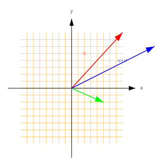
Lecture 1: VECTORS DOT PRODUCT **O. Knill, Math21a**

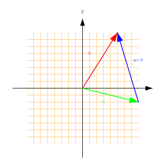
HOMEWORK: Section 10.1: 42,60: Section 10.2: 4,16

VECTORS. Two points $P_1 = (x_1, y_1)$, $Q = P_2 = (x_2, y_2)$ in the plane determine a **vector** $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. It points from P_1 to P_2 and we can write $P_1 + \vec{v} = P_2$.
COORDINATES. Points P in space are in one to one correspondence to vectors pointing from 0 to P. The numbers $\vec{v}_i$ in a vector $\vec{v} = (v_1, v_2)$ are also called **components** or of the vector.
REMARKS: vectors can be drawn **everywhere** in the plane. If a vector starts at 0, then the vector $\vec{v} = \langle v_1, v_2 \rangle$ points to the point $\langle v_1, v_2 \rangle$. That's is why one can identify points $P = (a, b)$ with vectors $\vec{v} = \langle a, b \rangle$. Two vectors which can be translated into each other are considered **equal**. In three dimensions, vectors have three components. In some Encyclopedias like Encylopedia Britannica define vectors as objects which have "both magnitude and direction". This is unprecise and strictly speaking incorrect because the zero vector is also a vector but has no direction.

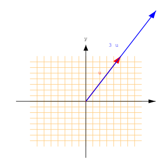
ADDITION SUBTRACTION, SCALAR MULTIPLICATION.



$\vec{u} + \vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$
$= \langle u_1 + v_1, u_2 + v_2 \rangle$



$\vec{u} - \vec{v} = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle$
$= \langle u_1 - v_1, u_2 - v_2 \rangle$

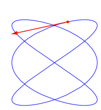


$\lambda \vec{u} = \lambda \langle u_1, u_2 \rangle$
$= \langle \lambda u_1, \lambda u_2 \rangle$

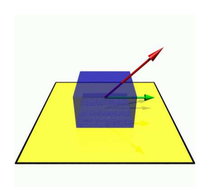
BASIS VECTORS. The vectors $\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$ are called **standard basis vectors** in the plane. In space, one has the basis vectors $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$.
Every vector $\vec{v} = (v_1, v_2)$ in the plane can be written as a sum of standard basis vectors: $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$. Every vector $\vec{v} = (v_1, v_2, v_3)$ in space can be written as $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$.

WHERE DO VECTORS OCCUR? Here are some examples:

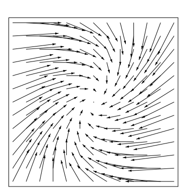
Velocity: if $(f(t), g(t))$ is a point in the plane which depends on time t, then $\vec{v} = \langle f'(t), g'(t) \rangle$ is the **velocity vector** at the point $(f(t), g(t))$.



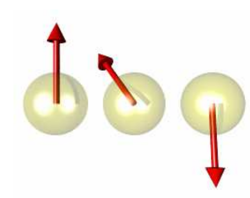
Forces: Some problems in statics involve the determination of a forces acting on objects. Forces are represented as vectors



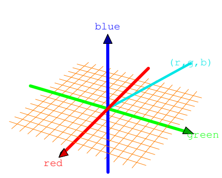
Fields: fields like electromagnetic or gravitational fields or velocity fields in fluids are described with vectors.



Qbits: in quantum computation, one does not work with bits, but with **qbits**, which are vectors.



Color can be written as a vector $\vec{v} = (r, g, b)$, where r is red, g is green and b is blue. An other coordinate system is $\vec{v} = (c, m, y) = (1 - r, 1 - g, 1 - b)$, where c is cyan, m is magenta and y is yellow.



SVG. Scalable Vector Graphics is an emerging standard for the web for describing two-dimensional graphics in XML.

VECTOR OPERATIONS: The addition and scalar multiplication of vectors satisfy "obvious" properties. There is no need to memorize them. We write $*$ here for multiplication with a scalar but usually, the multiplication sign is left out.

$\vec{u}+\vec{v}=\vec{v}+\vec{u}$	commutativity
$\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$	additive associativity
$\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{0}$	null vector
$r*(s*\vec{v})=(r*s)*\vec{v}$	scalar associativity
$(r+s)\vec{v}=\vec{v}(r+s)$	distributivity in scalar
$r(\vec{v}+\vec{w})=r\vec{v}+r\vec{w}$	distributivity in vector
$1*\vec{v}=\vec{v}$	the one element

LENGTH. The length $|\vec{v}|$ of $\vec{v}$ is the distance from the beginning to the end of the vector.

EXAMPLES. 1) If $\vec{v}=(3,4)$, then $|\vec{v}|=\sqrt{25}=5$. 2) $|\vec{i}|=|\vec{j}|=\vec{k}|=1$, $|\vec{0}|=0$.

UNIT VECTOR. A vector of length 1 is called a **unit vector**. If $\vec{v}\neq\vec{0}$, then $\vec{v}/|\vec{v}|$ is a unit vector.
EXAMPLE: If $\vec{v}=(3,4)$, then $\vec{v}=(2/5,3/5)$ is a unit vector, $\vec{i},\vec{j},\vec{k}$ are unit vectors.

PARALLEL VECTORS. Two vectors $\vec{v}$ and $\vec{w}$ are called **parallel**, if $\vec{v}=r\vec{w}$ with some constant r.

DOT PRODUCT. The **dot product** of two vectors $\vec{v}=(v_1,v_2,v_3)$ and $\vec{w}=(w_1,w_2,w_3)$ is defined as

$$\boxed{\vec{v}\cdot\vec{w}=v_1w_1+v_2w_2+v_3w_3}$$

Remark: in science, other notations are used: $\vec{v}\cdot\vec{w}=(\vec{v},\vec{w})$ (mathematics) $<\vec{v}|\vec{w}>$ (quantum mechanics) v_iw^i (Einstein notation) $g_{ij}v^iw^j$ (general relativity). The dot product is also called **scalar product**, or **inner product**.

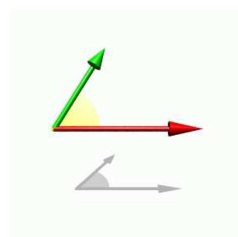
LENGTH. Using the dot product one can express the length of $\vec{v}$ as $|\vec{v}|=\sqrt{\vec{v}\cdot\vec{v}}$.

CHALLENGE. Express the dot product in terms of the length alone.

SOLUTION: $(\vec{v}+\vec{w},\vec{v}+\vec{w})=(\vec{v},\vec{v})+(\vec{w},\vec{w})+2(\vec{v},\vec{w})$ can be solved for $(\vec{v},\vec{w})$.

ANGLE. Because $|\vec{v}-\vec{w}|^2=(\vec{v}-\vec{w},\vec{v}-\vec{w})=|\vec{v}|^2+|\vec{w}|^2-2(\vec{v},\vec{w})$ is by the **cos-theorem** equal to $|\vec{v}|^2+|\vec{w}|^2-2|\vec{v}|\cdot|\vec{w}|\cos(\alpha)$, where α is the angle between the vectors $\vec{v}$ and $\vec{w}$, we get the important formula

$$\boxed{\vec{v}\cdot\vec{w}=|\vec{v}|\cdot|\vec{w}|\cos(\alpha)}$$

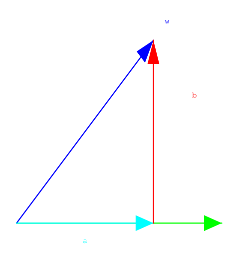


CAUCHY-SCHWARZ INEQUALITY: $|\vec{v}\cdot\vec{w}|\leq|\vec{v}||\vec{w}|$ follows from that formula because $|\cos(\alpha)|\leq 1$.
TRIANGLE INEQUALITY: $|\vec{u}+\vec{v}|\leq|\vec{u}|+|\vec{v}|$ follows from $|\vec{u}+\vec{v}|^2=(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})=\vec{u}^2+\vec{v}^2+2\vec{u}\cdot\vec{v}\leq\vec{u}^2+\vec{v}^2+2|\vec{u}\cdot\vec{v}|\leq\vec{u}^2+\vec{v}^2+2|\vec{u}|\cdot|\vec{v}|=(|\vec{u}|+|\vec{v}|)^2$.

FINDING ANGLES BETWEEN VECTORS. Find the angle between the vectors $(1,4,3)$ and $(-1,2,3)$.
ANSWER: $\cos(\alpha)=16/(\sqrt{26}\sqrt{14})\sim 0.839$. So that $\alpha=\arccos(0.839..)\sim 33^\circ$.

ORTHOGONAL VECTORS. Two vectors are called **orthogonal** if $v\cdot w=0$. The zero vector $\vec{0}$ is orthogonal to any vector. EXAMPLE: $\vec{v}=(2,3)$ is orthogonal to $\vec{w}=(-3,2)$.

PROJECTION. The vector $\boxed{\vec{a}=\text{proj}_{\vec{w}}(\vec{v})=\vec{w}(\vec{v}\cdot\vec{w}/|\vec{w}|^2)}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$.
The **scalar projection** is defined as $\boxed{\text{comp}_{\vec{w}(\vec{v})}=(\vec{v}\cdot\vec{w})/|\vec{w}|}$. (Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$.) The vector $\vec{b}=\vec{v}-\vec{a}$ is called the **component** of $\vec{v}$ orthogonal to the $\vec{w}$-direction.



EXAMPLE. $\vec{v}=(0,-1,1)$, $\vec{w}=(1,-1,0)$, $\text{proj}_{\vec{w}}(\vec{v})=(1/2,-1/2,0)$, $\text{comp}_{\vec{w}}(\vec{v})=1/\sqrt{2}$.