SCALAR LINE INTEGRALS.

If $f(r(t))$ is a function defined on a curve $\gamma : t \mapsto \vec{r}(t)$, then $\boxed{\int_a^b f(\vec{r}(t))|\vec{r}'(t)|\, dt}$ is called the **scalar line integral** of f along the curve γ.

NOTATION. The short-hand notation $\int_\gamma f\, ds$ is also used.
WRITTEN OUT. If $f(x, y)$ is the function and $\vec{r}(t) = (x(t), y(t))$, we can write $\int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt$.

In three dimensions, where $\vec{r}(t) = (x(t), y(t), z(t))$, we can write $\int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$.

EXAMPLE. Integrate $f(x, y, z) = x^2 + y^2 + z^2$ over the path $r(t) = (\cos(t), \sin(t), t)$ from $t = 0$ to $t = \pi$. The answer is $\int_0^\pi (1 + t^2)\sqrt{2}\, dt = \pi\sqrt{2} + \pi^3\sqrt{2}/3$.

EXAMPLE: Let $\mathbf{r}(t) = \{\cos(t), \sin(t), t^2/2\}$ be the path of a model plane. What is the average height of the plane?

This is not a very clearly formulated question. We want to know

$$\int_0^{2\pi} z(t)|r'(t)|\, dt = \int_0^{2\pi} \frac{t^2}{2}\sqrt{1 + t^2}\, dt$$

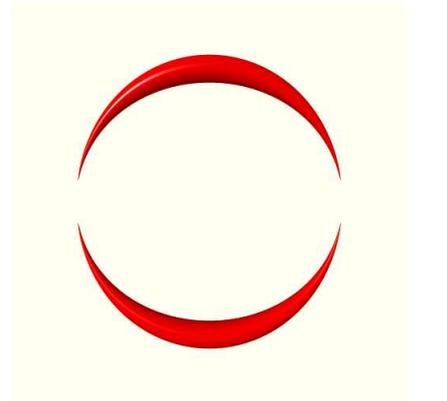if we want to know the average height of the path and

$$\int_0^{2\pi} z(t)\, dt = \int_0^{2\pi} t^2/2\, dt$$

if we want to know the average height per time.

EXAMPLE. A wire $r(t) = (\cos(t), 0, \sin(t))$ has thickness $f(r(t)) = \sin^2(t)$ and $t \in [0, 2\pi]$ What is the mass of this wire? The mass is, because $r'(t) = 1$:

$$M = \int_0^{2\pi} \sin^2(t)\, dt = \pi\ .$$

rode with her Kawasaki motorcycle and a camera through the deserted Chernobyl area and left an impressive document on the web. The URL is http://www.angelfire.com/extreme4/kiddofspeed/
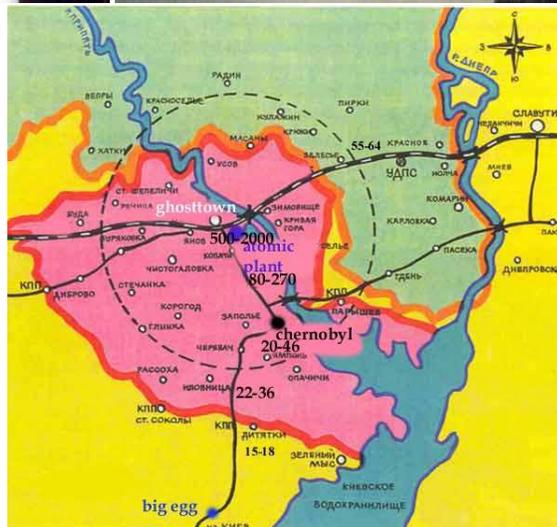

Assume Elena picks up radioactive radiation proportional to the radioactivity level $f(x,y)$ and the amount of path covered with the bike, then the total radiation obtained during the time $[0,T]$ is

$$\int_0^T f(r(t))|r'(t)|\, dt \ .$$

This is not realistic. If Elena stops, she would get no radiation increase. The correct integral would rather be

$$\int_0^T f(r(t))\, dt \ .$$

REMARK.

Scalar line integrals should be thought as a generalization of the length integral.

Do not mix it up with the line integral defined by a vector field which we cover next and which is infinitely more important.

The examples above show that dealing with scalar line integrals can be confusing. You are measuring quantities with it which are given **"per distance"** and not quantities **"per time"**. Scalar integrals hardly appear in applications. (They do for example in tomography where the problem is to reconctruct $f(x,y,z)$ from knowing all line integrals along all lines. But also there, it is possible and simpler to avoid them.)

- the application of computing mass is very artificial because mass is a triple integral. All solid bodies, even wires have three dimension.

- In general, for one dimensional situations, the density is constant so that the scalar line integral is actually a usual length integral. Nobody questions the importance of the length integral.

- For "center of mass" or moment of inertia" computations, one better uses triple integrals. Arcs or wires have a nonzero radius and the "simplification" done by computing it with one dimensional integrals **produces an error**. For mass computations, the error is zero by the **Pappus Centroid theorem**. Introducing artificial scalar line integrals is not only confusing, it is also not precise.

The topic has been introduced into calculus text books as a "bridge" to ease the transition from 1D integrals to line integrals but experience shows that this "bridge" unnecessarily complicates things because it introduces a new concept.

Just treat the topic as a "footnote" to the length integral.