definition of the flux integral, $\int\int_S F \cdot dS = \int\int_R F(r(u, v)) \cdot r_u \times r_v = \int\int_R (r_u \times r_v/|r_u \times r_v|) \cdot r_u \times r_v = \int\int_R |r_u \times r_v|\ dudv$ which is the area of the surface.
b) No, if F were the curl of an other field G, then the flux of F through the closed surface would be zero. But since it is the area, this is not possible.
c) From the divergence theorem follows that $\mathrm{div}(F)$ is nonzero somewhere inside the surface.

GREENS THEOREM AND LAPLACIAN.

Assume R is a region in the plane and let n denote the unit normal vector to the boundary C of R. For any function $u(x, y)$, we use the notation $\partial f/\partial u = \mathrm{grad}(u) \cdot \vec{n}$ which is the directional derivative of u into the direction $\vec{n}$ normal to C. We also use the notation $\Delta u = u_{xx} + u_{yy}$. Show that

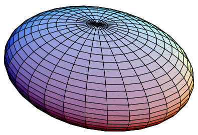
$$\int_C \partial u/\partial n \cdot dr = \int\int_R \Delta u\ dA$$



Answer: Define $F(x, y) = (-B, A)$ if $\partial u/\partial n = (A, B)$. The left integral is the line integral of F along C. The right integral is the double integral over $\Delta u = \mathrm{curl}(F)$.

STOKES THEOREM OR DIVERGENCE THEOREM

Find $\int\int_S \mathrm{curl}(F) \cdot dS$, where S is the ellipsoid $x^2 + y^2 + 2z^2 = 10$ and $F(x, y, z) = (\sin(xy), e^x, -yz)$.



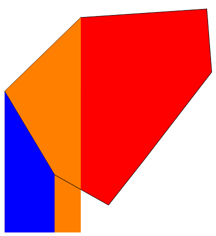
Answer. The integral is zero because the boundary of S is empty. This fact can be seen using Stokes theorem. It can also been seen by divergence theorem

$$\int\int_S \mathrm{curl}(F) \cdot dS = \int\int\int \mathrm{divcurl}(F)\ dV\ .$$

using $\mathrm{div}(\mathrm{curl})(F) = 0$.

AREA OF POLYGONS.

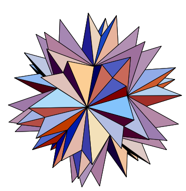
If $P_i = (x_i, y_i), i = 1, \dots, n$ are the edges of a polygon in the plane, then its area is $A = \sum_i (x_i - x_{i+1})(y_{i+1} + y_i)/2$.
The proof is an application of Green's theorem. The line integral of the vector field $F(x, y) = (-y, 0)$ through the side P_i, P_{i+1} is $(x_i - x_{i+1})(y_{i+1} + y_i)/2$, because $(x_{i+1} - x_i)$ is the projected area onto the x-axis and $(y_{i+1} + y_i)/2$ is the average value of the vector field on that side. Because $\mathrm{curl}(F)(x, y) = 1$ for all (x, y), the result follows from Greens theorem.



The result can also be seen geometrically: $(x_i - x_{i+1})(y_{i+1} + y_i)/2$ is the signed area of the trapezoid $(x_i, 0), (x_{i+1}, 0), (x_{i+1}, y_{i+1})$. In the picture, we see two of them. The second one is taken negatively.

VOLUME OF POLYHEDRA.

Verify with the divergence theorem: If $P_i = (x_i, y_i, z_i)$ are the edges of a polyhedron in space and $F_j = \{P_{i_1}, \dots P_{i_{k_j}}\}$ are the faces, then $V = \sum_j A_j \overline{z}_j$ where A_j is the area of the xy-projection (*) of the polygon F_j and $\overline{z}_j = (z_{i_1} + \dots + z_{k_j})/k_j$ is the average z value of the face F_j.

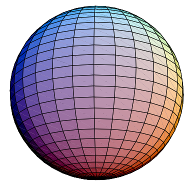


Solution. The vector field $F(x, y, z) = z$ has divergence 1. The flux through a face F is $|F_j|(z_{i_1} + \dots + z_{k_j})/k_j$. Gauss theorem assures that the volume is the sum of the fluxes $A_j \overline{z}_j$ through the faces.

(*) The projection of a polygon is the "shadow" when projecting from space along the z-axes onto the xy-plane. A triangle $(1, 0, 1), (1, 1, 0), (0, 1, 2)$ for example would be projected to the triangle $(1, 0), (1, 1), (0, 1)$.

STOKES AND GAUSS TOGETHER.

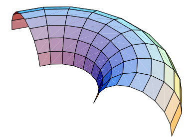
Can you derive $\mathrm{div}(\mathrm{curl}(F)) = 0$ using Gauss and Stokes theorem? Consider a sphere S of radius r around a point (x, y, z). It bounds a ball G. Consider a vector field F. The flux of $\mathrm{curl}(F)$ through S is zero because of Stokes theorem. Gauss theorem tells that the integral of $f = \mathrm{div}(\mathrm{curl}(F))$ over G is zero. Because S was arbitrary, f must vanish everywhere.



FUNDAMENTAL THEOREM AND STOKES.

Can you derive the identity $\mathrm{curl}(\mathrm{grad}(F)) = 0$ from integral theorems?

To see that the vector field $G = \mathrm{curl}(\mathrm{grad}(F)) = 0$ is identically zero, it is enough to show that the flux of G trough any surface S is zero. By Stokes theorem, the flux trough S is $\int_C \mathrm{grad}(F) \cdot dr$. By the fundamental theorem of line integrals, this is zero.



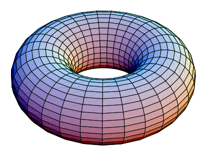
VOLUME COMPUTATION WITH GAUSS.

Calculate the volume of the torus $T(a, b)$ enclosed by the surface $r(u, v) = ((a + b\cos(v))\cos(u), (a + b\cos(v))\sin(u), b\sin(v))$ using Gauss theorem and the vector field $F(x, y, z) = (x, y, 0)/2$.

The vector field F has divergence 1. The parameterization of the torus gives

$$r_u \times r_v = b(a + b\cos(v))(\cos(u)\cos(v), \cos(v)\sin(u), \sin(v)) \ .$$

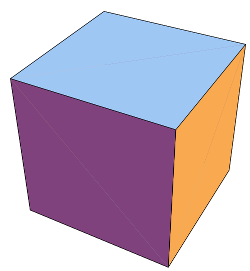
The flux of this vectorfield through the boundary of the torus is $\int_0^{2\pi} \int_0^{2\pi} b(a + b\cos(v))^2 \cos(v)\ dudv = 2\pi^2 ab^2$.



GAUSS OR STOKES?

You know that the flux of the vector field $G = \mathrm{curl}(F)(x, y, z)$ through 5 faces of a cube D is equal to 1 each. What is the flux of the same vector field G through the 6'th face?

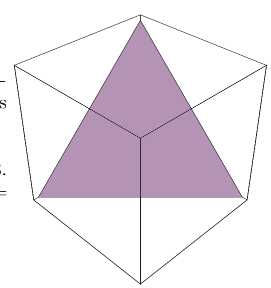
Solution: the problem is best solved with the divergence theorem: because the flux of G through the entire surface is zero, the flux through the 6'th face must cancel the sum of the fluxes 5 through the other 5 surfaces. The result is -5.



WORK COMPUTATION USING STOKES.

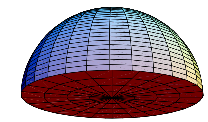
Calculate the work of the vector field $F(x, y, z) = (x - y + z, y - z + x, z - x + y)$, along the path C which connects the points $(1, 0, 0) \to (0, 1, 0) \to (0, 0, 1) \to (1, 0, 0)$ in that order.

Answer. The line integral over each part is each 1. The total is 3. $\mathrm{curl}(F) = (2, 2, 2)$ and $S : (u, v) \mapsto r(u, v) = (u, v, 1 - u - v)$ $r_u \times r_v = (1, 1, 1)$ $\int_S \mathrm{curl}(F) \cdot r_u \times r_v\ dudv = 6$ area of$S = 3$,



STOKES OR GAUSS?

Compute the flux of the vector field $F(x, y, z) = (x - x\sin(\sin(z)), 2y, 3z + \sin(\sin(z)))$ through the upper hemisphere $S = \{\ (x, y, z) |\ x^2 + y^2 + z^2 = 1, z \geq 0\}$.
Answer. We use Gauss: $\mathrm{div}(F) = 6$ and $\int\int\int_B \mathrm{div}(F)\ dV = 6\mathrm{Vol}(B) = 4\pi$. We can not easily compute the flux through the hemisphere. However, we can see that the flux through the floor of the region is zero because the normal component P of of the vector field $F = (M, N, P)$ is zero on $z = 0$. So: the result is $4\pi - 0 = 4\pi$.

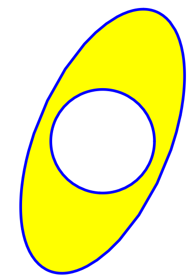


GREENS THEOREM.

Calculate the work of the vector field $F(x, y) = \frac{1}{x^2+y^2}(-y, x)$ along the boundary of the ellipse $r(t) = (3\cos(t) + \sin(t), 5\sin(t) + \cos(t))$.

Solution. Take an other curve $C : x^2 + y^2 \leq 4$ and apply Green's theorem to the region R bounded by the ellipse and the circle. Because $\mathrm{curl}(F)$ is zero in D, the line integral along the ellipse is the same as the line integral along the circle: $t \mapsto r(t) = (2\cos(t), 2\sin(t))$ with velocity $r'(t) = (-2\sin(t), 2\cos(t))$:

$$\int F \cdot dr = \int_0^{2\pi} \frac{(-2\cos(t), 2\sin(t))}{4} \cdot (-2\sin(t), 2\cos(t) = \int_0^{2\pi} 1\ dt = 2\pi \ .$$



TRUE/FALSE QUESTIONS ON INTEGRAL THEOREMS.

(TF) The flux of the curl of a vector field through the unit sphere is zero.

(TF) The line integral of the curl of a vector field along a closed curve is zero.

(TF) The line integral $\int_C F \cdot dr$ is independent of how a curve $C : t \mapsto r(t)$ is parametrized.

(TF) The maximal speed of a curve is independent on how the curve is parametrized.

(TF) The flux integral $\int_S F \cdot dS$ through a surface is independent on how the surface S is parametrized.

(TF) The area $\int_S dS$ of a surface is independent on how the surface S is parametrized.

(TF) The maximal value of $r_u \times r_v$ on a surface S is independent on how the surface is parametrized.

(TF) There exists a vector field in space which has zero divergence, zero curl but is not a constant field.

(TF) There exists a vector field in space which has zero gradient but is not a constant vector field $F(x, y, z) = (a, b, c)$.

(TF) There exists a function in space which has zero Laplacian $f_{xx} + f_{yy} + f_{zz} = 0$ but which is not constant.

(TF) $\mathrm{div}(\mathrm{grad}(F)) = 0$ and $\mathrm{div}(\mathrm{curl}(F)) = 0$ and $\mathrm{curl}(\mathrm{grad}(F)) = 0$.

(TF) The line integral of a gradient field along any part of a level curve $F = const$ is zero.

(TF) If $\mathrm{div}(F) = 0$, then the line integral along any closed curve is zero.

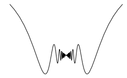
(TF) If $\mathrm{curl}(F) = 0$, then the line integral along any closed curve is zero.

(TF) If $\mathrm{div}(F) = 0$ then the flux integral along any sphere in space is zero.

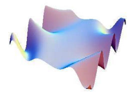
(TF) If $\mathrm{curl}(F) = 0$ then the flux integral along any sphere in space is zero.

OBJECTS $f : \mathbf{R}^n \to \mathbf{R}^m$ IN MULTIVARIABLE CALCULUS — Math21a, 2004

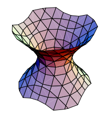
SCALAR FUNCTION (1D). ($n = 1, m = 1$). A function of one variable, where each $f(x)$ is a real number. The graph is a curve in the x, y plane. The derivative is $f'(x)$.



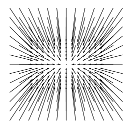
SCALAR FUNCTION (2D). ($n = 2, m = 1$). A function $f(x, y)$ defined in the plane is also called a scalar field. The graph of f is a curve in space (see figure). Level curves are curves in the plane. A derivative is $\nabla f(x, y)$, the gradient.



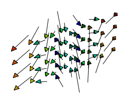
SCALAR FUNCTION (3D). ($n = 3, m = 1$). A function $f(x, y, z)$ defined in space is also called a scalar field. Its graph would be an object in 4D. Level surfaces are surfaces in space (see figure). A derivative is the gradient $\nabla f(x, y, z)$.



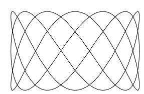
VECTOR FIELD (2D). ($n = 2, m = 2$) A function on the plane which attaches a vector $(P(x, y), Q(x, y))$ to each point (x, y). Derivatives are $\text{curl}(F) = Q_x - P_y$ or $\text{div}(F) = P_x + Q_y$ both scalar fields (real valued functions). Also **coordinate changes** $T(u, v) = (x(u, v), y(u, v))$ are the same mathematical object.



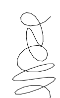
VECTOR FIELD (3D). ($n = 3, m = 3$) At each point in space, we attach a vector $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ at each point (x, y, z). Derivatives are the curl $\text{curl}(F)$, a vector field or the divergence $\text{div}(F) = P_x + Q_y + R_z$, a scalar field (real valued functions). Also **coordinate changes** $T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$ are the same mathematical object.



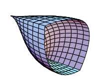
CURVE (2D). ($n = 1, m = 2$) For each t is defined a point $r(t) = (x(t), y(t))$ in the plane. A derivative is $r'(t) = (x'(t), y'(t))$, the velocity.



CURVE (3D). ($n = 1, m = 3$) For each t, we have a point $r(t) = (x(t), y(t), z(t))$ in space. A derivative is $r'(t) = (x'(t), y'(t), z'(t))$, the velocity.



SURFACE (2D). ($n = 2, m = 3$) For each (u, v) there is a point $r(u, v) = (x(u, v), y(u, v), z(u, v))$ in space. The normal vector at a point $r(u, v)$ is $r_u(u, v) \times r_v(u, v)$.



GLOSSARY CHECKLIST — Math 21a, Oliver Knill

Geometry of Space

- [ ] coordinates and vectors in the plane and in space
- [ ] $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$, $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
- [ ] dot product $v.w = v_1w_1 + v_2w_2 + v_3w_3 = |v||w|\cos(\alpha)$
- [ ] cross product, $v.(v \times w) = 0$, $w.(v \times w) = 0$, $|v \times w| = |v||w|\sin(\alpha)$
- [ ] triple scalar product $u \cdot (v \times w)$ volume of parallelepiped
- [ ] parallel vectors $v \times w = 0$, orthogonal vectors $v \cdot w = 0$
- [ ] scalar projection $\text{comp}_w(v) = v \cdot w/|w|$
- [ ] vector projection $\text{proj}_w(v) = (v \cdot w)w/|w|^2$
- [ ] completion of square: example $x^2 - 4x + y^2 = 1$ is equivalent to $(x-2)^2 + y^2 = -3$
- [ ] distance $d(P,Q) = |\vec{PQ}| = \sqrt{(P_1-Q_1)^2 + (P_2-Q_2)^2 + (P_3-Q_3)^2}$

Lines, Planes, Functions

- [ ] symmetric equation of line $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{z-z_0}{c}$
- [ ] plane $ax + by + cz = d$
- [ ] parametric equation for line $\vec{x} = \vec{x}_0 + t\vec{v}$
- [ ] parametric equation for plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$
- [ ] switch from parametric to implicit descriptions for lines and planes
- [ ] domain and range of functions $f(x,y)$
- [ ] graph $G = \{(x, y, f(x,y))\}$
- [ ] intercepts: intersections of G with coordinate axes
- [ ] traces: intersections with coordinate planes
- [ ] generalized traces: intersections with $\{x = c\}, \{y = c\}$ or $\{z = c\}$
- [ ] quadrics: ellipsoid, paraboloid, hyperboloids, cylinder, cone, hyperboloid paraboloid
- [ ] plane $ax + by + cz = d$ has normal $\vec{n} = (a, b, c)$
- [ ] line $\frac{(x-x_0)}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ contains $\vec{v} = (a, b, c)$
- [ ] sets $g(x,y,z) = c$ describe surfaces, example graphs $g(x,y,z) = z - f(x,y)$
- [ ] linear equation like $2x + 3y + 5z = 7$ defines plane
- [ ] quadratic equation like $x^2 - 2y^2 + 3z^2 = 4$ defines quadric surface
- [ ] distance point-plane: $d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$
- [ ] distance point-line: $d(P, L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$
- [ ] distance line-line: $d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$
- [ ] finding plane through three points P, Q, R: find first normal vector

Curves

- [ ] plane and space curves $\vec{r}(t)$
- [ ] velocity $\vec{r}'(t)$, Acceleration $\vec{r}\,''(t)$
- [ ] unit tangent vector $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$
- [ ] unit normal vector $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$
- [ ] $\vec{r}\,'(t)$ is tangent to the curve
- [ ] $\vec{v} = \vec{r}\,'$ then $\vec{r} = \int_0^t \vec{v}\, dt + \vec{c}$
- [ ] $\vec{r}(t) = (f(t)\cos(t), r(t)\sin(t))$ polar curve to polar graph $r = f(\theta)$.
- [ ] $r(t) = r_0(1+\epsilon)/(1+\epsilon\cos(t))$ ellipse.
- [ ] The three Kepler laws.

Surfaces

- [ ] polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$
- [ ] cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$
- [ ] spherical coordinates $(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$
- [ ] $g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
- [ ] $g(r, \theta, z) = 0$ cylindrical surface, i.e. $r = f(z, \theta)$ or $r = f(z)$ surface of revolution
- [ ] $g(\rho, \theta, \phi) = 0$ spherical surface especially $\rho = f(\theta, \phi)$
- [ ] $f(x,y) = c$ level curves of $f(x,y)$
- [ ] $g(x,y,z) = c$ level surfaces of $g(x,y,z)$
- [ ] circle: $x^2 + y^2 = r^2$, $\vec{r}(t) = (r\cos t, r\sin t)$.
- [ ] ellipse: $x^2/a^2 + y^2/b^2 = 1$, $\vec{r}(t) = (a\cos t, b\sin t)$
- [ ] sphere: $x^2 + y^2 + z^2 = r^2$, $\vec{r}(u,v) = (r\cos u\sin v, r\sin u\sin v, r\cos v)$
- [ ] ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $\vec{r}(u,v) = (a\cos u\sin v, b\sin u\sin v, c\cos v)$
- [ ] line: $ax + by = d$, $\vec{r}(t) = (t, d/b - ta/b)$
- [ ] plane: $ax + by + cz = d$, $\vec{r}(u,v) = \vec{r}_0 + u\vec{v} + v\vec{w}, (a, b, c) = \vec{v} \times \vec{w}$
- [ ] surface of revolution: $r(\theta, z) = f(z)$, $\vec{r}(u,v) = (f(v)\cos(u), f(v)\sin(u), v)$
- [ ] graph: $g(x,y,z) = z - f(x,y) = 0$, $\vec{r}(u,v) = (u, v, f(u,v))$

Partial Derivatives

- [ ] $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$ partial derivative
- [ ] partial differential equation PDE: $F(f, f_x, f_t, f_{xx}, f_{tt}) = 0$
- [ ] $f_t = f_{xx}$ heat equation
- [ ] $f_{tt} - f_{xx} = 0$ wave equation
- [ ] $f_x - f_t = 0$ transport equation
- [ ] $f_{xx} + f_{yy} = 0$ Laplace equation
- [ ] $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$ linear approximation
- [ ] tangent line: $L(x,y) = L(x_0,y_0)$, $ax + by = d$ with $a = f_x(x_0,y_0), b = f_y(x_0,y_0)$, $d = ax_0 + by_0$
- [ ] tangent plane: $L(x,y,z) = L(x_0,y_0,z_0)$
- [ ] estimate $f(x,y,z)$ by $L(x,y,z)$ near (x_0,y_0,z_0)
- [ ] $|f(x,y) - L(x,y)|$ in box R around (x_0,y_0) is $\leq M(|x - x_0| + |y - y_0|)^2/2$, where M is the maximal value of $|f_{xx}(x,y)|, |f_{xy}(x,y)|, |f_{yy}(x,y)|$ in R.
- [ ] $f(x,y)$ called differentiable if f_x, f_y are continuous
- [ ] $f_{xy} = f_{yx}$ Clairot's theorem
- [ ] $\vec{r}_u(u,v)$, $\vec{r}_v$ tangent to surface $\vec{r}(u,v)$

Gradient

- [ ] $\nabla f(x,y) = (f_x, f_y)$, $\nabla f(x,y,z) = (f_x, f_y, f_z)$, gradient
- [ ] $D_v f = \nabla if \cdot v$ directional derivative
- [ ] $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ chain rule
- [ ] $\nabla f(x_0,y_0,z_0)$ is orthogonal to the level surface $f(x,y,z) = c$ which contains (x_0,y_0,z_0).
- [ ] $\frac{d}{dt} f(\vec{x} + t\vec{v}) = D_v f$ by chain rule
- [ ] $\frac{x-x_0}{f_x(x_0,y_0,z_0)} = \frac{y-y_0}{f_y(x_0,y_0,z_0)} = \frac{z-z_0}{f_z(x_0,y_0,z_0)}$ normal line to surface $f(x,y,z) = c$ at (x_0,y_0,z_0)
- [ ] $(x - x_0)f_x(x_0,y_0,z_0) + (y - y_0)f_y(x_0,y_0,z_0) + (z - z_0)f_z(x_0,y_0,z_0) = 0$ tangent plane at (x_0,y_0,z_0)
- [ ] directional derivative is maximal in the $\vec{v} = \nabla f$ direction
- [ ] $f(x,y)$ increases, if we walk on the xy-plane in the ∇f direction
- [ ] partial derivatives are special directional derivatives
- [ ] if $D_v f(\vec{x}) = 0$ for all $\vec{v}$, then $\nabla f(\vec{x}) = \vec{0}$
- [ ] implicit differentiation: $f(x, y(x)) = 0$, $f_x 1 + f_y y'(x) = 0$ gives $y'(x) = -f_x/f_y$

Extrema

$\nabla f(x,y) = (0,0)$, critical point or stationary point
$D = f_{xx}f_{yy} - f_{xy}^2$ discriminant or Hessian determinant
$f(x_0,y_0) \geq f(x,y)$ in a neighborhood of (x_0,y_0) local maximum
$f(x_0,y_0) \leq f(x,y)$ in a neighborhood of (x_0,y_0) local minimum
$\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, λ Lagrange multiplier
two constraints: $\nabla g = 0$ or $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$
second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0,0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0,0), D < 0$ **saddle**

Double Integrals

$\int\int_R f(x,y)\ dA$ double integral
$\int_a^b \int_c^d f(x,y)\ dydx$ integral over rectangle
$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$ type I region
$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dxdy$ type II region
$\int\int_R f(r,\theta)\boxed{r}\ drd\theta$ polar coordinates
$\int\int_R |\vec{r}_u \times \vec{r}_v|\ dudv$ surface area
$\int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$ Fubini
$\int\int_R 1\ dxdy$ area of region R
$\int\int_R f(x,y)\ dxdy$ volume of solid bounded by graph(f) xy-plane

Triple Integrals

$\int\int\int_R f(x,y,z)\ dV$ triple integral
$\int_a^b \int_c^d \int_u^v f(x,y,z)\ dydx$ integral over rectangular box
$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\ dzdydx$ type I region
$f(r,\theta,z)\ \boxed{r}\ dzdrd\theta$ cylindrical coordinates
$\int\int\int_R f(\rho,\theta,z)\ \boxed{\rho^2\sin(\phi)}\ dzdrd\theta$ spherical coordinates
$\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\ dxdydz$ Fubini
$V = \int\int\int_R 1\ dV$ volume of solid R
$M = \int\int\int_R \rho(x,y,z)\ dV$ mass of solid R with density ρ
$(\int\int\int_R x\ dV/V, \int\int\int_R y\ dV/V, \int\int\int_R z\ dV/V)$ center of mass

Line Integrals

$F(x,y) = (P(x,y), Q(x,y))$ vector field in the plane
$F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ vector field in space
$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\ dt$ line integral
Use also notation $\int_C F \cdot Tds$ and $\int Mdx + Ndy + Pdz$.
$F(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative

Fundamental theorem of line integrals

FTL: $F(x,y) = \nabla f(x,y)$, $\int_a^b F(r(t)) \cdot r'(t))\ dt = f(r(b)) - f(r(a))$
Closed loop property $\int_C F\ dr = 0$, for all closed curves C
Always equivalent are: closed loop property, conservativeness and gradient field
Mixed derivative test $\mathrm{curl}(F) \neq 0$ assures F is not a gradient field.
In simply connected domains, $\mathrm{curl}(F) = 0$ implies conservativeness.

Green's Theorem

$F(x,y) = (P,Q)$, $\mathrm{curl}(F) = Q_x - P_y = \nabla \times F$.
Green's theorem: C boundary of R, then $\int_C F \cdot dr = \int\int_R \mathrm{curl}(F)\ dxdy$
Area computation: Take F with $\mathrm{curl}(F) = N_x - M_y = 1$ like $F = (-y,0)$ or $F = (0,x)$ or $F = (-y,x)/2$.
Greens theorem is useful to compute difficult line integrals or difficult 2D integrals.

Flux integrals

$F(x,y,z)$ vector field, $S = r(R)$ parametrized surface
$r_u \times r_v$ normal vector, $\vec{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$ unit normal vector
$r_u \times r_v\ dudv = \vec{dS} = \vec{n}\ dS$ normal surface element
$\int\int_S F\ \vec{dS} = \int\int_S F(r(u,v)) \cdot (r_u \times r_v)\ dudv$ flux integral
Use also notation $\int\int_S F \cdot nd\sigma$

Stokes Theorem

$F(x,y,z) = (P,Q,R)$, $\mathrm{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \nabla \times F$
Stokes's theorem: C boundary of surface S, then $\int_C F \cdot dr = \int\int_S \mathrm{curl}(F) \cdot dS$
Stokes theorem is useful to compute difficult flux integrals of $\mathrm{curl}(F)$ or difficult line integrals.

Div Grad Curl

$\nabla = (\partial_x, \partial_y, \partial_z)$, $\mathrm{grad}(F) = \nabla f$, $\mathrm{curl}(F) = \nabla \times F$, $\mathrm{div}(F) = \nabla \cdot F$
$\mathrm{div}(\mathrm{curl}(F)) = 0$
$\mathrm{curl}(\mathrm{grad}(F)) = \vec{0}$
$\mathrm{div}(\mathrm{grad}(f)) = \Delta f$.

Divergence Theorem

$\mathrm{div}(P,Q,R) = P_x + Q_y + R_z = \nabla \cdot F$
Divergence theorem: E bounded by S then $\int\int_S F \cdot dS = \int\int\int_E \mathrm{div}(F)\ dV$
The divergence theorem is useful to compute difficult flux integrals or difficult 3D integrals.

Some topology

Simply connected region D: can deform any closed curve within D to a point on curve.
Interior of a region D: points in D for which small neighborhood is still in D.
Boundary of a curve: the end points of the curve if they exist.
Boundary of a surface S are curves which bound the surface, points in the surface which correspond to parameters (u,v) which are not in the interior of the parametrization domain.
Boundary of a solid D: the surfaces which bound the solid, points in the solid which are not in the interior of D.
Closed surface: a surface without boundary like for example the sphere.
Closed curve: a curve with no boundary like for example a knot.

Some surface parametrizations

Sphere of radius ρ: $r(u,v) = (\rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v))$
Graph of function $f(x,y)$: $f(u,v) = (u,v,f(u,v))$
Graph of function $f(\phi,r)$ in polar: $f(u,v) = (v\cos(u), v\sin(u), f(u,v))$
Plane containing P and vectors $\vec{u},\vec{v}$: $r(u,v) = P + u\vec{u} + v\vec{v}$
Surface of revolution: distance $g(z)$ of $z - axes : r(u,v) = (g(v)\cos(u), g(v)\sin(u), v)$
Cylinder: $r(u,v) = (r\cos(u), r\sin(u), v)$.

COMMERCIAL SCANNERS. There are scanners on the market which can scan 60'000 points per second. The model on the photo to the right costs currently 410,000 dollars (www.cyberware.com).

THE PROBLEM. Given a surface in space. Take a cup for example. How do we model the surface mathematically? Taking pictures of the cup only gives us two dimensional representations. Reconstructing the surface from two dimensional projections is called **3D scanning**.

SOLUTION. One method is to make two or more projections. Each point in the photo determines a line in space. If we have at least two photos which contain both points, the point is fixed: the intersection of these lines is the point we are looking for. The next problem is then construct a surface from these points.

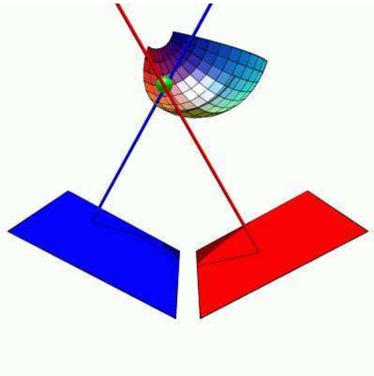
To see how this works in detail, we split up the problem into three subproblems.

PROBLEM 1) Given the camera position $C = (a, b, c)$ and the center $P = (x, y, z)$ of the object as well as a point $p = (u, v)$ on the film: find a line line $r \mapsto C + rU$ which passes through the point P, where $U = C - O$.

SOLUTION. If $U = C - O = (u_1, u_2, 0)$ and $N = (0, 0, 1)$. A unit vector orthogonal to u and N is $V = U \times N = (-u_2, u_1, 0)/\sqrt{u_1^2 + u_2^2}$. Then $P = C + Vu + Nv$.

PROBLEM 2) Given the location $C_1 = (x_1, y_1, z_1)$, $C_2 = (x_2, y_2, z_2)$ of the two cameras and the location $O = (0, 0, 0)$ of the center of the object. Given a point (u_1, v_1) on the first film and a point (u_2, v_2) on the second film. Find the point P on the surface.

SOLUTION. $p = (u, v)$ on the first film determines by PROBLEM 1) a line $P + rU$. Similarly, a point q on the second film determins a line $Q + sV$ in space. Theoretically, these two lines meet in a point $X = (x, y, z)$. However, due to errors, these two lines will miss in general. We assign the best possible choice, the point closest to both points. Because we know $u_3 = u_1 \times u_2$ the direction of the closest connection between the two lines, we have to solve $C_1 + su_1 + tu_3 = C_2 + ru_2$ to get the point $X = C_1 + su_1 + (t/2)u_3$. This is a system of three equations for the three unknowns t_1, t_2, t_3. We can solve this system (in Lagrange multiplier problems, we solved more complicated nonlinear systems of equations).

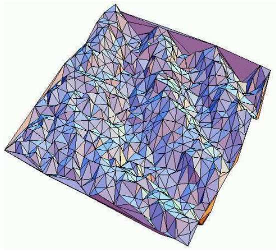


FAMILIAR EXAMPLE. If the camera C_1 is on the x axes and takes a shot of the $y - z$ plane, then the point (x, y, z) will be the point (y, z) in the photo. Similarly, if C_2 is on the y axes, then a point (x, y, z) will be the point (x, z) on the photo. Ideally, if the point is from both sides, then (x, y, z) is determined.

PROBLEM 3) Given a set of points $P_i = (x_i, y_i, z_i)$ on a surface. Model the surface as a set of triangles.

SOLUTION. The solution to this problem is provided by an algorithm called **Delaunay triangularization**. It produces a desired set of triangles.

Fortunately, Mathematica has already built in a function "TriangularSurfacePlot" which is contained in the DiscreteMath'ComputationalGeometry' module. The figure to the right shows a triangularsation applied to a random surface.



STEP 1. DATA INPUT (C PROGRAM). A GUI xwindows C program 3dscan.c helps us to enter the data with a mouse. The two pictures are scaled, concatenated to a single picture and displayed. The program 3dscan.c allows to enter pairs of points with the mouse. The data are written onto a text file "result.dat".

STEP 2. DATAPROCESSING (MATHEMATICA). Given the object $O = (0, 0, 0)$, the camera positions $C_1 = (C_{1x}, C_{1y}, C_{1z})$,$C_2 = (C_{2x}, C_{2y}, C_{2z})$ and the points (u_1, v_1) on the first film and (u_2, v_2) on the second film. The formula for the point $P(x, y, z)$ are obtained from solving $A_1 + t_1 * C_1 + t_3 * N = A_2 + t_2 * C_2$, where $A_i = C_i + u_i(C_i - O) \times N/|C_i - O| + v_i N$.
A mathematica program "stereo3d.m" takes the data in "result.dat" and writes a new file "3d.dat" which contains the points P_i in space.

Also the triangularisation is done with Mathematica. The program is called "triangular.m". It takes as in input the file "3d.dat" and outputs a Povray file "surface.inc" which is essentially a set of triangles written in form, Povray can read.

STEP 3. RENDERING (POVRAY). Finally, with all the triangles known, we run the data through a ray tracing program "surface.pov". This Povray program takes the file "result.inc" as an input and produces a picture "surface.ppm".

POVRAY. Povray is a free raytracing program. It allows to render three dimensional objects. To the right, you see a simple Povray program. It simply shows a red sphere on a white background.

```
camera { up y right x location <0,1,-3> look_at <0,-1,3> }
light_source  { <0,300, -100>  colour rgb <1,1,1> }
#background { rgb <1,1,1> }

#declare r = texture {
  pigment { rgb <1,0,0> }
  finish { phong 1.0 ambient 0.5 diffuse 0.8}
}

sphere {<0,0,0>,1 texture{ r } }
                                        9,0-1         Top
```

THE LANGUAGE POVRAY.

$< a, b, c >$	point or vector (a,b,c)
sphere$\{P, r\}$	sphere centered of radius r centered at (a,b,c)
box$\{P, Q\}$	rectangular box with diagonal corners P,Q
cylinder$\{P, Q, r\}$	cylinder of radius r over segment P Q
plane$\{N, d\}$	plane with normal vector N, in distance d from origin
triangle { P,Q,R }	triangle with endpoints P,Q,R

One can take intersections, unions, differences of objects. Furthermore, every object can be scaled, rotated and translated.
In the example above, you see how to apply textures. Colors are represented as vectors $< r, g, b >$, where r is "red" and g is "green" and b is "blue". The addition of colors is "additive". For example $< 1, 1, 0 >$ is yellow. The best way to learn the language is to look at examples. There are thousands of examples available on the web. The official distribution of Povray includes many examples and contains a nice manual.

5/7/2004 CALCULUS BEYOND CALCULUS Math21a, O. Knill

INTRODUCTION. Topics beyond multi-variable calculus are usually labeled with special names like "linear algebra", "ordinary differential equations", "numerical analysis", "partial differential equations", "functional analysis" or "complex analysis". Where one would draw the line between calculus and non-calculus topics is not clear but if calculus is about learning the basics of limits, differentiation, integration and summation, then multi-variable calculus is the "black belt" of calculus. Are there other ways to play this sport?

HOW WOULD ALIENS COMPUTE? On an other planet, calculus might be taught in a completely different way. The lack of fresh ideas in the current textbook offerings, where all books are essentially clones (*) of each other and where innovation is faked by ejecting new editions every year (which of course has the main purpose to prevent the retail of second hand books), one could think that in the rest of the universe, multi-variable calculus would have to be tought in the same way, and where chapter 12 is always the chapter about multiple integrals. Actually, as we want to show here, even the human species has come up with a wealth of different ways to deal with calculus. It is very likely that calculus textbooks would look very different in other parts of our galaxy. This week broke the news that one has discovered a 5'th arm of the milky way galaxy. It is 77'000 light years long and should increase the chance that there are other textbooks in our home galaxy. In this text, we want to give an idea that the calculus topics in this course could be extended or built completely differenty. Actually, even **numbers** can be defined differenty. John Conway introduced once numbers as pairs $\{L|R\}$ where L and R are sets of numbers defined previously. For example $\{\emptyset|\emptyset\} = 0$ and $\{0|\emptyset\} = 1$, $\{\emptyset|0\} = -1$, $\{0|1\}$. The advantage of this construction is that it allows to see "numbers" as part of "games". Donald E. Knuth, (the giant of a computer scientist, who also designd "TeX", a typesetting system in which this text is written), wrote a book called "surreal numbers" in which two students find themselves on an iland. They find a stone with the axioms for a new number system is written and develop from that an entirely new number system which contains the real line and more. The book is a unique case, where mathematical discovery is described as a novel. This is so totally different from what we know traditionally about numbers that one could expect Conway to be an alian himself if there were not many other proofs of his ingenious creativity.

(*) One of the few exceptions is maybe the book of Marsden and Tromba, which is original, precise and well written. It is unfortunately slightly too mathematical for most calculus consumers and suffers from the same disease that other textbooks have: **a scandalous prize**. A definite counterexample is the book "how to ace calculus" which is funny, original and contains the essential stuff. And it comes as a **paperback**. Together with a Schaum outline volume (also in paperback), it would suffice as a rudimantary textbook combination (and would cost together half and weight one fourth) of the standard doorstoppers.

NONSTANDARD CALCULUS. At the time of Leonard Euler, people thought about calculus in a more intuitive, but less formal way. For example $\boxed{(1 + x/n)^n = e^x}$ with **infinitely large** n would be perfectly fine. A modern approach which catches this spirit is "nonstandard analysis", where the notion of "infinitesimal" is given a precise meaning. The simplest approach is to extend the language and introduce **infinitesimals** as objects which are smaller than all **standard** objects. We say $x \sim y$ if $|x - y|$ is infinitesimal. All numbers are traditionally defined like π or $\sqrt{2}$ are "standard". The notion which tells that every bounded sequence has an accumulation point is expressed by the fact that there exits a finite set A such that all $x \in D$ are infinitesimally close to an element in A. The fact that a continuous function on a compact set takes its maximum is seen by taking $M = \max_{a \in A} f(a)$.

What impressed me as an undergraduate student learning nonstandard calculus (in a special course completely devoted to that subject) was the elegance of the language as well as the compactness in which the entire calculus story could be packed. For example, to express that a function f is continuous, one would say $x \sim y$ then $f(x) \sim f(y)$. This is more intuitive than the Weierstrass definition $\forall \epsilon > 0 \exists \delta > 0 |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ which is understood today primarily by intimidation. To illustrate this, Ed Nelson, the founder of one of the nonstandard analysis flavors, asks the meaning of $\boxed{\forall \delta > 0 \exists \epsilon > 0 |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon}$ in order to demonstrate how unintuitive this definition really is. The derivative of a function is defined as the standard part of $f'(x) = (f(x + dx) - f(x))/dx$, where dx is infinitesimal. Differentiability means that this expression is finite and independent of the infinitesimal dx chosen. Integration $\int_a^x f(y)\, dy$ is defined as the standard part of $\sum_{x_j \in [a,x]} f(x_i)dx$ where $x_k = kdx$ and dx is an infinitesimal. The fundamental theorem of calculus is the triviality $F(x + dx) - F(x)) = f(x)dx$.

The reasons that prevented nonstandard calculus to go mainstream were lack of marketing, bad luck, being below a certain critical mass and the initial believe that students would have to know some of the foundations of mathematics to justify the game. It is also quite a sharp knife and its easy to cut the finger too and doing mistakes. The name "nonstandard calculus" certainly was not fortunate too. People call it now "infinitesimal calculus". Introducing the subject using names like "hyper-reals" and using "ultra-filters" certainly did not help to promote the ideas (we usually also dont teach calculus by introducing Dedekind cuts or completenes) but there are books like by Alain Robert which show that it is possible to teach nonstandard calculus in a natural way.

DISCRETE SPACE CALCULUS. Many ideas in calculus make sense in a discrete setup, where space is a graph, curves are curves in the graph and surfaces are collections of "plaquettes", polygons formed by edges of the graph. One can look at functions on this graph. Scalar functions are functions defined on the vertices of the graphs. Vector fields are functions defined on the edges, other vector fields are defined as functions defined on plaquettes. The gradient is a function defined on an edge as the difference between the values of f at the end points.

Consider a network modeled by a planar graph which forms triangles. A **scalar function** assigns a value f_n to each node n. An **area function** assigns values f_T to each triangle T. A **vector field** assign values F_{nm} to each edge connecting node n with node m. The **gradient** of a scalar function is the vector field $F_{nm} = f_n - f_m$. The **curl** of a vector field F is attaches to each triangle (k, m, n) the value $\text{curl}(F)_{kmn} = F_{km} + F_{mn} + F_{nk}$. It is a measure for the circulation of the field around a triangle. A curve γ in our discrete world is a set of points $r_j, j = 1, \dots, n$ such that nodes r_j and r_{j+1} are adjacent. For a vector field F and a curve γ, the **line integral** is $\sum_{j=1}^n F_{r(j)r_{j+1}}$. A **region** R in the plane is a collection of triangles T. The **double integral** of an area function f_T is $\sum_{T \in R} f_T$. The **boundary** of a region is the set of edges which are only shared by one triangle. The orientation of γ is as usual. Greens theorem is now almost trivial. Summing up the curl over a region is the line integral along the boundary.

One can push the discretisation further by assuming that the functions take values in a finite set. The integral theorems still work in that case too.

QUANTUM MULTIVARIABLE CALCULUS. Quantum calculus is "calculus without taking limits". There are indications that space and time look different at the microscopic small, the Planck scale of the order $\hbar$. One of the ideas to deal with this situation is to introduce quantum calculus which comes in different types. We discuss q-calculus where the derivative is defined as $\boxed{D_q f(x) = d_q f(x)/d_q(x)}$ with $\boxed{d_q f(x) = f(qx) - f(x)}$. You can see that $D_q x^n = [n]x^{n-1}$, where $[n] = \frac{q^n - 1}{q - 1}$. As $q \to 1$ which corresponds to the deformation of quantum mechanics with $\hbar \to 0$ to classical mechanics, we have $[n] \to n$. There are quantum versions for differentiation rules like $D_q(fg)(x) = D_q f(x)g(x) + f(qx)D_q g(x)$ but quantum calculus is more friendly to students because **there is no simple chain rule**.

Once, we can differentiate, we can take anti-derivatives. It is denoted by $\int f(x)\, d_q(x)$. As we know the derivative of x^n, we have $\int x^n\, dq(x) = \sum_n a_n x^{n+1}/[n+1] + C$ with a constant C. The anti-derivative of a general function is a series $\boxed{\int f(x)\, d_q(x) = (1 - q)x \sum_{j=0}^{\infty} q^j f(q^j x)}$ For functions $f(x) \leq M/x^\alpha$ with $0 \leq \alpha < 1$, the integral is defined. With an anti-derivative, there are definite integrals. The **fundamental theorem of q-calculus** $\int_a^b f(x) d_q(x) = F(b) - F(a)$ where F is the anti-derivative of f still holds. Many results generalize to q-calculus like the Taylor theorem. A book of Kac and Cheung is a beautiful read about that.

Multivariable calculus and differential equations can be developed too. A handicap is the lack of a chain rule. For example, to define a line integral, we would have to define $\int F\, d_q r$ in such a way that $\int \nabla f\, dr_q = f(r(b)) - f(r(a))$. In quantum calculus, the naive definition of the length of a curve depends on the parameterization of the curve. Surprises with **quantum differential equations** QODE $d_q f = f$ which is $f(qx) - f(x) = (q-1)f(x)$ simplifies to $f(qx) = qf(x)$. A classic differentiation gives $f'(qx) = f'(x)$ showing that the QODE has **many solutions** $f(x) = Cx + g(\log(x)/\log(q))$, where $g(x) = g(x + 1)$ is periodic.

INFINITE DIMENSIONAL CALCULUS. Calculus in infinite dimensions is called **functional analysis**. Functions on infinite dimensional spaces which are also called **functionals** for which the gradient can be defined. The later is the analogue of $D_u f$. One does not always have $D_u f = \nabla f \cdot u$. An example of an infinite dimensional space is the set X of all continuous functions on the unit interval. An example of a two dimensional surface in that space would be $r(u, v) = (\cos(u)\sin(v)x^2 + \sin(u)\sin(v)\cos(x) + \cos(v)/(1 + x^2)$. This surface is actually a two dimensional sphere. On this space X one can define a dot product $f \cdot g = \int f(x)g(x)\, dx$.

The theory which deals with the problem of extremizing functionals in infinite dimensions is called **calculus of variations**. There are problems in this field, which actually can be answered within the realm of multivariable calculus. For example: a classical problem is to find among all closed regions with boundary of length 1 the one which has maximal area. The solution is the circle. One of the proofs which was found more than 100 years ago uses Greens theorem. One can also look at the problem to find the polygon with n edges and length 1 which has maximal area. This is a Lagrange extremization problem with regular polygons as solutions. In the limit when the number of points go to infinity, one obtains the **isoperimetric inequality**. Other problems in the calculus of variations are the search for the shortest path between two points in a hilly region. This shortest path is called a **geodesic**. Also here, one can find approximate solutions by considering polygons and solving an extremization problem but direct methods in that theory are better.

GENERALIZED CALCULUS. In physics, one wants to deal with objects which are more general than functions. For example, the vector field $F(x,y) = (-y,x)/(x^2+y^2)$ has its curl concentrated on the origin $(0,0)$. This is an example of an object which is a **distribution**. An example of such a Schwartz distribution is a "function" f which is infinite at 0, zero everywhere else, but which has the property that $\int f d\ dx = 1$. It is called the **Dirac delta function**. Mathematically, one defines distributions as a linear map on a space of smooth "test functions" ϕ which decay fast at infinity. One writes (f,ϕ) for this. For continuous functions one has $(f,\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\ dx$, for the Dirac distribution one has $(f,\phi) = \phi(0)$. One would define the derivative of a distribution as $(f',\phi) = -f(\phi')$. For example for the Heavyside function $H(x) = \left\{ \begin{array}{l} 0\ x<0 \\ 1\ x>0 \end{array} \right.$ one has $(H,\phi) = \int_{-\infty}^{\infty} \phi(x)\ dx$ and because $(H,\phi') = \int_{-\infty}^{\infty} \phi'(x)\ dx = -\phi'(0)$ one has $(H',\phi) = \phi(0)$ the Dirac delta function. The Dirac delta function is still what one calls a **measure**, an object which appears also in probability theory. However, if we differentiate the Dirac delta function, we obtain $(D',\phi) = -(D,\phi') = -\phi'(0)$. This is an object which can no more be seen as a probability distribution and is a new truly "generalized" function.

COMPLEX CALCULUS Calculus in the complex is called **complex analysis**. Many things which are a bit mysterious in the real become more transparent when considered in the complex. For example, complex analysis helps to solve some integrals, it allows to solve some problems in the real plane better. Here are some places where complex analysis could have helped us: to find harmonic functions, one can take a nice function in the complex like $z^4 = (x+iy)^4 = x^4 - x^2y^2 + y^4 + i(x^3y + xy^3)$ and look at its real and complex part. These are harmonic functions. Differentiation in the complex is defined as in the real but since the complex plane is two dimensional, one asks more: $f(z+dz) - f(z)/dz$ has to exist and be equal for every $dz \to 0$. One writes $\partial_z = (\partial_x - i\partial_y)/2$ and **complex differentiation** satisfies all the known properties from differentiation on the real line. For example $d/dz z^n = nz^{n-1}$. Multivariable calculus very much helps also to integrate functions in the complex. Again, because the complex plane is two dimensional, one can integrate along paths and the **complex integral** $\int_C f(z)\ dz$ is actually a line integral. If $z(t) = x(t) + iy(t)$ is the path and $f = u + iv$ then $\boxed{\int_C f(z)\ dz = \int_a^b (ux' - vy')dt + i\int_a^b (uy' + vx')\ dt}$. Greens theorem for example is the easiest way to derive the Cauchy integral theorem which says that $\int_C f(z)\ dz = 0$ if C is the boundary of a region in which f is differentiable. Integration in the complex is useful for example to compute definite integrals.
One can ask, whether calculus can also be done in other number systems besides the reals and the complex numbers. The answer is yes, one can do calculus using **quaternions**, **octonions** or over finite fields but each of them has its own difficulty. Quaternion multiplication already does no more commute $AB \neq BA$ and octonions multiplication is even no more associative $(AB)C \neq A(BC)$. In order to do calculus in finite fields, differentiation will have to be replaced by differences. A Finnish mathematician Kustaanheimo promoted around 1950 a finite geometrical approach with the aim to do real physics using a very large prime number. These ideas were later mainly picked up by philosophers. It is actually a matter of fact that the **natural numbers** are the most complex and the **complex numbers** are the **most natural**.

CALCULUS IN HIGHER DIMENSIONS. When extending calculus to higher dimensions, the concept of vectorfields, functions and differentiations grad,curl,div are reorganized by introducing **differential forms**. For an integer $0 \leq k \leq n$, define **k-forms** as objects of the form $\boxed{\alpha = \sum_I a_I dx_I = \sum_{i_1<i_2<\ldots<i_k} a_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}}$, where $dx_i \wedge dx_j$ is a multiplication called **exterior product** which is anti-commutative: $\alpha \wedge \beta = -\beta \wedge \alpha$. Then, one defines an **exterior derivative** as $d\sum_I a_I dx_I = da_I \wedge dx_I$ In three dimensions, where we write $dx = dx_1, dy = dx_2$ and $dz = dx_3$:

$k=$	form $\alpha=$	$d\alpha=$
0	f	$f_x dx + f_y dy + f_z dz$
1	$Mdx + Ndy + Pdz$	$(N_x - M_y)dx \wedge dy + (P_x - M_z)dx \wedge dz + (P_y - N_z)dy \wedge dz$
2	$Adx_1 \wedge dx_2 + Bdx_1 \wedge dx_3 + Cdx_2 \wedge dx_3$	$(A_z - B_y + C_x)dx \wedge dy \wedge dz$
3	$gdx_1 \wedge dx_2 \wedge dx_3$	$0.$

(The computation of the curl was $N_x dx \wedge dy + P_x dx \wedge dz + M_y dy \wedge dx + P_y dy \wedge dz + M_z dz \wedge dx + N_z dz \wedge dy = (N_x - M_y)dx \wedge dy + (P_x - M_z)dx \wedge dz + (P_y - N_z)dy \wedge dz$.) After an identification of 0 with 3 forms and 1 with 2 forms (called **Hodge *operation**) one can see the exterior derivatives as gradient, curl and divergence:

It is useful also to clarify the notion of a surface by defining **manifolds**. Many definitions which we have seen for curves or surfaces can be extended to manifolds. The dot product can be defined more generally on manifolds and leads to a theory called **Riemannian geometry** used in the theory of general relativity. When learning relativity, one deals with 4-dimensional manifolds which incoorporate both space and time. There, the language of differential forms is already mandatory. The exterior derivative of a 1 form is a 2 form which has 6 component. An example of such a 2-form is the electromagnetic field $F = dA$ combining 3 electric and 3 magnetic components. The **Maxwell equations** are $dF = 0, d^*F = I$, where $d^* = *d*$ is defined using the Hodge * operation. The consequence $d^*dA = I$ becomes the wave equation $d^*dA = 0$ in the absense of I a vector incoorporating both electric charge and current i.

FRACTAL CALCULUS. We have dealt in this courses with 0-dimensional objects (points), 1-dimensional objects (curves), 2-dimensional objects (surfaces) as well as 3-dimensional objects (solids). Since more than 100 years, mathematicians also studied **fractals**, objects with noninteger dimension are called fractals. An exampls is the Koch snowflake which is obtained as a limit by repeated stellation of an equilateral triangle of initial arclength $A = 3$. After one stellation, the length has increased by a factor $4/3$. After n steps, the length of the curve is $A(4/3)^n$. The dimension is $\log(4/3) > 1$. There are things which do no more work here. For example, the length of this curve is infinite. Also the curve has no defined velocity at all places. One can ask whether one could apply Greens theorem still in this case. In some sense, this is possible. After every finite step of this construction one can compute the line integral along the curve and Greens theorem tells that this is a double integral of curl(F) over the region enclosed by the curve. Since for larger and large n, less and less region is added to the curve, the integral $\int\int \mathrm{curl}(F)\ dxdy$ is defined in the limit. So, one can define the line integral along the curve. Closely related to fractal theory is **geometric measure theory**, which is a generalization of differential geometry to surfaces which are no more smooth. Merging in ideas from generalized functions and differential forms, one defines **currents**, functionals on smooth differential forms. The theory is useful for studiing minimal surfaces. Fractals appear naturally in differential equations as attractors. The most infamous fractal is probably the Mandelbrot set, which is defined as the set of complex numbers c for which the iteration of the map $f(z) = z^2 + c$ starting with 0 leads to a bounded sequence $0 \to c \to c^2 + c \to (c^2+c)^2 + c....$ The boundary of the Mandelbrot set is actually not a fractal. It is so wiggly that its demension is actually 2. One can modify the Koch snowflake to get dimension 2 too. If one adds new triangles each time so that the length of the new curve is doubled each time, then the dimension of the Koch curve is 2 too.

THE FOUNDATIONS OF CALCULUS. Would it be possible that aliens somewhere else would build up mathematics radically different, by starting with a different axiom system? It is likely. The reason is that already we know that there is not a single way to build up mathematical truth. We have some choice: it came as a shock around the middle of the last century that for any strong enough mathematical theory, one can find statements which are **not provable** within that system and which one can either accept as a new axiom or accept the negation as a new axiom. Even some of the respected axioms are already independent of more elementary ones. One of them is the **axiom of choice** which says that for any collection C of nonempty set, one can choose from each set an element and form a new set. A consequence of this axiom which is close to calculus is a "compactness property" for functions on the interval. If we take the distance $d(f,g) = \max(f(x) - g(x))$ between two functions and take a sequence of differentiable functions satisfying $|f_n(x)| \leq M$ and $|f_n'(x)| \leq M$ then there exists a subsequence f_{n_k} which converges to a continuous function.

A rather counterintuitive consequence of the Axiom of choice is that one can decompose the unit ball into 5 pieces E_i, move those pieces using rotations and translations and reassemble them to form two copies of the unit ball. We can not integrate $\int\int\int_{E_i}\ dV$ over this regions: the sum of their volumes would be either $4\pi/3$ or $8\pi/3$ depending on whether we integrate before or after the arrangement. This is such a paradoxical construction that one calls it a **paradox**: the **Banach-Taski** paradox.

An other axiom for which is not clear, whether it matters in calculus is the continuum hypothesis which tells that there is no cardinality between the "infinity" of the natural numbers $\aleph_0$ and the "infinity" of the real numbers α_1.

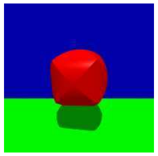
It had been realized in the 19'th century by Cantor that these two infinities are different. Cantors argument is to assume that one could enumerate all numbers between 0 and 1: then look at the diagonal number, in which each digit is altered. For example: $\boxed{x = 0.35285....}$. This real number x disagrees with all the numbers in the list and was therefore not accounted for in the counting.

1	$0.\boxed{2}42314234123412341 34...$
2	$0.3\boxed{4}42234982734139041 73...$
3	$0.69\boxed{1}43410741473467388 74...$
4	$0.999\boxed{7}42833824642001041 31...$
5	$0.3620\boxed{4}47473892389342111 47...$

The possibilities to question or alter the mathematical builup does not stop on the set theoretical level. People even tried to change **logic**. Among things which have been proposed are **fuzzy logic**, where truth can take a value between 0 and 1 or quantum logic or a logic in which one has three possibilites, true, not true or "undecided".
One not only has the possibility to extend mathematics differently. A rather rich playground can be covered by **restricting tools**. One can for example ask, what part of calculus can still be done in without the axiom of choice. One can also ask that one only is allowed to talk about objects which can be constructed explicitely. There were people, **strict finitists**, who went even further and suggested to disqualify too large numbers like $10^{10^{10}}$. Calculus teachers have even to go further and produce problems so that the answer is 1 or 0 or π or $1/3$. Problems in final exams, where the answer is $1234/12$ is unthinkable. The philosophical direction teachers are forced to follow is called **ultra strict finitism**...

A SCENE. The y axes is up. We place the object (the intersection of three cylinders) at the origin and the camera slightly above on the z axes.

```
camera{ up y right x location <0,2,-5> look_at <0,0,0> }
light_source { <0,300,-100> colour rgb <1,1,1> }
#background { rgb <1,1,1> }

#macro r(c)
  pigment { rgb c }
  finish { phong 1.0 ambient 0.5 diffuse 0.5 }
#end

intersection {
  cylinder { <-1,0,0>,<1,0,0>, 1 }
  cylinder { <0,-1,0>,<0,1,0>, 1 }
  cylinder { <0,0,-1>,<0,0,1>, 1 }
  texture { r(<1,0,0>) }
  rotate <10,10,5>
}

plane {<0,1,0>,-2 texture { r(<0,1,0>) } }
plane {<0,0,1>, 2 texture { r(<0,0,1>) } }
```



RUNNING POVRAY. If the povray file is called test.pov, run povray as "povray good.ini +i test.pov". The .ini file contains instructions like size of the picture, quality of rendering etc. With "povray anim.ini +i test.pov", a sequence of pictures is rendered. The .ini file contains now instructions how many frames the movie should have.

```
Width=200
Height=200
Quality=11
Antialias_Depth=4
Antialias=On
Antialias_Threshold=0.1
Jitter_Amount=0.5
Jitter=On
```

```
Width=200
Height=200
Quality=11
Antialias_Depth=4
Antialias=On
Antialias_Threshold=0.1
Jitter_Amount=0.5
Jitter=on
Initial_Frame = 1
Final_Frame   = 10
Initial_Clock = 0.0
Final_Clock   = 1.0
```

WHAT IS RAYTRACING? The objects, the camera and lights are in place. If photons leave the light source, they will reflect at the objects, change color and intensity and some of them will reach the camera. Adding up the light intensities of all these photons gives the picture. Raytracers work more efficient: instead of shooting photons from the light source and wasting most of the photos because they don't reach the film, it is better to start the light ray at the film and compute backwards.

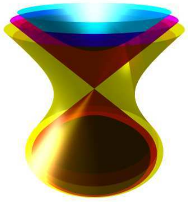
COLORS. Every color is given by a vector $< r, g, b >$ in the color cube $[0,1] \times [0,1] \times [0,1]$. Color vectors can be added like usual vectors.

PROGRAMMING. Povray is a programming language. Macros are procedures which can be reused and called within the the macro. This allows recursion like in this rendering of the 3D fractal to the right. You see also how one can play with colors. Identifying them with vectors in space colors the pyramid according to the color cube.

```
camera {location <1,1.2,-1.7> up y right x look_at <0,0,0> }
light_source {< 100, 80,  100> colour rgb <1,1,1> }
light_source {< 100, 80, -100> colour rgb <1,1,1> }
light_source {< 100, 80, -100> colour rgb <1,1,1> }
light_source {<-100, 80, -100> colour rgb <1,1,1> }
background   { rgb <1,1,1> }

#macro te(c)
  texture { pigment { rgb c }
  finish  { phong 1.0 phong_size 10 ambient 0.1 diffuse 0.7 } }
#end

#declare maxrecursion=4;
#macro shirpinski(p,w,n)
   #local hw=w/2;   #local fw=w/4;
   #if (n>=maxrecursion)
      union {
      triangle { p,p+< hw,-w,-hw>,p+< hw,-w, hw> }
      triangle { p,p+< hw,-w, hw>,p+<-hw,-w, hw> }
      triangle { p,p+<-hw,-w, hw>,p+<-hw,-w,-hw> }
      triangle { p,p+<-hw,-w,-hw>,p+< hw,-w,-hw> }
      triangle { p+< hw,-w, hw>,p+<-hw,-w, hw>,p+<-hw,-w,-hw> }
      triangle { p+<-hw,-w,-hw>,p+< hw,-w,-hw>,p+< hw,-w, hw> }
      te(p) }
   #else
      shirpinski(p,hw,n+1)
      shirpinski(p+< fw,-hw,-fw>,hw,n+1)
      shirpinski(p+< fw,-hw, fw>,hw,n+1)
      shirpinski(p+<-fw,-hw, fw>,hw,n+1)
      shirpinski(p+<-fw,-hw,-fw>,hw,n+1)
   #end
#end

union { shirpinski(<0,1,0>,1.3,0)
  translate -y*0.5 rotate y*30 translate y*0.5
}
```

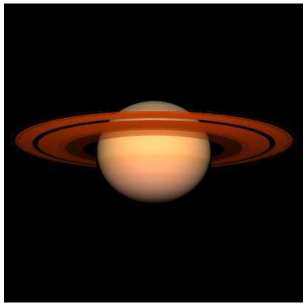
OBJECTS. Many objects like quadrics $ax^2 + bxy + cy^2 + dx + ey + fz + g = 0$ are predefined in Povray. The most basic objects are boxes, spheres, cylinders, lathes (extruded curves). By joining, intersecting or taking differences of such objects, one can build more complicated objects. Additionally, every object can be rotated and scaled.

```
camera        { location  <1,2,-4> up  <0, 1,0> right <1,0,0> look_at <
light_source  { <-100, 30, -10>    color rgb <1,1,1>
light_source  { <0, 2000, 0>  color rgb <1,1,1>  area_light <-6,0,-6>, 
background    { rgb <1.0,1.0,1.0>

#macro te(c) texture { pigment { rgb c} finish { phong 1.0 phong_size 10

poly {2,<1,0,0,0,-1,0,0,1,0,-0.3> scale 1.5  hollow  clipped_by { sphere
poly {2,<1,0,0,0,-1,0,0,1,0, 0.3> scale 1.5  hollow  clipped_by { sphere
poly {2,<1,0,0,0,-1,0,0,1,0, 0.0> scale 1.5  hollow  clipped_by { sphere
```



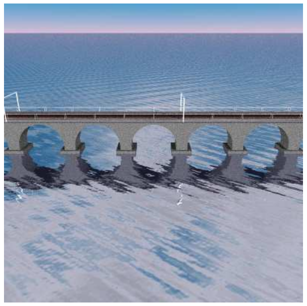
TEXTURES. Textures to planets of the Solar system are available on the web. The texture is mapped onto the sphere using the map $X(u, v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ In the source code, the units are chosen so that 1=1km. Having natural units allows to model things more easily. In the scene an artificial light has been added from the camera in order to see the rings better.

```
camera { location <0,50000,-320000> up z right x look_at 0 }
#declare sun= light_source {
  0 color rgb <1,1,1> looks_like { sphere { 1, 696265
  pigment { color rgb <1,1,0> } finish { ambient 1} }}
  translate -z*142600000
}
light_source {<0,20000,-250000> color rgb <0.3,0.1,0>}
#declare saturn_planet = sphere {0, 60368
  pigment {image_map {gif "saturn.gif" map_type 1 interpolate 2}}
  finish  {ambient 0 diffuse 1}
}
#declare C_ring = disc { <0, 0, 0>  y,       92000, 74500
  pigment {color rgb<0.99,0.88,0.79> filter 0.6}
  finish {ambient 0 diffuse 5}
}
#declare inner_B_ring = disc { <0, 0, 0>  y, 98390, 92000
  pigment {color rgb<0.99,0.88,0.79> filter 0.2}
  finish {ambient 0 diffuse 5}
}
#declare outer_B_ring = disc { <0, 0, 0> y,  117500, 98390
   pigment {color rgb<0.99,0.88,0.79> filter 0.05}
   finish {ambient 0 diffuse 5}
}
#declare inner_A_ring = disc { <0, 0, 0>  y, 133570, 122200
   pigment {color rgb<0.99,0.88,0.79> filter 0.05}
   finish {ambient 0 diffuse 5}
}
#declare outer_A_ring = disc { <0, 0, 0>  y, 136800, 133895
   pigment {color rgb<0.99,0.88,0.79> filter 0.2}
   finish {ambient 0 diffuse 5}
}
#declare F_ring = disc { <0, 0, 0> y,        140460, 140210
   pigment {color rgb<0.99,0.88,0.79> filter 0.05}
   finish {ambient 0 diffuse 5}
}
union {
   object {sun}           object {saturn_planet scale <1,0.9,1>}
   object {C_ring      }  object {inner_B_ring}
   object {outer_B_ring}  object {inner_A_ring}
   object {outer_A_ring}  object {F_ring}
   rotate x*20
}
```



WATER AND SKY. Water and sky are frequenly added, often by "artistic" reasons.

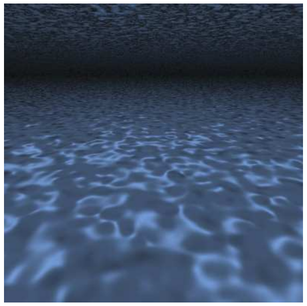
```
#include "colors.inc"
#include "skies.inc"
#include "sbb.inc"

light_source { <-200, 10, -200> color rgb <1,1,1>  }
light_source { < 200,100, -200> color rgb <1,1,1>  }

#declare p=<0,40,-100>;
camera { location p up y right x look_at 0 rotate y*360*clock}

#declare WaterTexture = texture {
    pigment { color rgb<0.39, 0.41, 0.83> }
    finish  { ambient 0.15 brilliance 5 diffuse 0.6 specular 0.80
              roughness 1/100 reflection 0.65 }
}

#declare Water = plane { y, 0.0
   texture { WaterTexture
      normal { waves 0.05 frequency 5000.0 scale 3000.0 }
   }
}

object{ SBBBruecke translate -x*100}
object{ Water }
sky_sphere { S_Cloud2 }
```



REFRACTION. When light passes from one medium to an other, the path of the ray of light is bent because the speed in different media is different and the path choses the path traversed in shortest time. This is called **refraction**. For example air and water have different densities and thus refract differently.

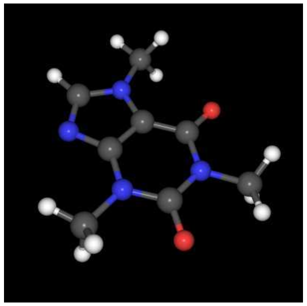
```
light_source { <0, 50, 0> color rgb <1,1,1>          }
camera { direction z location <0,6,-15> look_at <0,2,0> }

plane { y,0
  pigment { rgb <0.6,0.6,0.6> }
  finish { ambient 0.1 diffuse 0.7 }
  normal { bumps 0.8 }
}

plane { y, 12 hollow on
  pigment { rgb <0.5,0.7,1.0> filter 0.93 }
  finish  { reflection 0.7 }
  interior { ior 1.33 caustics 1.0 }
  translate <5, 0, -10>
  normal { bumps 0.8 }
}
```

The index of refraction **ior value** is used to describe the relative density of substances. The default ior value 1.0 (air) gives no refraction. Water has ior 1.33, glass is 1.5 and diamond 2.4. In the picture to the right, the camera is under water, the light source above the water. You see caustics like at the bottom of a swimming pool.

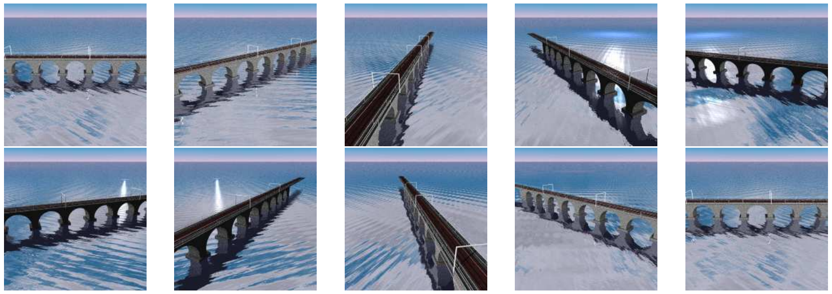


EXTERIOR PROGRAMS. Many organic or an-organic molecules are available on the web in .pdb format. There are viewers which allow to see the molecule and translators, which export a povray file, which can then be rendered. To the right, you see the caffeine molecule.

```
COMPND    caffeine
AUTHOR    Created by Dave Woodcock at Okanagan University College
AUTHOR    email:woodcock@okanagan.bc.ca
AUTHOR    Date revised: Fri Sep 29 14:53:27 2000  GENERATED BY BABEL 1.6
HETATM    1  C           1       0.000   0.000   0.000  1.00  0.00
HETATM    2  C           1       1.392   0.000   0.000  1.00  0.00
HETATM    3  N           1       2.076   1.164   0.000  1.00  0.00
HETATM    4  C           1       1.373   2.321  -0.003  1.00  0.00
HETATM    5  O           1       1.978   3.365  -0.017  1.00  0.00
HETATM    6  N           1       0.017   2.344   0.003  1.00  0.00
HETATM    7  C           1      -0.710   1.202   0.002  1.00  0.00
HETATM    8  O           1      -1.915   1.218  -0.006  1.00  0.00
HETATM    9  N           1      -0.404  -1.287  -0.019  1.00  0.00
HETATM   10  N           1       1.830  -1.279  -0.020  1.00  0.00
HETATM   11  C           1       0.715  -2.048  -0.031  1.00  0.00
HETATM   12  C           1      -1.795  -1.761  -0.044  1.00  0.00
HETATM   13  C           1       3.546   1.178  -0.016  1.00  0.00
HETATM   14  C           1      -0.690   3.634  -0.013  1.00  0.00
HETATM   15  H           1       0.720  -3.138  -0.055  1.00  0.00
HETATM   16  H           1      -1.813  -2.850  -0.090  1.00  0.00
HETATM   17  H           1      -2.307  -1.428   0.860  1.00  0.00
HETATM   18  H           1      -2.302  -1.352  -0.918  1.00  0.00
HETATM   19  H           1       3.894   1.455  -1.011  1.00  0.00
HETATM   20  H           1       3.929   0.190   0.239  1.00  0.00
HETATM   21  H           1       3.911   1.904   0.710  1.00  0.00
HETATM   22  H           1      -1.557   3.583   0.645  1.00  0.00
HETATM   23  H           1      -0.027   4.428   0.329  1.00  0.00
HETATM   24  H           1      -1.020   3.851  -1.029  1.00  0.00
CONECT    1    2    2    7    9
CONECT    2    1    1    3   10
CONECT    3    2    4   13
CONECT    4    3    5    5    6
CONECT    5    4    4
CONECT    6    4    7   14
CONECT    7    1    6    8    8
CONECT    8    7    7
CONECT    9    1   11   12
CONECT   10    2   11   11
CONECT   11    9   10   10   15
CONECT   12    9   16   17   18
CONECT   13    3   19   20   21
CONECT   14    6   22   23   24
CONECT   15   11
CONECT   16   12
CONECT   17   12
CONECT   18   12
CONECT   19   13
CONECT   20   13
CONECT   21   13
CONECT   22   14
CONECT   23   14
CONECT   24   14
MASTER        0    0    0    0    0    0    0    0   24    0   24    0
END
```



ANIMATION. When doing animations, the camera moves along a curve. The velocity and acceleration vector can help to determine, in which direction the camera should look. The frames below were made with a bridge built after the Rheinfall bridge in Switzerland.



MODELS. There are converters to produce povray files from other formats. For example, models created with Poser can be transformed into a form readable by Povray.

```
camera { location <120,11,70> up y right x direction <0,0,7> look_at <30,18,16>}

light_source { <-213, 157, -10> color rgb<1,1,1> }
light_source { <210, 157, -160> color rgb<0.5,0.5,0.5> }
light_source { <-538, 180,481> color rgb<1,1,1> }
light_source { <180, 74, -102> color rgb<1,1,1> }

#include "liberty.inc"
object {statue_of_liberty texture { pigment { color rgb<1,1,1> }
  finish { ambient 0.3 diffuse 0.4 phong 0.7 reflection 0 phong_size 32 } }}
```

The converted file is usually included as a .inc file. The beginning of the file liberty.inc (obtained from the web at http://www.multimania.com/froux/modeles) is displayed blow. There are many models available on the web (good and less good).

```
#declare statue_of_liberty =
union{
mesh{
 smooth_triangle{<1.98,27.1,1.5><9.77,-1.99,-0.8><1.97,27.1,1.02><9.82,-1.3,1.4><1.93,27.34,1.1><9.55,1.41,-2.61>}
 smooth_triangle{<2.1,27.11,1.46><-6.71,1.7,-7.21><2.1,27.49,1.53><-6.33,0.1,-7.74><2.21,27.33,1.36><-7.91,1.6,-5.91>}
 smooth_triangle{<2.02,25.82,1.19><2.3,-4.59,-8.58><2.01,25.82,1.19><-0.5,-4.63,-8.85><2.01,25.86,1.17><0,-5.1,-8.6>}
....
}
```

RAY TRACING COMPETITION. There is a annual ray tracing comptetion going on. Here is an example of an entry from the 1999 competition The authors of the scene are Ian and Ethel MacKay. The source file for this scene is a single povray file.

RENDERING. Raytracing is a CPU time intensive task. The software has to compute the light ray paths bouncing around in a virtual world, compute reflections or refractions. Tracing an image can take from a few seconds to days. A single frame in movies like ”Toy Story” takes several hours to render. To get the large number of frames needed for a movie companies like ”Pixar” (recently again visible with Monsters inc.) use ”computer farms” a huge number of workstations now mostly running linux.

THE ROLE OF MATH 21a. Many topics you learned appear in the area of ray tracing. For example, to compute the normal vector to a polygon, one uses an area formula, which we have proven in a homework using Green's theorem (see below). Normal vectors play an important role because they are needed to compute the reflection of rays. Just look inside a book or article on 3D graphics and many now familiar objects will pop up.

EXAMPLE 1). The method of computing normals to a surface by just looking at three points is error prone. It is often better to consider a polygon $P_i = (x_i, y_i, z_i)$ on the surface. What is the normal to such a polygon? Note that the points P_i are not necessarily on a plane. Here is what people do in 3D graphics: consider the xy projection of the polygon. This gives a polygon (x_i, y_i) in the plane. Now compute the area of that projected polygon with the formula $1/2 \sum_k (x_{k+1} + x_k)(y_{k+1} - y_k)$ (see page 275 in your book). This formula was proven using Green's theorem but can be seen geometrically. This area will be the third component c of the normal vector. Analoguously, compute the other components.

The normal vector to a not necessarily planar polygon $P_i = (x_i, y_i, z_i)$ in space is defined as

$$n = \begin{bmatrix} 1/2 \sum_k (y_{k+1} + y_k)(z_{k+1} - z_k) \\ 1/2 \sum_k (z_{k+1} + z_k)(x_{k+1} - x_k) \\ 1/2 \sum_k (x_{k+1} + x_k)(y_{k+1} - y_k) \end{bmatrix} .$$

EXAMPLE 2) Snells law of refraction is the problem to determine the fastest path between two points, if the path crosses a border of two media and the media have different indices of refraction. This is a Lagrange multiplier problem:

A light ray travles from $A = (-1, 1)$ to the point $B = (1, -1)$ crossing a boundary between two media (air and water). In air (y¿0) the speed of the ray is $v_1 = 1$ (in units of speed of light). In the second medium (y¡0) the speed of light is $v_2 = 0.9$. The light ray travels on a straight line from A to a point $P = (x, 0)$ on the boundary and on a straight line from P to B. Verify Snell's law of refraction $\sin(\theta_1)/\sin(\theta_2) = v_1/v_2$. where θ_1 is the angle the ray makes in air with the y axes and where θ_2 is the angle, the ray makes in water with the y axes.

Solution: Minimize

$$F(x, y) = \sqrt{(-1-x)^2 + y^2}/v1 + \sqrt{(1-x)^2 + y^2}/v2 = l_1/v1 + l_2/v2$$

under the constraint $G(x, y) = y = 0$. The Lagrange equations show that $F_x(x, y) = 0$. This is already Snells law $F_x = v_1 2(x+1)/(2l_1) + v_2 2(1-x)/(2l_2) = 0$ means $v_1/v_2 = \sin(\theta_1)/\sin(\theta_2)$.

URL'S		
	http://www.povray.org	official povray site
	http://www.povworld.org	collection of objects

OTHER RAYTRACING PROGRAMS OR MODELERS. Note that Povray provides the source code to the software in contrast to commercial software which can be very expensive.

http://www.blender.nl	Blender (free 3D modeling)
http://www.aliaswavefront.com	Alias Wave Front (Maya)
http://www.lightwave.com	Lightwave 3D
http://www.corel.com	Bryce
http://www.curiouslabs.com	Poser (3D figure design)
http://www.eovia.com	Carrara (Ray Dream studio)
http://www.artifice.com	Radiance (architecture)
http://www.3dlinks.com	Links

MATHEMATICA — Math21a, O. Knill

VECTORS. Vectors $v = (x, y, z)$, the dot product $v \cdot w$ or the vector product $v \times w$.

```
{3,4,5}+2.0*{-4,5,6}              Cross[{3,4,5},{-4,5,6}]
{3,4,5}.{-4,5,6}                  l[v_]:=Sqrt[v.v]; l[{3,4,5}]
```

PROJECTION of v onto w is $w(v \cdot w)/|w|^2$.
SCALAR PROJECTION: length of projection.

```
l[v_]:=Sqrt[v.v]; proj[v_,w_]:=w (v.w)/(w.w);
scalarproj[v_,w_]:=l[proj[v,w]];
v0={3,4,5}; w0={4,5,1}; scalarproj[v0,w0]
```

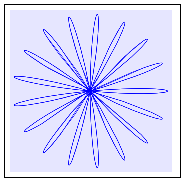
DISTANCES.
$d(P, Q) = |P - Q|$
$d(P, Q + tv) = |(P - Q) \times v)|/|v|$
$d(P, Q + tv + sw) = \frac{|(P-Q)\cdot(v\times w)|}{|v\times w|}$
$d(P + tv, Q + sw) = \frac{|(P-Q)\cdot(v\times w)|}{|v\times w|}$

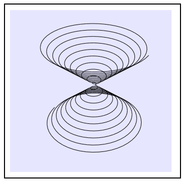
```
P0={2,3,4}; Q0={3,4,6}; v0={1,1,1}; w0={2,4,-1};
l[v_]:=Sqrt[v.v];

PP[P_,Q_]:=Sqrt[(P-Q).(P-Q)];
PL[P_,{Q_,v_}]:=Module[{n=Cross[P-Q,v]/l[v]},l[n]];
PS[P_,{Q_,v_,w_}]:=Module[{n=Cross[v,w]},(P-Q).n/l[n]];
LL[{P_,v_},{Q_,w_}]:=Module[{n=Cross[v,w]},m=l[(P-Q).n]/l[n];

PP[P0,Q0]                              PL[P0,{Q0,v0}]
PS[P0,{Q0,v0,w0}]                      LL[{P0,v0},{Q0,w0}]
```

PLANE CURVES. $r(t) = (x(t), y(t))$.
SPACE CURVES. $r(t) = (x(t), y(t), z(t))$.

```
ParametricPlot[{Cos[t] Cos[17t],Sin[t] Cos[17t]},{t,0,2Pi}]
```

```
ParametricPlot3D[{t Cos[10t],t Sin[10t],t},{t,-2Pi,2Pi}]
```

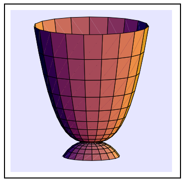


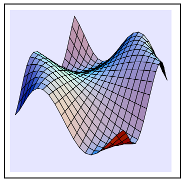


SURFACES. $X(u, v) = (x(u, v), y(u, v), z(u, v))$.
GRAPHS. Graph of map $z = f(x, y)$.

```
ParametricPlot3D[{v Cos[u],v Sin[u],5 v^3},{u,0,2 Pi},{v,-5,10}]
```

```
S=Plot3D[Sin[x*y],{x,-2,2},{y,-2,2}]
```





INTEGRAL. Antiderivative $\int f(x)\, dx$ and definite integral $\int_a^b f(x)\, dx$.

```
Integrate[Cos[Sqrt[x]],x]          NIntegrate[Cos[Sqrt[x]]/x^2,{x,1,Infinity}]
```

LENGTH OF CURVE. $\int_a^b \sqrt{\dot{x} + \dot{y}^2 + \dot{z}^2}\, dt$.
Explicit solutions with "Integrate".

```
r[t_]:={Cos[t],Sin[t],t}; v[t_]:=r'[s] /. s->t;
NIntegrate[Sqrt[v[t].v[t]],{t,0,2Pi}]
```

DOUBLE INTEGRAL. $\int_a^b \int_c^d f(x, y)\, dxdy$.

```
Integrate[ Sin[x] x^2+y^3,{x,0,2 Pi},{y,0,3 Pi} ]
```

TRIPLE INTEGRAL. $\int_a^b \int_c^d \int_e^f f(x, y, z)\, dxdydz$.

```
Integrate[ Sin[x] x^2+y^3+z^4,{x,0,2 Pi},{y,0,3 Pi},{z,0,Pi} ]
```

LINE INTEGRAL. $\int_a^b F(r(t)) \cdot r'(t)\, dt$.

```
r[t_]:={Cos[t],Sin[t],t}; v[t_]:=r'[s] /. s->t;
F[{x_,y_,z_}]:={x^2,y,z}; NIntegrate[F[r[t]].v[t],{t,0,2Pi}]
```

FLUX INTEGRAL.
$\int_a^b \int_c^d F(u, v) \cdot (X_u \times X_v)(u, v)\, dvdu$.

```
X[u_,v_]:={Cos[u],Sin[u],v}; F[{x_,y_,z_}]:={x^2,y,z};
Xu[u_,v_]:=D[X[s,v],s] /. s->u; Xv[u_,v_]:=D[X[u,t],t] /. t->v;
R[u_,v_]:=Cross[Xu[u,v],Xv[u,v]];
Integrate[F[X[u,v]].R[u,v],{u,0,2Pi},{v,0,1}]
```

LAGRANGE MULTIPLIERS. Extremize $F(x, y, z)$ under constraint $G(x, y, z) = c$. Solve system $\nabla F(x, y, z) = \lambda \nabla G(x, y, z), G(x, y, z) = c$ for the x, y, z, λ.

```
F[x_,y_,z_]:=-x^2-y^2-z^2;
G[x_,y_,z_]:=x+y+z;
Solve[{D[F[x,y,z],x]== L*D[G[x,y,z],x],
       D[F[x,y,z],y]== L*D[G[x,y,z],y],
       D[F[x,y,z],z]== L*D[G[x,y,z],z],
       G[x,y,z]==1},{x,y,z,L}]
```

GRAD CURL DIV.
$\mathrm{grad}(f) = (f_x, f_y, f_z)$
$\mathrm{rurl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$
$\mathrm{div}(P, Q, R) = P_x + Q_y + R_z$.

```
f[x_,y_,z_]:=x^2 y z; F[x_,y_,z_]:={-y,x,z^2 x};
grad[U_]:={D[U[x,y,z],x],D[U[x,y,z],y],D[U[x,y,z],z]};
curl[F_]:={D[F[x,y,z][[3]],y]-D[F[x,y,z][[2]],z],
           D[F[x,y,z][[1]],z]-D[F[x,y,z][[1]],x],
           D[F[x,y,z][[2]],x]-D[F[x,y,z][[3]],y]};
div[F_]:=D[F[x,y,z][[1]],x]
        +D[F[x,y,z][[2]],y]
        +D[F[x,y,z][[3]],z];
```

LINEAR APPROXIMATION.
$L(x, y, z) = f(x, y, z) + \mathrm{grad}(f)(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)$

```
f[x_,y_,z_]:=Sin[x*y]*z-x^2*y;
Dx[f_][x_,y_,z_]:=D[f[u,v,w],u] /. {u->x,v->y,w->z}
Dy[f_][x_,y_,z_]:=D[f[u,v,w],v] /. {u->x,v->y,w->z}
Dz[f_][x_,y_,z_]:=D[f[u,v,w],w] /. {u->x,v->y,w->z}
grad[f_][x_,y_,z_]:={Dx[f][x,y,z],Dy[f][x,y,z],Dz[f][x,y,z]};
L[x_,y_,z_]:=f[1,1,1]+grad[f][1,1,1].{x-1,y-1,z-1}
```

QUADRATIC APPROXIMATION.
$Q(x, y) = L(x, y) + [f''(x_0, y_0)(x - x_0, y - y_0)] \cdot (x - x_0, y - y_0)/2$.

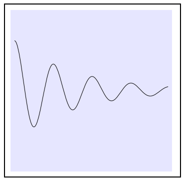
```
f[x_,y_]:=Sin[x*y]-x^2*y;
Dx[f_][x_,y_]:=D[f[u,v],u] /. {u->x,v->y}
Dy[f_][x_,y_]:=D[f[u,v],v] /. {u->x,v->y}
grad[f_][x_,y_]:={Dx[f][x,y,z],Dy[f][x,y,z]};
Dxx[f_][x_,y_]:=D[f[u,v],{u,2}]  /. {u->x,v->y}
Dxy[f_][x_,y_]:=D[D[f[u,v],u],v] /. {u->x,v->y}
Dyy[f_][x_,y_]:=D[f[u,v],{v,2}]  /. {u->x,v->y}
hess[f_][x_,y_]:={{Dxx[f][x,y],Dxy[f][x,y]},{Dxy[f][x,y],Dyy[f][x,y]}}
Q[x_,y_]:=f[1,1]+grad[f][1,1].{x-1,y-1}+(hess[f][1,1].{x-1,y-1}).{x-1,y-1}/2
```

ODE's. $p'[t] = f(p, t)$

```
DSolve[D[p[x],x]==p[x]*(1-p[x]),p[x],x]
```

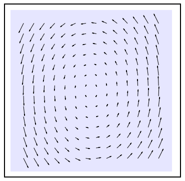
INITIAL VALUE PROBLEM. $x'' = x - \mu x', x[0] = a, x'[0] = b$

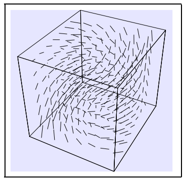
```
U=DSolve[{x''[t]==-x[t]-x'[t]/5,x[0]==1,x'[0]==0},{x},t];
S=Plot[Evaluate[x[t] /. U],{t,0,25}]
```



VECTOR FIELD. $F(x, y) = (P(x, y), Q(x, y))$

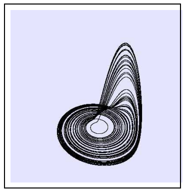
```
Needs["Graphics`PlotField`"]
PlotVectorField[{-y,2 x},{x,-1,1},{y,-1,1}]
PlotGradientField[x^2 y,{x,-1,1},{y,-1,1}]

Needs["Graphics`PlotField3D`"]
PlotVectorField3D[{-y,2 x,z},{x,-1,1},{y,-1,1},{z,-1,1}]
PlotGradientField3D[x^2 y z,{x,-1,1},{y,-1,1},{z,-1,1}]
```





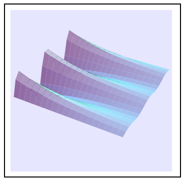
ODE's in space $\dot{x} = F(x)$. The example shows the Rössler attractor.

```
s=NDSolve[{x'[t]==-(y[t]+z[t]),y'[t]==x[t] +0.2*y[t],z'[t]==.2+x[t]*z[t]-5.7*z[t],
x[0]==.3,z[0]==4.2,y[0]==1.3},{x,y,z},{t,0,400},MaxSteps->12000];
S=ParametricPlot3D[Evaluate[{x[t],y[t],z[t]}/.s],{t,0,400}]
```



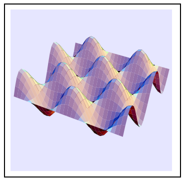
HEAT EQUATION $u_t = \mu u_{xx}$.

```
s=NDSolve[{D[u[x,t],t]==D[u[x,t],{x,2}],
    u[x,0]==Sin[5*Pi*x],u[0,t]==0, u[1,t]==0},u,{x,0,1},{t,0,.01}];
Plot3D[Evaluate[u[x,t]/. s[[1]]],{t,0,.01},{x,0,1}]
```



WAVE EQUATION $u_{tt} = c^2 u_{xx}$.

```
s=NDSolve[{D[u[x,t],{t,2}] == D[u[x,t],{x,2}],
         u[x,0] == Sin[4*Pi*x], Derivative[0,1][u][x,0]==0,
         u[0,t] == u[1,t]}, u, {x,0,1}, {t,0,1}];
Plot3D[Evaluate[u[x,t]/. s[[1]]],{t,0,1},{x,0,1}]
```



ORDINARY DIFFERENTIAL EQUATIONS **Math21a, O. Knill**

Differential equations are laws which determine how a system evolves in time. An example of a system is our solar system. The motion of the planets is described by Newton equations, an ordinary differential equation (ODE). An other example is the electromagnetic field for a fixed charge and current distribution. The motion of this system is described by Maxwell equations. In vacuum, they reduce to the wave equation, describing electromagnetic waves like light. The Maxwell equations or the wave equations are examples of partial differential equations (PDE's).

ODE. An **ordinary differential equation** (ODE) is an equation relating derivatives $f'(t), f''(t), ...$ of a function f or a curve $t \mapsto f(t)$. Only derivative with respect to one variable can appear. Otherwise, the equation is called a **partial differential equation** (PDE). If derivatives up to the order n appear, the equation is called a **differential equation of order** n. If $f(t)$ is a vector in $\mathbf{R}^d$, then the ODE is in d dimensions:

Equation	ODE?
$f''(t) + \cos(f(t)\sin(t)) = 0$	yes
$f_x(x,y) + f_y(x,y) = 0$	no
$\ddot{x}(t) = \sin(x(t))$	yes
$(\dot{x}, \dot{y}) = (-y, x)$	yes
$\mathrm{curl}(F) = A$	no

Equation	order	dim
$f''(x) = f(x)$	2	1
$\sin(f'''(x)) = f'(x)$	3	1
$(f'(x), g'(x), h'(x)) = F(f(x), g(x), h(x))$	1	3
$(\ddot{x}, \ddot{y}) = (-y, x)$	2	2
$(\dot{x})^5 = x$	1	1

While for usual equations like $x^2 + 2x + 1 = 0$, the unknown x is a **number**, for a differential equation like $\ddot{x} = x$ it is a **function** $x(t)$ respectively a **curve** $t \mapsto \vec{x}(t)$. Think always of t as **time**.

SOLVING AN ODE. Solving an ODE means to find all functions or curves which satisfy the equation. Examples.

$f'(x) = f(x)$ has solutions e^x or $5e^x$ and $f''(x) = -f(x)$ has solutions $\cos(x)$ or $4\sin(x)$ and $(x'(t), y'(t)) = (2, x(t))$ has solutions $(2t, t^2)$.

FACT. Every ODE can be written as $d/dt\vec{x}(t) = \vec{F}(\vec{x}(t))$, where $\vec{F}$ is a vector field and $\vec{x}$ is a vector:
PROOF. Solve $F(f, f', f'', \dots, f^{(n)}) = 0$ for $f^{(n)} = G(f, f', f'', \dots, f^{(n-1)})$ and introduce the vector $\vec{x} = (x_1, x_2, \dots, x_n) = (f, f', f'', \dots, f^{(n-1)})$. Then $\vec{x}' = (x_2, x_3, \dots, G(x_1, \dots, x_{n-1})) = F(\vec{x})$.

Example: $f'''(t) - \sin(f''(t)) + f(t) = 0$. Define $\vec{x} = (x(t), y(t), z(t)) = (f''(t), f'(t), f(t))$. Then $d/dt\vec{x} = (f'''(t), f''(t), f'(t)) = (\sin(f''(t)) - f(t), f''(t), f'(t)) = (\sin(y) - x, y, x) = F(\vec{x})$. By introducing t as a separate variable, one can have F time-independent. F is then called autonomous. **By increasing the dimension, an ODE can be written as a first order autonomous system** $\dot{x} = F(x)$.

EXAMPLES OF ODE's.

$\dot{x}(t) = ax(t)$.	Population models for $a > 0$, radioactive decay for $a < 0$
$\dot{x} = ax(1-x)$.	Logistic equation: population model
$\ddot{x} = -ax$.	Harmonic oscillator: describes periodic oscillations
$m_j\ddot{x}_j = \sum_{i \neq j} m_i/\lvert x_i - x_j\rvert^2$	Newton equations. Example: x_j positions of planets of mass m_j.

SOLVING DIFFERENTIAL EQUATIONS. The different approaches are illustrated with $\dot{x} = x$.

1) **Integration.** To find explicit solutions of $dx/dt = x$ we write $dx/x = dt$ or $\int \frac{1}{x} dx = t + c$. Integration on both sides gives $\log(x) = t + c$ or $x(t) = e^t e^c = Ce^t$.
2) **Expansion**. For example by Taylor expansion: if $x(t) = a_0 + a_1 t + a_2 t^2 + ...$, then $\dot{x}(t) = a_1 + 2a_2 t + 3a_3 t^2 + ...$ Comparing coefficients gives $a_1 = a_0, a_2 = a_1/2 = a_0/2, a_3 = a_2/3 = a_0/6$ etc. so that $x(t) = a_0(1 + t/2 + t^2/3! + ...)$.
3) **Approximation**. Simplest numerical method is the linear approximation $x(t+dt) = x(t) + dt\dot{x}(t) = x(t) + dtx(t) = (1+dt)x(t)$ and $x(t+kdt) = (1+dt)^k x(t)$. If $dt = t/n$ and $k = n$, then $x(0+t) = x(0+nt/n) = (1+t/n)^n x(0)$.

In the example, all three methods led to the correct solution $x(t) = Ce^t$. In 1), C was an integration constant, in 2), the zeroth Taylor coefficient a_0, in 3), $C = x(0)$. The explicit expression e^t came in 1) from anti-derivation, in 2) from $(1 + t/2 + t^2/3! + ...) = e^t$, in 3) from $\lim_{n\to\infty}(1+t/n)^n = e^t$. For most ODE's, there are no closed form solutions. One can work however with truncated Taylor series or approximating polygons $x(dt), x(2dt), \dots$ for the path $x(t)$.

EXISTENCE OF SOLUTIONS of $\dot{x} = F(x)$. Given $x(0)$, there exists a solution $x(t)$ for some time $t \in [0, a]$ if F is smooth. Solutions don't always exist for all times: $\dot{x} = x^2$ can be solved by integrating $dx/x^2 = dt$ so that $-1/x = t + c$ or $x(t) = -1/(t+c)$. From $x(0) = -1/c$ we get $c = -1/x(0)$ and $x(t) = -1/(t - 1/x(0))$. The solutions exists for $t \leq 1/x(0)$.

THE EQUATION $\dot{x} = ax$. For $a > 0$, it models a explosive chain-reaction or population growth (the number of newborn people is proportional to the number of people which exist already). For $a < 0$, $\dot{x} = ax$ has solutions which decay (example is radioactive decay: as more atoms there are, as more atoms decay). The explicit solution is $x(t) = x(0)e^{at}$. For $a = 0$, there are only solutions x which are constant in time.

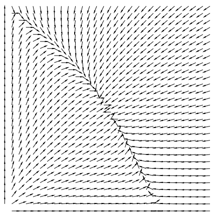
LINEAR EQUATIONS. A ODE is **linear** if it can be written as $\dot{x} = Ax$, where A is a matrix. It can be solved by $x(t) = e^{At}x(0)$, where the matrix e^B is defined as $e^B = 1 + B + B^2/2! + B^3/3!...$ and for example $B^3 = BBB$ is obtained by matrix multiplication. To understand the solutions, it is helpful to choose coordinates in which A is simple, for example diagonal. (See Math21b).

EXAMPLE. The ODE $\ddot{x} = -x$ is with $y = \dot{x}$ equivalant to $(\dot{x}, \dot{y}) = (y, -x)$ and can be written as $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$. Because $A^2 = -1, A^3 = -A, A^4 = 1, A^5 = A$ etc. we have $e^{At} = \begin{bmatrix} 1 - t^2/2! + t^4/3! - ... & t - t^3/3! + t^5/5! - ... \\ -t + t^3/3! - t^5/5! + ... & 1 - t^2/2! + t^4/3! - ... \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$ leading to an explicit solution $x(t) = \cos(t)x(0) + \sin(t)y(0), y(t) = -\sin(t)x(0) + \cos(t)y(0)$. Indeed, the circular curves $r(t) = (x(t), y(t))$ solve the differential equation.

SEPARATION OF VARIABLES. A system $\dot{x} = g(x,t)$ (equivalently $\dot{x} = g(x,t), \dot{t} = 1$) can sometimes be solved by **separation of variables**: $dx = f(x)g(t)dt$ or $dx/f(x) = g(t)dt$ can be integrated:
- If $\dot{x} = g(t)$, then $x(t) = \int_0^t g(t)\, dt$.
- If $\dot{x} = h(x)$, then $dx/h(x) = dt$ and so $t = \int_0^x dx/h(x) = H(x)$ so that $x(t) = H^{-1}(t)$.
- If $\dot{x} = g(t)/h(x)$, then $H(x) = \int_0^x h(x)\, dx = \int_0^t g(t)\, dt = G(t)$ so that $x(t) = H^{-1}(G(t))$.

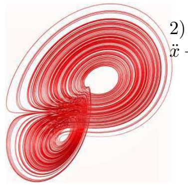
In general, there are no closed form solutions (i.e. $\dot{x} = e^{-t^2}$ has a solution $x(t) = \int_0^t e^{-t^2}\, dt$ which can not be expressed by $\exp, \sin, \log, \sqrt{\cdot}, etc.$. The anti-derivative of $e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \dots$ is $x(t) = t - t^3/3 + t^4/(3 \cdot 2!) - t^7/(7 \cdot 3!) + \dots$).

2D SYSTEMS. A two dimensional ODE is of the form $\dot{x} = P(x,y), \dot{y} = Q(x,y)$. If $r(t) = (x(t), y(t))$ is the solution curve, then the ODE tells that the velocity vector $r'(t)$ is equal to $F(r(t))$. Often a good understanding is achieved by drawing vector field and some solution curves.
Examples 2) and 3) below are 2D systems. Example 1) shows a system in three dimensions and also the timedependent system 4) became a 3D system when adding t as the z variable.



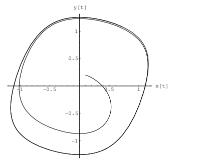
1) LORENTZ SYSTEM.

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= -xz + 28x - y \\ \dot{z} &= xy - \frac{8z}{3} \end{aligned}$$



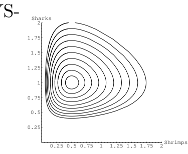
2) VAN DER POL EQUATION.
$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ or

$$\begin{aligned} \dot{x} &= y - (\frac{x^3}{3} - x) \\ \dot{y} &= -x \end{aligned}$$



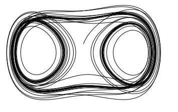
3) VOLTERRA-LODKA SYSTEMS.

$$\begin{aligned} \dot{x} &= -\frac{y}{10} + \frac{2xy}{5} \\ \dot{y} &= \frac{2x}{5} - \frac{2xy}{5} \end{aligned}$$



4) THE DUFFING SYSTEM.
$\ddot{x} + \dot{x}/10 - x + x^3 - 12\cos(t) = 0$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{y}{10} - x + x^3 - 12\cos(z) \\ \dot{z} &= 1 \end{aligned}$$



PARTIAL DIFFERENTIAL EQUATIONS I — Math21a, O. Knill

PDE. If an equation for a function contains derivatives with respect to different variables, it is called PDE. We will see examples of PDES, see how they are derived from physics and how one can solve them.

EXAMPLES of partial differential equations:

- $u_t = u_x$ **advection equation**, models transport
- $u_{tt} = \Delta u = u_{xx} + u_{yy}$ **wave equation** in 2D.
- $u_t = \kappa u_{xx}$ **heat equation** in 1D
- $u_t = \kappa \Delta u = \kappa(u_{xx} + u_{yy})$ **heat equation** in 2D.
- $u_{tt} = \Delta u - m^2 u$ **Klein-Gordon** equation, appears in quantum theory.
- $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$ **Laplace equation** in 3D.
- $\mathrm{grad}(U) = F$ search of **potential** U for vector field F. (Need $\mathrm{curl}(F) = 0$).
- $\mathrm{div}(B) = 0$, $\mathrm{div}(E) = 0$, $\dot{B} = -c\ \mathrm{curl}(E)$, $\dot{E} = c\ \mathrm{curl}(B)$. **Maxwell equations** in vacuum.
- $\mathrm{curl}(A) = F$ search of **vector potential** for vector field F. (Need $\mathrm{div}(F) = 0$).
- $\Delta u = 4\pi\rho$ **Poisson equation**. ρ charge, $E = \mathrm{grad}(u)$ is electric field.
- $u_t + u_{xxx} + 6uu_x = 0$ **KdV equation** has soliton solutions famous nonlinear waves.
- $i\hbar\ \dot{u} = \hbar^2 \Delta u/2m + Vu$ **Schrödinger equation** in quantum mechanics.
- $u_{tt} - u_{xx} + \sin(u) = 0$ **Sine-Gordon equation** has "kink-antikink" solutions.
- $\dot{u} + u \cdot \nabla u = \nu \Delta u - \nabla p + f, \mathrm{div}(u) = 0$ **Navier Stokes** equations. Win 1 Mio prove existence!
- $\dot{u} = uu_x$ **Burger equation**, models shocks, breaking waves.
- $\dot{u} = (u_x)^2/2 + V(x)$ Hamilton-Jacobi equation. Classical mechanics.
- $\det(f'') = \rho$ **Monge-Ampere equation**, f'' Hessian of $f = f(x,y,z)$. Appears in geometry.

DERIVATION OF HEAT EQUATION. Conservation of mass tells that the flux of a fluid with density ρ and speed j through a surface S bounding a region G is equal to the change of mass in the region $\int\int_S j \cdot dS = \int\int\int_G \rho\ dV$. By Gauss theorem, $\int\int_S j \cdot dS = \int\int\int_G \mathrm{div}(j)\ dV$ so that $\int\int\int_G (\mathrm{div}(j) - \rho)\ dV = 0$. Because the spherical region is arbitrary, we have $f = \mathrm{div}(j) - \rho = 0$ (if the function f were not zero, we could chose a region G contained in $f > 0$). The density and the current are therefore linked by the **continuity equation** $\dot{\rho} = \mathrm{div}(j)$. If particles move randomly, they tend to move more likely from places with high density ρ to places with low density. This means $j = \kappa \mathrm{grad}(\rho)$. Plugging this into the continuity equation gives $\dot{\rho} = \kappa \mathrm{div}(\mathrm{grad})(\rho) = \kappa \Delta \rho$. This is the **diffusion equation**, also called **heat equation**.

DERIVATION OF THE WAVE EQUATION. The identity $\Delta B = \mathrm{grad}(\mathrm{div}(B)) - \mathrm{curl}(\mathrm{curl}(B))$ gives together with $\mathrm{div}(B) = 0$ and $\mathrm{curl}(B) = \dot{E}/c$ the relation $\Delta B = -d/dt \mathrm{curl}(E)/c$ which leads with $\dot{B} = -c\ \mathrm{curl}(E)$ to the wave equation $\Delta B = d^2/dt^2 B/c^2$. This equation $B_{tt} = c^2\ \Delta B$ (as well the equation $E_{tt} = c^2\ \Delta B$ derived in the same way) describes the propagation of electric and magnetic fields in vacuum.

DERIVATION OF THE TRANSPORT EQUATION. Start again with the continuity equation $\dot{u} = \mathrm{div}(j) = j_x$ in one dimensions. If the current j is proportional to u_x we get $u_t = au_x$.

DERIVATION OF BURGER'S EQUATION. If $j = (u^2)_x$, this means that the current j depends nonlinearly on u, becoming large for large u. The corresponding equation $u_t = 2uu_x$ is a model for waves near the beach.

SUPERPOSITION. A PDE is called **linear** if it is linear in each variable, as well as in u, u_x, u_t etc. The heat, Laplace, wave, advection or Maxwell equations in vacuum are linear. The Burger equation, Sin-Gordon or KdE equation are examples of nonlinear equations.

If u and v are solutions of a linear equation, then λu and the superposition $u + v$ are solutions too.

THE ADVECTION EQUATION. A general solution of $u_t = cu_x - ru$ is

$$u(t,x) = e^{-rt} f(x + ct) ,$$

here $f(x) = u(0,x)$. This formula provides an explicit solution of a PDE. The initial function $u(0,x)$ determines $u(t,x)$ at later time in an explicit way.

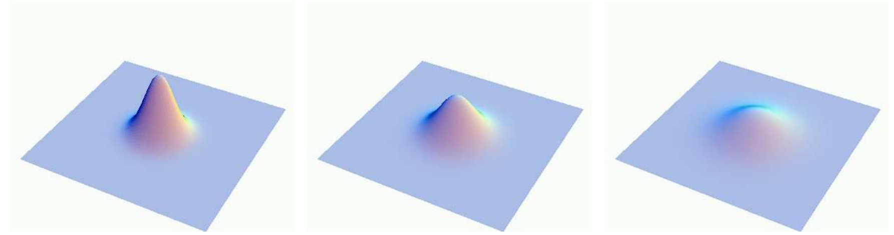
DIFFUSION EQUATION. A special solution to the diffusion equation $u_t = \mu u_{xx} - ru$ is

$$u(x,t) = a \frac{1}{\sqrt{4\pi\mu t}} e^{-rt} e^{-x^2/(4\mu t)} .$$

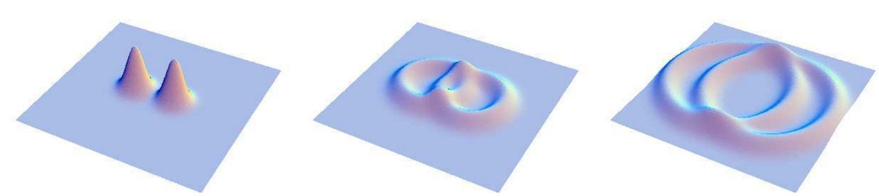
(Check $u_t/u = -r + (x^2 - 2\mu t)/(4\mu t^2), u_{xx}/u = (x^2 - 2\mu t)/(4\mu^2 t^2)$). Also translations $u(x - s, t)$ are solutions. By superposition, sums of such functions are solutions and by passing to limits of Riemann sums, also integrals

$$u(x,t) = \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{\infty} a(s) e^{-rt} e^{-(x-s)^2/(4\mu t)}\ ds$$

is a solution. By taking the limit $t \to 0$, we see that $a(s) = u(0,s)$. We have therefore an explicit formula for the solution of the diffusion equation. The initial heat distribution determines the distribution at time t. It is considered an explicit formula. It can be generalized to arbitrary dimensions. The three pictures show an evolution in the plane.



WAVE EQUATION. Write $u_{tt} - u_{xx} = 0$ as $(\partial_t - \partial_x)(\partial_t + \partial_x)u$ or $(\partial_t + \partial_x)(\partial_t - \partial_x)u$. You see that if u satisfies the transport equation $(\partial_t + \partial_x)u = 0$ or $(\partial_t - \partial_x)u = 0$, then u solves the wave equation. Solutions are $f(x - t)$ or $g(x + t)$. Now, if $u(0,x) = F(x)$ and $u_x(u,x) = G(x)$ are known and $u(t,x) = f(x - t) + g(x + t)$, then $F(x) = u(0,x) = f(x) + g(x)$ and $G(x) = u_t(0,x) = -f'(x) + g'(x)$ determine f and g: from $F = f + g, F' = f' + g', G = g' - f'$ we get $f' = (F' - G)/2$ and $g' = (F' + G)/2$ and so f, g. The pictures show an evolution of the wave equation in two dimensions.



SOLUTIONS BY SEPARATIONS OF VARIABLE. Some linear equations in one dimension can be solved by starting with an "Ansatz" $u(t,x) = T(t)X(x)$.
TRANSPORT EQUATION. $u_t = \dot{T}X$, $u_x = TX'$. Because $\dot{T}/T = \dot{X}/X$ does not depend on x (look at LHS) and not depend on t (look at RHS), it is a constant C and we have $T(t) = e^{Ct}$ and $X(x) = e^{Cx}$ and therefore $u(t,x) = T(t)X(x) = e^{Ct}e^{Cx} = e^{C(t+x)}$. By summing up such solutions one build any solution $f(t + x)$.
DIFFUSION EQUATION. $u_t = \ddot{T}X$, $u_{xx} = TX''$. Again, $\dot{T}/T = X''/X$ is constant $C = -k^2$ so that $T(t) = e^{-k^2 t}$ and $X(x) = e^{ikx}$ and $u(t,x) = e^{-k^2 t}e^{ikx}$. (If X is a bounded function, then C can not be positive so that we can write $C = -k^2$.) A general solution is a superposition $u(t,x) = \int a(k) e^{-k^2 t} e^{ikx} dx$ of such solutions. One can determine $a(k)$ from the initial condition $u(0,x) = \int a(k) e^{ikx} dx$ using Fourier theory.
WAVE EQUATION. Also the wave equation can be treated like that. We get $\ddot{T}/T = X''/X = C$. Again $C = -k^2$ and $X(x) = e^{\pm ikx}, T(t) = e^{\pm ikt}$ so that $u(t,x) = e^{\pm ikx}e^{\pm ikt}$ is a solution. We see that $a(k)e^{ik(x-t)}$ and $b(k) = e^{ik(x+t)}$ are solutions. Superposition leads to arbitrary solutions $u(t,x) = f(x - t) + g(x + t)$.

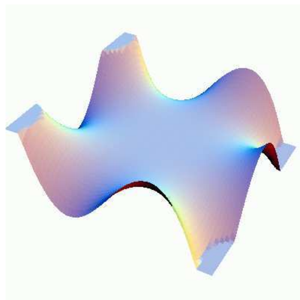
BEYOND PDE's. A computer which solves ODE's or PDE's numerically discretizes the system. u_t is replaced by the difference $u_{n+1} - u_n$, u_{tt} by $u_{n+1} - 2u_n - u_{n-1}$. The equation $\dot{x} = ax$ for example is discretizes as $x_{n+1} - x_n = ax_n$ or $x_{n+1} = (1 + a)x_n$. Instead of a **curve** $x(t)$, the solution is a **sequence** x_n. Time is discrete. Also PDE's can be discretized. The diffusion equation becomes $u_{n+1,m} - u_{n,m} = u_{n,m+1} - 2u_{n,m} + u_{n,m-1}$ which is $u_{n+1,m} = u_{n,m+1} - u_{n,m} + u_{n,m-1}$. This is also called a **coupled map lattice**. In each time step, a lattice configuration $\{u_{n,m}\}_{m=-\infty}^{\infty}$ is mapped into a new configuration. If also the values of $u_{n,m}$ are discretizes, one gets a **cellular automaton**. Both can be used to simulate partial differential equations numerically.

PARTIAL DIFFERENTIAL EQUATIONS II — Math 21a, O. Knill

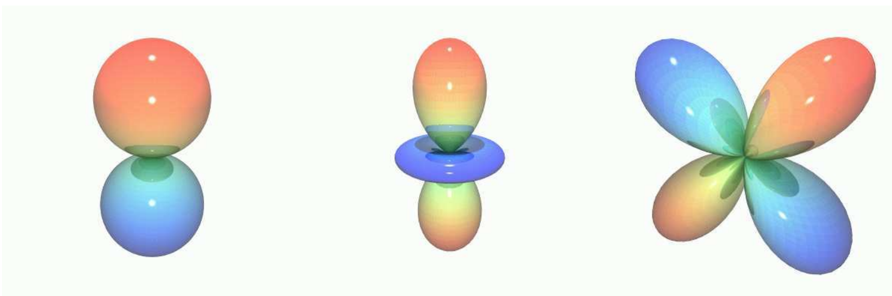
A great deal of my work is just playing with equations and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just looking for beautiful mathematical relations which maybe don't have any physical meaning at all. Sometimes they do.
Paul A. M. Dirac

LAPLACE EQUATION. In the plane, the equation is $\Delta u = \text{divgrad}u = u_{xx} + u_{yy} = 0$. Solutions are called **harmonic**. You have found such solutions already in an earlier homework assignment.

Here is a general way to produce solutions: if $z = x + iy$ is a complex number, then the real or imaginary part of $(x+iy)^n = r^n e^{in\theta}$ are solutions of the Laplace equation. For example, $u(x,y) = \text{Re}((x+iy)^3) = x^3 - 3xy^2 = r^3\cos(3\theta)$ or $u(x,y) = \text{Im}((x+iy)^3) = 3x^2y - y^3 = r^3\sin(3\theta)$ solve the Laplace equation.



In one dimension, only linear functions $u(x) = ax+b$ are harmonic: just solve $u_{xx} = 0$. In three dimensions, harmonic functions are constructed in spherical coordinates $u(r,\phi,\theta) = \text{Re}(r^l P_l^m(\cos(\theta))e^{im\phi})$ with special polynomials P_l. The angular portions of such functions are called **spherical harmonics** and denoted by Y_{lm}. The pictures show three examples. These functions are important for quantum mechanics because there are functions R_{nl} of r such that $u_{n,l,m}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\phi,\theta)$ define stationary electron distributions in the hydrogen atom. Knowing about these functions allows to understand of the hydrogen atom and makes the construction of the periodic system of elements transparent.



You can play with spherical harmonics in a Java applet on *http://www.sct.gu.edu.au/research/laserP/livejava/spher_harm.html* Harmonic functions have the property that they are stationary solutions to the heat equation $u_t = \mu\Delta u$ or the wave equation $u_{tt} = c^2\Delta u$ or the Schrödinger equation of a free particle.

H																	He
Li	Be											B	C	N	O	F	Ne
Na	Mg											Al	Si	P	S	Cl	Ar
K	Ca	Sc	Ti	V	Cr	Mn	Fe	Co	Ni	Cu	Zn	Ga	Ge	As	Se	Br	Kr
Rb	Sr	Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd	Ag	Cd	In	Sn	Sb	Te	I	Xe
Cs	Ba	La	Hf	Ta	W	Re	Os	Ir	Pt	Au	Hg	Tl	Pb	Bi	Po	At	Rn
Fr	Ra	Ac	Unq	Unp	Unh	Uns	Uno	Une									
Ce	Pr	Nd	Pm	Sm	Eu	Gd	Tb	Dy	Ho	Er	Tm	Yb	Lu				
Th	Pa	U	Np	Pu	Am	Cm	Bk	Cf	Es	Fm	Md	No	Lr				

POISSON EQUATION. A generalization of the Laplace equation is the **Poisson equation** $\Delta u = \rho$, where ρ is a scalar function. If ρ is a **charge distribution**, then u is the **electric potential**: The vector field $E = \text{grad}(u)$ satisfies the Maxwell equation $\text{div}(E) = \rho$. The trick to solve this Maxwell equation is to solve the Poisson equation. There are solution formulas using **Green's functions** g_d, depending on the dimension d:

1D	$u(x) = 1/2\int_R \rho(s)\|x-s\|\,ds.$	$g_1 \star \rho$
2D	$u(x) = \frac{-1}{2\pi}\int\int_{R^2}\rho(s)\log\|x-s\|\,dA(s)$	$g_2 \star \rho$
3D	$u(x) = \frac{-1}{4\pi}\int\int\int_{R^3}\rho(s)/\|x-s\|\,dV(s)$	$g_3 \star \rho$

On the right, the notation $f \star b = \int f(x-y)g(y)\,dy$ is used. $*$ is a "multiplication" of functions called **convolution**. Fourier theory allow to prove these formulas with no effort: the Poisson equation $\Delta u = \rho$ becomes after Fourier transform $\sum_{i=1}^d x_i^2\hat{u} = \hat{\rho}$ so that $\hat{u} = \hat{g}_d\hat{\rho}$, where $\hat{g}_d = 1/\sum_{i=1}^d x_i^2$. Reversing the Fourier transform gives $u = g_d \star \rho$ (there is a general rule $\hat{fg} = \hat{f}\star\hat{g}$. Because the Fourier transform of $1/x^2$ is $g_1(x) = |x|$, the Fourier transform of $1/(x^2+y^2)$ is $g_2(x) = \frac{1}{4\pi}\log|x^2+y^2|$ and the Fourier transform of $1/(x^2+y^2+z^2)$ is $\frac{1}{2\pi}1/\sqrt{(x^2+y^2+z^2)}$, the solution formulas become obvious.
UNIQUENESS OF SOLUTIONS? Note that if you take a solution u of the Poisson equation and add a harmonic function u satisfying $\Delta u = 0$, you get an other solution.
ABOUT THE GREEN FUNCTIONS. The Green functions g_d are the natural **Newton potentials** in $\mathbf{R}^d$. The one dimensional potential is studied because two infinite planes attract each other with force $|x|$. The n-body problem with such potentials attractive (gravitational) or repelling (electric) is studied a lot.

EXISTENCE OF SOLUTIONS. Having explicit solutions to partial differential equations is rather exceptional. In general one has to work hard to establish **existence of solutions**. Existence can be subtle. Even for one of the simplest nonlinear PDE's $\text{grad}(u) = F$, there are not necessarily solutions. Only if $\text{curl}(F) = 0$, then a potential u can exist. For nonlinear PDE's, existence can be very hard to prove. One million dollars have been written out by the Clay institute for a proof in the case of the Navier Stokes equation. Anyhow, as the following story shows, mathematicians are crazy about existence:

> An engineer, a chemist and a mathematician are staying in three adjoining cabins at an old motel. First the engineer's coffee maker catches fire. He smells the smoke, wakes up, unplugs the coffee maker, throws it out the window, and goes back to sleep. Later that night the chemist smells smoke too. She wakes up and sees that a cigarette butt has set the trash can on fire. She says to herself, "Hmm. How does one put out a fire? One can reduce the temperature of the fuel below the flash point, isolate the burning material from oxygen, or both. This could be accomplished by applying water." So she picks up the trash can, puts it in the shower stall, turns on the water, and, when the fire is out, goes back to sleep. The mathematician, of course, has been watching all this out the window. So later, when he finds that his pipe ashes have set the bed-sheet on fire, he is not in the least taken aback. He says:
>
> "Aha! A solution exists!"
>
> and goes back to sleep ...

PDE's IN MULTIVARIABLE CALCULUS. Where did PDE's matter in multivariable calculus?

- when doing partial derivatives, we checked that functions are solutions of PDE's. Examples were the **continuity equation**, the **Laplace equation**, the **wave equation**, the **Burger equation**.
- we have seen the meaning of the **heat equation** $T_t = \Delta T$ using the divergence theorem.
- The **Laplace equation** $\Delta f = 0$ defines harmonic functions.
- We have seen relations between certain PDE's and the topology of the space:
  $curl(F) = 0$ implied $F = \text{grad}(f)$ if simply connected
  $div(F) = 0$ implied $F = \text{curl}(G)$ if every sphere can be collapsed to a point
  $grad(f) = 0$ implied $f = const$ if region is connected
- the **Maxwell equations** illustrate the divergence theorem as well as Stokes theorem.
- we have seen the **Poisson equation** $\text{div}(F) = \rho$ describing gravity as well as electrostatics.