Homework for Wednesday October 3:

- pg 46 number 4
- pgs. 56-58 number 14,20,22
- pgs. 69-74 number 10,16,20,22
- pgs. 82-85 number 16,22,34
- pgs. 90-93 number 10,22,26
- Find all vector functions of time $r(t)$, that obey the differential equation $r'(t) =$ r(t). (Hint: All function f(t) of time that obey $f'(t) = f(t)$ have the form $f(t) = Ce^t$, where C can be any constant.)

Suggested Problems:

- pgs 69-74 number 3, 7, 9, 11, 13, 17, 25, 31
- pg 82 number 5
- pg 71 number 17
- Find the work done by a force $F = (3, 0, 0)$ in moving an object along the line segment from the origin $(0, 0, 0)$ in $\mathbf{R}^3$ to the point $(1, 1, 1)$.

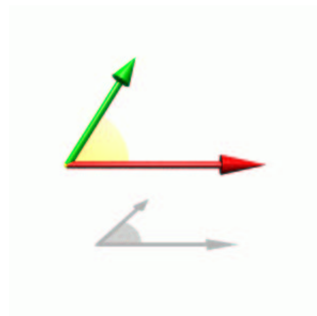
DOT PRODUCT. The **dot product** of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is defined as $v \cdot w = v_1w_1 + v_2w_2 + v_3w_3$. Other notations are $v \cdot w = (v, w)$ or $< v|w >$ (quantum mechanics) or v_iw^i (Einstein notation) or $g_{ij}v^iw^j$ (general relativity). The dot product is also called **scalar product**, or **inner product**.

LENGTH. Using the dot product one can express the length of v as $||v|| = \sqrt{v \cdot v}$.

CHALLENGE. Express the dot product in terms of length only!
SOLUTION: $(v + w, v + w) = (v, v) + (w, w) + 2(v, w)$ can be solved for (v, w).

ANGLE. Because $||v - w||^2 = (v - w, v - w) = ||v||^2 + ||w||^2 - 2(v, w)$ is by the **cos-theorem** equal to $||v||^2 + ||w||^2 - 2||v|| \cdot ||w|| \cos(\phi)$, where ϕ is the angle between the vectors v and w, we have the formula

$$v \cdot w = ||v|| \cdot ||w|| \cos(\phi)$$



FINDING ANGLES BETWEEN VECTORS. Find the angle between the vectors $(1, 4, 3)$ and $(-1, 2, 3)$.
ANSWER: $\cos(\phi) = 16/(\sqrt{26}\sqrt{14}) \sim 0.839$. So that $\phi = \arccos(0.839..) \sim 33°$.

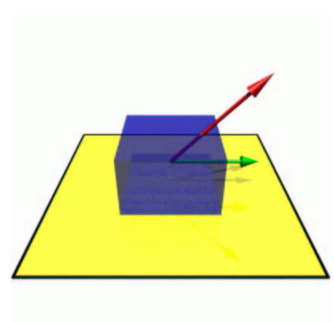
ORTHOGONALITY. Two vectors are **orthogonal** if and only if $v \cdot w = 0$.

PROJECTION. The vector $a = w(v{\cdot}w/||w||^2)$ is called the **projection** of v onto w. Its length $||a|| = (v{\cdot}w/||w||^2)$ is called the **scalar projection**. The vector $b = v - a$ is the **component** of v orthogonal to the w-direction.

WORK. If $r(t)$ is a path and $F(t)$ is the force, then

$$W = \int_a^b r'(t) \cdot F(t)\ dt$$

is the **work** done on the body. It is the antiderivative of the **power** $r'(t) \cdot F(t)$.



148**Watt** $= 148kg\ m^2/s^3$ and is at rest at time $t = 0$. What is the speed at time t?

SOLUTION. The work done until time t is $\int_0^t P dt = Pt$ which is the kinetic energy $mv^2/2$. Therefore $v(t) = \sqrt{2tP/m} = 2\sqrt{t}$. The acceleration is $a = v' = 1/\sqrt{t}$, the jerk $a' = v'' = -3/(2t^{3/2})$. Jerk and acceleration are infinite at $t = 0$!

A PARADOX? A car of mass $m = 100kg$ moving along $r(t) = (x(t), 0, 0)$ accelerates from 0 to $10m/s$. The kinetic energy it gained was $m||r'(t)||^2/2 = 10^2m/2 - 0^2/2 = 5000J$. Now look at this situation in a moving coordinate system $R(t) = (X(t), 0, 0) = (x(t) + vt, 0, 0)$ with $v = -10$. In that coordinate system, the car accelerates from 10 to 20. The energy it gained is $100(20^2/2 - 10^2/2) = 15'000J$. Why this discrepancy in energies? Physical laws should be invariant under **Galileo transformations** $\vec{x} \mapsto \vec{X} = (x + ut, y + vt, z + wt), \dot{\vec{x}} \mapsto \dot{\vec{X}} = (\dot{x} + u, \dot{y} + v, \dot{z} + w)$.

PRODUCT RULE. We have $(v, w)' = (v', w) + (v, w')$ as you can check by expanding both sides.

APPLICATION. If a curve $r(t)$ is located on a sphere, then r' is orthogonal to r. Proof. $0 = \frac{d}{dt}||r(t)||^2 = r' \cdot r + r \cdot r' == 2r' \cdot r = 0$. That means that r' is orthonogal to r.

ENERGY. $K(t) = m||r'(t)||^2/2$ is called the **kinetic energy** of a body moving along $r(t)$ with having mass m. .

ENERGY CONSERVATION. If the energy $K(t)$ is differentiated with respect to time, we get $K'(t) = m(r''(t) \cdot r'(t))$. Therefore, by Newton's law $mr''(t) = F(t)$ and using the fundamental theorem of calculus, we have

$$K(b) - K(a) = \int_a^b K'(t)\, dt = \int_a^b r'(t) \cdot mr''(t)\, dt = \int_a^b r'(t) \cdot F(t)\, dt = W$$

Energy conservation: the energy difference is the amount of work which has been used.

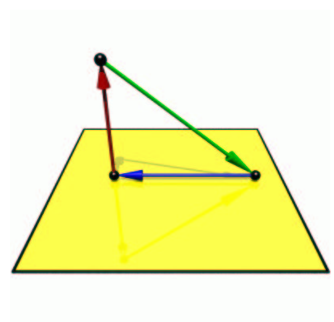
PLANES. A surface $ax + by + cz = d$ in space is called a **plane**. Using the dot product we can say $n \cdot x = d$, where $n = (a, b, c)$ and $x = (x, y, z)$.

NORMAL VECTOR If x, y are two points on the plane, then $v \cdot x = d$ and $v \cdot y = d$ which means $v \cdot (x - y) = 0$. Every vector inside the plane is orthogonal to v, which is called the **normal vector** to the plane.

DISTANCE POINT-PLANE IN SPACE. Let P be a point and $n \cdot x = d$ a plane which contains the point Q. Then

$$|(P - Q) \cdot n|/||n||$$

is the **distance** from P to the plane. A formula worthwile to remember.



DISTANCE POINT-LINE IN THE PLANE. Let P be a point in the plane and $n \cdot x = d$ a **line** which contains a point Q. Then Then $|(P - Q) \cdot n|/||n||$ is the distance from P to the line.

REMARK. Sometimes, distance formulas are given without the absolute sign: $(P - Q) \cdot n/||n||$ is then a **signed distance** which also shows on which side of the plane (rsp. line) we are. That is useful in computer graphics or games, where it can make a difference on which side of of a wall you are.

PROBLEM. A plane has distance $d = 4$ from the origin and is orthogonal to the vector $(2, 4, 5)$. Find the equation of the plane.

SOLUTION. $\vec{n} = (2, 4, 5)$ is a normal vector to the plane. Therefore, the equation has the form $2x + 4y + 5z = \vec{n} \cdot \vec{x} = d$. The point $Q = (d/2, 0, 0)$ is on the plane for example. The distance of $P = (0, 0, 0)$ to the plane is $|(P - Q) \cdot n|/||n|| = d/||n|| = d/\sqrt{29} = 4$ so that the still unknown constant is $d = 4\sqrt{29}$.